Notes on Relativistic Quantum Mechanics

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Note: We are using the West Coast convention, i.e. $+---$ metric signature, and setting $c = 1$ and $\hbar = 1$.

Contents

1 One Particle Systems: Mathematical Formalism

The simplest system to consider is a single particle. The function space used to model quantum-mechanical states is a Hilbert Space $\mathcal{H}$ of square integrable functions on the physical space (denoted by $\mathcal{C}$):

$$L^2(\mathcal{C}) = \left\{ f \;:\; \int_{\mathcal{C}} |f(\overline{x})|^2 d^3\overline{x} < \infty \right\} \tag{1.1}$$

Note that in all fairness, $\mathcal{H}$ can be written in either position coordinates $\overline{x}$ or momentum coordinates $\overline{p}$. The relationship between the position-space and momentum-space is precisely the familiar Fourier transform:

$$\mathcal{F}(f)(\overline{p}) \stackrel{\text{def}}{=} \int e^{i\overline{x}\cdot\overline{p}} f(\overline{x}) d^3\overline{x} \tag{1.2}$$

Despite the change of variables, $\mathcal{F}$ sends $\mathcal{H}$ to itself, so both f and its Fourier transform $\mathcal{F}(f)$ are in $\mathcal{H}$.

Remark 1.1. It should be emphasized that if f is square-integrable, then $e^{i\overline{x}\cdot\overline{p}} f(\overline{x})$ is square-integrable *but not necessarily integrable!* That is, we have no guarantee that $e^{i\overline{x}\cdot\overline{p}} f(\overline{x}) \in L^1(\mathcal{C})$.

To define the Fourier transform on $\mathcal{H}$, we should first define it on some suitably nice subspace of $\mathcal{H}$ (e.g. the space of smooth functions with "compact support" — i.e. they are zero outside of a compact subset of their domain). Then we observe that the Fourier transform is an isometry (up to some scale factor) on our nice subspace, so we extend this isometry from our nice subspace to all of $\mathcal{H}$.

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We represent the observables by operators. More relevantly, the position operators $\widehat{x}_m$ and momentum operators $\widehat{p}_m$ are represented in position-space by multiplication by the coordinate fuhnctions x_m and the partial derivative operators $-i\partial_m$ (respectively). Observe also that the Fourier transform converts multiplication by x_m on functions of $\overline{x}$ into the differential operators $-i\partial_m$ on functions of $\overline{p}$:

$$\mathcal{F}(x_m f)(\overline{p}) = -i\partial_m \mathcal{F}(f)(\overline{p}). \tag{1.3}$$

The natural question to ask is "What are the eigenstates of these operators?" Well, in position space, we find the position eigenstates are just delta functions

$$\begin{aligned}(\widehat{x}_m \delta_{\overline{q}})(\overline{x}) &= \widehat{x}_m \delta(\overline{x} - \overline{q}) && (1.4a)\\ &= q_m \delta(\overline{x} - \overline{q}) && (1.4b)\\ &= (q_m \delta_{\overline{q}})(\overline{x}) && (1.4c)\end{aligned}$$

Similarly, for the eigenstates of the momentum operators $\widehat{p}_m$, we see that the eigenstates in position-space are $e_{\overline{p}}(\overline{x})$:

$$\begin{aligned}(\widehat{p}_m e_{\overline{p}})(\overline{x}) &= -i\partial_m \exp(i\overline{p}\cdot\overline{x}) && (1.5a)\\ &= p_m \exp(i\overline{p}\cdot\overline{x}) && (1.5b)\\ &= (p_m e_{\overline{p}})(\overline{x}). && (1.5c)\end{aligned}$$

But we have just two minor problems: 1. neither $\widehat{x}_m$ nor $\widehat{p}_n$ act on all of $\mathcal{H}$, and 2. $\mathcal{H}$ doesn't contain the eigenstates of either operators. We can solve the first problem fairly easily — we'll consider the subspace $S \subseteq \mathcal{H}$ where the operators map S to itself. Similarly, we resolve the second problem by defining the kets as elements of S^*, the space of continuous antilinear functionals on S. Since $\widehat{p}_n$ acts on all functions of S, these functions must be infinitely differentiable, and so S^* will contain the δ-functions and all their derivatives. Similarly, by taking the Fourier Transform, since $\widehat{x}_m$ acts on S, it follows that S^* will contain exponential functions $\exp(i\overline{p}\cdot\overline{x})$.

Instead of a single Hilbert space, we end up with a triple

$$S \subseteq \mathcal{H} \subseteq S^* \tag{1.6}$$

The physical states live in S, and the operator eigenstates live in S^*. With appropriate demands on the space S, this triple ends up being a *Rigged Hilbert Space* [de 05] [Mad06]. In this context "Rigged" *is not* in the sense of "This game is rigged" but rather in the sense of "equipped" — like how a boat is "rigged" or "equipped to sail".

In fact, the triple $S \subseteq \mathcal{H} \subseteq S^*$ is a rigged Hilbert space if S is a nuclear subspace of $\mathcal{H}$. See Gelfand [GfS64] or Maurin [Mau68] for rigorous details about the notion of nuclear spaces. We'll discuss one such criteria for S to be nuclear. Specifically,

1. there exists a countable family $\|\cdot\|_k$ of norms on S with respect to which convergence is defined by
$$f_n \to f \quad\iff\quad \|f_n - f\|_k \to 0\ \ \forall k \geq 0; \tag{1.7}$$
2. S is complete with respect to this notion of convergence; and
3. there exists a Hilbert-Schmidt operator on S with a continuous inverse.

We'll leave the interested reader to refer to the cited sources.

In a rigged Hilbert Space we have eigenfunction expansions. More precisely, consider a state $|f\rangle$ represented by the function f, let $|\overline{x}\rangle$ be the position eigenstate represented by the distribution $\delta_{\overline{x}}$. We assume the relationship between the functions and the kets is such that

$$f(\overline{x}) = \langle\overline{x}|f\rangle. \tag{1.8}$$

We can then expand the state $|f\rangle$ in terms of the position eigenstate $|\overline{x}\rangle$ which should be of the form

$$|f\rangle = \int |\overline{x}\rangle \langle\overline{x}|f\rangle \, d^3\overline{x} = \int f(\overline{x}) \, |\overline{x}\rangle \, d^3\overline{x}. \tag{1.9}$$

The conditions on S in a rigged Hilbert Space ensure that $f(\overline{x})|\overline{x}\rangle$ is integrable for all $f \in S$.

2 One Particle Systems: Physical Aspects

We're interested in a toy model of relativistic quantum mechanics, so we begin with a single particle. All we really need, truth be told, is a state space plus a Hamiltonian operator. We should remember, from Special Relativity, the energy-momentum four-vector $\widehat{p}_\mu$ has as its time component the Hamiltonian $\widehat{p}_0 = \widehat{H}$. For convenience, we'll work in the momentum space with the momentum operator eigenbasis

$$\widehat{p}_m|\overline{k}\rangle = k_m|\overline{k}\rangle \tag{2.1}$$

We assume the states are normalized thus

$$\langle\overline{k}|\overline{k}'\rangle = \delta^{(3)}(\overline{k} - \overline{k}'). \tag{2.2}$$

This means that the length of a ket is undefined. It is, nonetheless, a normalization suitable for integration over momentum. As an added bonus, we also get the resolution of the identity

$$\mathbf{1} = \int |\overline{k}\rangle \, \langle\overline{k}| \, d^3\overline{k} \tag{2.3}$$

Since energy-momentum is a four-vector, we demand that

$$\widehat{p}^\mu\widehat{p}_\mu = \widehat{H}^2 - |\widehat{p}_m\widehat{p}^m| \tag{2.4}$$

needs to be constant on the orbits of the Poincaré group. Further if $|\overline{k}\rangle$ and $|\overline{k}\,'\rangle$ are two states of a single particle, then there exists a Lorentz boost from one to the other (up to scale). Hence we assume the existence of a scalar quantity μ (the particle mass) which satisfies

$$(\widehat{H}^2 - \widehat{p}_m\widehat{p}^m)|\overline{k}\rangle = \mu^2|\overline{k}\rangle \tag{2.5}$$

This implies that the Hamiltonian operator $\widehat{H}$ is diagonal in the momentum eigenbasis (i.e. the basis of eigenstates of the momentum operator):

$$\widehat{H}|\overline{k}\rangle = \left(\|\overline{k}\|^2 + \mu^2\right)^{1/2} |\overline{k}\rangle \tag{2.6}$$

The eigenvalues of the Hamiltonian operator come up enough times that we introduce the shorthand for it:

$$\omega(\overline{k}) \stackrel{\text{def}}{=} \left(\|\overline{k}\|^2 + \mu^2\right)^{1/2} \tag{2.7}$$

(This should be vaguely reminiscent of the de Broglie relations $E = \hbar\omega$.)

Remark 2.1. Observe that this entire scheme we've devised is equivalent to taking the limit of the state space for a cube of side L under periodic boundary conditions, i.e. the particle in a box situation. In such a cube, we should recall the spectrum of the momentum operator is discrete and the normalization is given by the Kronecker delta:

$$\overline{k} = \frac{2\pi}{L}(n_x, n_y, n_z), \quad \text{and} \quad \langle\overline{k}|\overline{k}\,'\rangle = \delta_{\overline{k},\overline{k}\,'} \tag{2.8}$$

This observation is taken advantage of when deriving the differential transition probability per unit time for particle scattering.

3 Unitary Representation of Poincaré Group

3.1 Action of Translation on States

The Lorentz transformation is usually "represented" by a matrix Λ which, when written explicitly, is

$$(\Lambda x)^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{3.1}$$

where Einstein convention is used (implicit sum over ν occurs). We have that the matrix $\Lambda^{\mu}{}_{\nu}$ must satisfy

$$\Lambda^{\lambda}{}_{\mu} \Lambda_{\lambda\nu} = \eta_{\mu\nu} \tag{3.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric (metric for flat spacetime).

Now, the Poincaré group is the set of Lorentz transformations and space-time translations, so the element of the group would be (Λ, a) such that

$$x^{\mu} \to y^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}. \tag{3.3}$$

The group multiplication law is then just

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2). \tag{3.4}$$

We are interested in irreducible unitary representations $U(\cdot)$ of our group, all we need to worry about are the generators.

The translations, rotations, and boosts of the Poincaré group must act on the space of states. A Poincaré group element g acts as a unitaruy operator $U(g)$ on the state space. The action must satisfy a multipication condition

$$U(gh) = U(g)U(h) \tag{3.5}$$

for all g, h in the Poincaré group.

Translation of spacetime by a four-vector a^{μ} is defined by

$$\Delta_a(x) = x + a. \tag{3.6}$$

Translation of a state ψ, on the other hand, should be moving the graph by a. This means that $\Delta_a \psi(x) = \psi(x - a)$. The unitary representation $U(\Delta_a)$ of Δ_a must thus be defined by

$$U(\Delta_a)|\psi\rangle = |\Delta_a \psi\rangle. \tag{3.7}$$

We'd like to find an expression for $U(\Delta_a)$ in terms of the energy-momentum four-vector $\widehat{p}_{\mu}$.

Evolution in time is translation of the observer forward in time, or (equivalently) translation of the system backwards in time:

$$\exp(-it\widehat{H})|\psi(x)\rangle = |\psi(x_0 + t, \overline{x})\rangle. \tag{3.8}$$

Let $\tau^{\mu} = (-t, \vec{0})$ be a four-vector, then we can rewrite our translation in time as

$$\exp(i\tau^{\mu}\widehat{p}_{\mu})|\psi\rangle = |\Delta_{\tau}\psi\rangle. \tag{3.9}$$

Lorentz invariance implies that this equation is true whenever τ is timelike, and the additivity of translations then shows this to be true for all four-vectors τ. From this definition of $U(\Delta_a)$ we can therefore deduce that

$$U(\Delta_a) = \exp(ia^{\mu}\widehat{p}_{\mu}). \tag{3.10}$$

Although this unitary representation is derived in the position-space formulation of quantum mechanics, it works equally well in the momentum-space formulation. We can deduce that the unitary representation of translations on momentum eigenstates is given by

$$U(\Delta_a)|\overline{k}\rangle = \exp(ia^{\mu}\widehat{p}_{\mu})|\overline{k}\rangle = \exp(ia^{\mu}k_{\mu})|\overline{k}\rangle \tag{3.11}$$

where $k_0 = \omega(\overline{k})$.

Remark 3.1. Recall Taylor's theorem in real analysis can be formulated as

$$f(x+h) = \left(\sum_{n=0}^{\infty} h^n \frac{d^n}{dx^n}\right) f(x) = \exp\left(h\frac{d}{dx}\right) f(x) \tag{3.12}$$

which should look familiar: we just deduced the unitary representation of spacetime translations should be

$$\exp(i\tau^{\mu}\widehat{p}_{\mu})|\psi\rangle = U(\Delta_{\tau})|\psi\rangle. \tag{3.13}$$

If we don't distinguish $|\psi\rangle$ from $\psi(x)$, we see that Taylor's theorem guarantees our representation to be of spacetime translations.

3.2 Action of the Lorentz Group

The space of particle states is three dimensional. The energy k_0 of a particle with momentum $\overline{k}$ is constrained by

$$k_0 \geq 0 \tag{3.14}$$

and

$$k^2 = k_{\mu}k^{\mu} = \mu^2. \tag{3.15}$$

Therefore the possible energy-momentum vectors lie on a hyperbolic sheet in k-space, the mass hyperboloid. We need an integration measure on this hyperboloid if we want to do Lorentz invariant computations.

Let $\theta(t)$ be the Heaviside step function

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases} \tag{3.16}$$

Define an integration $d\lambda(k)$ on the positive hyperboloid as follows:

$$d\lambda(k) \stackrel{\text{def}}{=} d^4k\delta(k^2-\mu^2)\theta(k_0) \tag{3.17}$$

The Lebesgue measure d^4k is Lorentz invariant due to the Lorentz transformation having unit determinant. Here since $k^2-\mu^2$ is Lorentz invariant, the δ function is Lorentz invariant. Similar reasoning holds for $\theta(k_0)$ being Lorentz invariant.

We can take advantage of the identity

$$\delta(f(k)) = \sum_{\{k:f(k)=0\}} \frac{1}{\|f'(k)\|}\delta(k) \tag{3.18}$$

and the fact that

$$\begin{aligned} k^2-\mu^2 &= (k_0^2 - \|\overline{k}\|^2) - \mu^2 && (3.19\text{a}) \\ &= k_0^2 - (\|\overline{k}\|^2 + \mu^2) && (3.19\text{b}) \\ &= k_0^2 - \omega(\overline{k})^2 && (3.19\text{c}) \\ &= (k_0 - \omega(\overline{k}))(k_0 + \omega(\overline{k})) && (3.19\text{d}) \end{aligned}$$

to deduce that

$$\begin{aligned} \delta(k^2-\mu^2)\theta(k_0) &= \delta\left((k_0 - \omega(\overline{k}))(k_0 + \omega(\overline{k}))\right)\theta(k_0) && (3.20\text{a}) \\ &= \frac{1}{2\omega(\overline{k})}(\delta(k_0 - \omega(\overline{k}))\theta(k_0) + \delta(k_0 + \omega(\overline{k}))\theta(k_0)) && (3.20\text{b}) \\ &= \frac{1}{2\omega(\overline{k})}\delta(k_0 - \omega(\overline{k}))\theta(k_0) && (3.20\text{c}) \end{aligned}$$

since $\delta(k_0 + \omega(\overline{k}))$ requires $k_0 < 0$ which then demands that $\theta(k_0) = 0$, so that term drops out completely. Observe that this means we can effectively eliminate k_0 from any integral with respect to $\omega(\overline{k})$ as follows:

$$\int f(k) d\lambda(k) = \int f(k) \left(\frac{\delta(k_0 - \omega(\overline{k}))}{2\omega(\overline{k})} \theta(k_0) d^3\overline{k} dk_0 \right) \tag{3.21a}$$

$$= \int f\left(\omega(\overline{k}), \overline{k}\right) \frac{d^3\overline{k}}{2\omega(\overline{k})} \tag{3.21b}$$

This integral and the arbitrary function f are commonly eliminated from this result, leaving an equality of measures

$$d\lambda(k) = \frac{d^3\overline{k}}{2\omega(\overline{k})} \tag{3.22}$$

and

$$k_0 = \omega(\overline{k}). \tag{3.23}$$

If we define Lorentz-normalized kets $|k\rangle$ by

$$|k\rangle = \left(2\omega(\overline{k})\right)^{1/2} (2\pi)^{3/2} |\overline{k}\rangle \tag{3.24}$$

with $k_0 = \omega(\overline{k})$, then the new normalization conditions is

$$\langle k|k'\rangle = 2\omega(\overline{k})(2\pi)^3 \delta^{(3)}(\overline{k} - \overline{k}') \tag{3.25}$$

and the resolution of the identity is based on the Lorentz invariant measure:

$$\mathbf{1} = \int |k\rangle\langle k| \frac{d^3\overline{k}}{(2\pi)^3 2\omega(\overline{k})}. \tag{3.26}$$

With these Lorentz-normalized states, we can define the unitary representation of the Lorentz group simply:

Theorem 3.2. *If we define $U(\Lambda)$ by $U(\Lambda)|k\rangle = |\Lambda k\rangle$, then U is a unitary representation of the Lorentz group.*

Proof. The multiplications property $U(\Lambda\Lambda') = U(\Lambda)U(\Lambda')$ follows immediately from definition. To show that the representation is unitary, we use the resolution of the identity

$$U(\Lambda)U(\Lambda)^\dagger = \int U(\Lambda)|k\rangle\langle k|U(\Lambda)^\dagger \frac{d^3\overline{k}}{(2\pi)^3 2\omega(\overline{k})} \tag{3.27a}$$

$$= \int |\Lambda k\rangle\langle \Lambda k| \frac{d^3\overline{k}}{(2\pi)^3 2\omega(\overline{k})} \tag{3.27b}$$

$$= \mathbf{1} \tag{3.27c}$$

since the measure is Lorentz-invariant. □

It is mildly surprising that $U(\Lambda)$ defined in our theorem is a unitary operator due to $|k\rangle$ and $|\Lambda k\rangle$ appear to have different lengths when Λ is a boost. *However,* $\delta^{(3)}(0)$ is undefined, so the normalization of the kets does not determine a length. We regard the uniformly unlocalized state described by $|k\rangle$ as *unphysical.* The physical states have the form

$$|\psi\rangle \stackrel{\text{def}}{=} \int \psi(\overline{k})|k\rangle \frac{d^3\overline{k}}{(2\pi)^3 2\omega(\overline{k})} \tag{3.28}$$

where the measure is structured so $\langle k|\psi\rangle = \psi(\overline{k})$. We can check that the length of $|\psi\rangle$ is well defined whenever $\psi(\overline{k})$ is square-integrable and that our definition of $U(\Lambda)$ makes the representation unitary on the space of physical states.

3.3 Representing the Poincaré Group

We really want to find a unitary representation of the Poincaré group, which is the Lorentz group plus spacetime translations (i.e. rotations, Lorentz boosts, and space-time translations). We have the representation condition $U(gh) = U(g)U(h)$ must hold for all g, h in the Poincaré group. We've seen what happens when both g, h are in the Lorentz group, and when both g, h are space-time translations. We now need to ask: what happens when one is a translation and the other is a boost?

We can uniquely factor any element g of the Poincaré group as the product

$$g = \Delta_a \Lambda \tag{3.29}$$

where Λ is in the Lorentz group, and Δ_a is a translation. Multiplication in the Poincaré group depends on multiplication in the Lorentz group and addition of translations through an interchange in the order of the two facts:

$$\begin{aligned} gh &= \Delta_a \Lambda \Delta_b M && \text{(3.30a)} \\ &= \Delta_a (\Lambda \Delta_b \Lambda^{-1}) \Lambda M && \text{(3.30b)} \\ &= \Delta_a \Delta_{\Lambda b} \Lambda M && \text{(3.30c)} \end{aligned}$$

where we have used the identity

$$\Lambda \Delta_b \Lambda^{-1} = \Delta_{\Lambda b} \tag{3.31}$$

a relation trivially verified when we act on a 4-vector x.

Our definition of U so far covers translations and Lorentz group elements only; when we extend to the Poincaré group, we do so through the definition

$$U(\Delta_a \Lambda) \stackrel{\text{def}}{=} U(\Delta_a) U(\Lambda) \tag{3.32}$$

We can now see that U is a representation of the Poincaré group if and only if U preserves the action $\Lambda \Delta_b \Lambda^{-1} = \Delta_{\Lambda b}$ of Lorentz group elements on translations:

$$\begin{aligned} & U(\Delta_a \Lambda) U(\Delta_b M) = U(\Delta_a \Delta_{\Lambda b} \Lambda M) && \text{(3.33a)} \\ \iff & U(\Delta_a) U(\Lambda) U(\Delta_b) U(M) = U(\Delta_a) U(\Delta_{\Lambda b}) U(\Lambda) U(M) && \text{(3.33b)} \\ \iff & U(\Lambda) U(\Delta_b) = U(\Delta_{\Lambda b}) U(\Lambda) && \text{(3.33c)} \\ \iff & U(\Lambda) U(\Delta_b) U(\Lambda)^{\dagger} = U(\Delta_{\Lambda b}) && \text{(3.33d)} \end{aligned}$$

We verify the final condition by evaluating both sides on some test state $|k\rangle$. From the right hand side, we have

$$U(\Delta_{\Lambda b})|k\rangle = \exp(i \Lambda b^{\mu} k_{\mu})|k\rangle \tag{3.34}$$

and from the left hand side

$$\begin{aligned} U(\Lambda) U(\Delta_b) U(\Lambda)^{\dagger} |k\rangle &= U(\Lambda) U(\Delta_b) |\Lambda^{-1} k\rangle && \text{(3.35a)} \\ &= U(\Lambda) \exp(i b^{\mu} \Lambda_{\mu}{}^{\nu} k_{\nu}) |\Lambda^{-1} k\rangle && \text{(3.35b)} \\ &= \exp(i b^{\mu} \Lambda_{\mu}{}^{\nu} k_{\nu}) |k\rangle. && \text{(3.35c)} \end{aligned}$$

The equality of the two sides follows from the Lorentz-invariance of the inner product.

We can now summarize our results of U in the following theorem:

Theorem 3.3. *The map U from the Poincaré group to operators on the state space defined by*

$$\begin{aligned} U(\Delta_a)|k\rangle &= e^{i a^{\mu} k_{\mu}} |k\rangle && \text{(3.36a)} \\ U(\Lambda)|k\rangle &= |\Lambda k\rangle && \text{(3.36b)} \\ U(\Delta_a \Lambda) &= U(\Delta_a) U(\Lambda) && \text{(3.36c)} \end{aligned}$$

is a unitary representation of the Poincaré group.

The unitary representation U is often boasted to successfully combines the principle (as represented by the Poincaré group) with the principles of quantum mechanics (as represented by unitary operators and state-space formalisms). This combined structure of a one-particle state space provides the foundation for the many-particle state space used in all quantum field theories.

4 Notes on a Position Operator

The astute reader would probably have realized by now we "implemented" relativity in the momentum space. The question that naturally presents itself is "Why not try to implement relativity in position-space, as we usually do when introducing relativity classically?" In this section, we'll answer that question.

The short answer is that it turns out to be inconsistent. We can sketch out the general scheme and its problem in this paragraph too. Consider putting a particle (of mass m) into a box whose sides are small compared to the Compton wavelength λ, then the uncertainty in position satisfies

$$\Delta x \lll \lambda \tag{4.1}$$

and the uncertainty in momentum satisfies

$$\Delta p \ggg m. \tag{4.2}$$

But this makes the range of energies so large that pair production becomes possible. Hence, from first principles, the position of a one-particle system is not so well defined. We'll show (slightly more rigorously) that the notion of Lorentz causality is violated by measuring the position operator.

We first set up the axioms for (properties satisfied by) the position operator $\widehat{x}^m$. We want:

Axiom 1 $\widehat{x} = \widehat{x}^\dagger$ (i.e. it's self-adjoint, so it has real eigenvalues);

Axiom 2 If Δ_a is a spatial translation, then $U(\Delta_a)^\dagger \widehat{x}^m U(\Delta_a) = \widehat{x}^m + a^m$

Axiom 3 If R is a spatial rotation, then $U(R)^\dagger \widehat{x}^m U(R) = R^{m'}{}_m \widehat{x}^m$.

From axiom 2 and $U(\Delta_a) = \exp(ia^m \widehat{P}_m)$, we deduce

$$e^{ia^m \widehat{p}_m} \widehat{x}^n e^{-ia^m \widehat{p}_m} = \widehat{x}^n + a^n. \tag{4.3}$$

(Note that the sign in the exponent reflects the relationship between the Lorentz dot product and the Euclidean dot product of 3-vectors.) Differentiating both sides with respect to the component a^n of a then setting $a^m = \vec{0}$, we recover the usual commutation relations:

$$[i\widehat{p}_n, \widehat{x}^m] = \delta^m{}_n. \tag{4.4}$$

Remark 4.1. The position operator is "essentially" unique. That is to say, it's unique up to unitarity. Suppose we have two operators $\widehat{y}^{m'}$, $\widehat{x}^m$ that satisfy our axioms. We'll demonstrate that there exists a unitary operator U such that $\widehat{y}^m = U^\dagger \widehat{x}^m U$.

Assume that $\widehat{y}^m$ is the position operator with respect to the basis $|\overline{k}\rangle$. The canonical commutation relations eq (4.4) shows that $\widehat{p}_n$ commutes with $\widehat{x}^n - \widehat{y}^n$. Therefore, supposing any operator can be expressed using $\widehat{x}^m$ and $\widehat{p}_n$, we have

$$\widehat{y}^m = \widehat{x}^m + f^m(\widehat{p}). \tag{4.5}$$

Axiom 3 however implies that $f^m(\widehat{p}) \sim g(\|\widehat{p}\|^2)\widehat{p}_m$. This vector-valued function of a vector has zero curl and thus may be written as the gradient of a scalar function. Lets denote this scalar function as $\phi(\|\widehat{p}\|^2)$ where

$$\phi(\xi) \stackrel{\text{def}}{=} \int_0^\xi g(\eta)\, d\eta \tag{4.6}$$

If we define our new kets using a unitary operator U to change phases

$$|\overline{k}\rangle_{\text{new}} \stackrel{\text{def}}{=} U|\overline{k}\rangle \stackrel{\text{def}}{=} \exp(-i\phi(\|\overline{k}\|^2))|\overline{k}\rangle, \tag{4.7}$$

then since

$$\langle\psi'|\widehat{y}^m|\psi\rangle = {}_{\text{new}}\langle\psi'|U\widehat{y}^m U^\dagger|\psi\rangle_{\text{new}} \tag{4.8}$$

the new operators are $U\widehat{y}^m U^\dagger$. Writing $U^\dagger = e^A$ we find

$$\begin{aligned} U\widehat{y}^m U^\dagger &= U\left(U^\dagger \widehat{y}^m + [\widehat{y}^m, U^\dagger]\right) && (4.9a)\\ &= \widehat{y}^m + U[\widehat{y}^m, 1 + A + \frac{1}{2}A^2 + \cdots] && (4.9b)\\ &= \widehat{y}^m + U(1 + A + \frac{1}{2}A^2 + \cdots)[\widehat{y}^m, A] && (4.9c)\\ &= \widehat{y}^m + [\widehat{y}^m, i\phi(\|\widehat{p}\|^2)] && (4.9d)\\ &= \widehat{y}^m - g(\|\widehat{p}\|^2)\widehat{p}_m && (4.9e)\\ &= \widehat{x}^m. && (4.9f) \end{aligned}$$

We therefore conclude that any two sets of position operators $\widehat{x}^m$, $\widehat{y}^n$ are related by a change of basis. We also note since U is a function of the momentum operators, the new momentum operators $U\widehat{p}_m U^\dagger$ are precisely the old ones $\widehat{p}_m$. This shows that the axioms determining the position operator uniquely up to a choice of phase in the momentum eigenstates, and this concludes our remark.

The simplest inconsistency emerges when we consider a state initially localized at the origin and see whether it can be detected outside the forward lightcone of the origin.

Suppose we have a position operator $\widehat{x}^m$. Let $|\overline{x}\rangle$ be a basis of position eigenstates. Then, form our knowledge of nonrelativistic quantum mechanics, we can choose the normalization of these kets to be such that

$$\langle\overline{x}|\overline{k}\rangle = \exp(i\overline{x}\cdot\overline{k}). \tag{4.10}$$

Now consider the evolution $|\psi\rangle$ of a state $|\psi_0\rangle$ initially localized at the origin:

$$\psi_0(\overline{x}) \stackrel{\text{def}}{=} (2\pi)^3\delta^{(3)}(\overline{x}) \;\Rightarrow\; \widehat{\psi}_0(\overline{k}) = 1 \;\Rightarrow\; |\psi_0\rangle = \int |k\rangle d^3\overline{k}, \tag{4.11}$$

where $\widehat{\psi}_0$ is the Fourioer transform of ψ_0. The evolution of this state is given by:

$$\begin{aligned} \psi(t,\overline{x}) &= \langle\overline{x}|e^{-iHt}|\psi_0\rangle && (4.12a)\\ &= \int \langle\overline{x}|e^{-iHt}|\overline{k}\rangle d^3\overline{k} && (4.12b)\\ &= \int \langle\overline{x}|e^{-i\omega(\overline{k})t}|\overline{k}\rangle d^3\overline{k} && (4.12c)\\ &= \int e^{-i\omega(\overline{k})t} e^{i\overline{x}\cdot\overline{k}} d^3\overline{k}. && (4.12d) \end{aligned}$$

If the theorey is relativistic, then a state initially localized at the origin should have zero amplitude outside the lightcone (otherwise, there is a positive probability that something could travel faster than light). We therefore proceed to estimate $\psi(t,\overline{x})$ outside the light cone. Using spherical coordinates, letting $k = \|\overline{k}\|$, $r = \|\overline{x}\|$, we find that

$$\begin{aligned} \psi(t,\overline{x}) &= \int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\phi \int_0^\infty k^2 e^{-it\sqrt{k^2+\mu^2}} e^{ikr\cos\theta} dk && (4.13a)\\ &= \frac{2\pi}{ir}\int_0^\infty k e^{-it\sqrt{k^2+\mu^2}}(e^{ikr} - e^{-ikr}) dk && (4.13b) \end{aligned}$$

$$= \frac{2\pi}{ir} \int_{-\infty}^{\infty} k e^{-it\sqrt{k^2+\mu^2}} e^{ikr} dk. \tag{4.13c}$$

We can use complex analysis to evaluate this integral when $r > t$, we deform the contour of integration from $\mathbb{R}$ to the first principal branch cut from $i\mu$ to $i\infty$. Substituting $k = iz$, we find

$$\psi(t, \overline{x}) = \frac{4\pi i}{r} \int_{\mu}^{\infty} z \sinh(t\sqrt{z^2 - \mu^2}) e^{-zr} dz \tag{4.14}$$

which is clearly nonzero.

Remark 4.2. The integral we've been manipulating is actually divergent. This is a consequence of the extreme nature of the initial state $|\psi_0\rangle$. If we had started with a physical state instaead of a position eigenstate, there would be no convergence problem. The moral of the story is to treat integrals which arise in such situations as defining distributions.

The outcome is that a position operator is inconsistent with relativity. This compels us to find another way of modeling localization of events. In field theory, we do this by making observable operators dependent on position in spacetime.

5 Conclusion

We've reviewed some notions from quantum mechanics, such as the Rigged Hilbert Space and using unitary operators for observables. When using representation theory, we need a unitary representation of a group for use in quantum theory.

We've introduced various aspects of making quantum mechanics relativistic. The main approach is to take advantage of the fact that special relativity is basically "just" the Poincaré group. We then proceeded to find a unitary representation of the Lorentz group and the group of spacetime translations, then combined them in a suitably nice way.

We've considered the situation of making the position operator relativistic, and concluded after a few naive attempts that it wouldn't work.

The interested reader is free to peruse the resources cited in the bibliography for further reading (specifically, the notion of measurement relative to an observer is tackled beautifully in Gambini and Porto [GP02]).

References

[Boh] A. Bohm. "The Rigged Hilbert Space and Quantum Mechanics." CPT-205-TEXAS.

[Cos96] R. Cosgrove. "A Single Particle Interpretation of Relativistic Quantum Mechanics." `arXiv:gr-qc/9602051`.

[de 05] R. de la Madrid. "The role of the rigged Hilbert space in Quantum Mechanics." *EUR.J.PHYS.* **26** (2005) 287. `quant-ph/0502053`.

[GfS64] I. Gel-fand, G. Shilov. *Generalized functions*, volume 4. Academic Press (1964).

[GP02] R. Gambini, R. A. Porto. "Relational Reality in Relativistic Quantum Mechanics." *Phys. Lett.* **A294** (2002) 129–133. `doi:10.1016/S0375-9601(02)00043-9`. `arXiv:quant-ph/0105146`.

[Mad06] R. de la Madrid. "The rigged Hilbert space of the algebra of the one- dimensional rectangular barrier potential." *J. Phys.* **A37** (2006) 8129–8157. `doi:10.1088/0305-4470/37/33/011`. `arXiv:quant-ph/0407195`.

[Mau68] K. Maurin. *Generalized Eigenfunction Expansions and Unitary Representations of Topological Groups*. Polish Scientific Publishers (1968).

[Str04] F. Strocchi. "Relativistic quantum mechanics and field theory." *Found. Phys.* **34** (2004) 501–527. `doi:10.1023/B:FOOP.0000019625.30165.35`. `arXiv:hep-th/0401143`.

[Tic99] R. Ticciati. *Quantum Field Theory for Mathematicians*. Cambridge University Press (1999).