Quantum Gravity

Alex Nelson*

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Introduction

These are my notes on quantum gravity taken from Steve Carlip's course during Spring Quarter of 2010 at UC Davis. Lectures were on Wednesdays and Fridays. Any errors or typos are mine. I have opted to include the inline citations, which Carlip gave in class, and collected them in the end in the references section.

I have tried to correct some small idiosyncracies in my notes, like referring to the wave functional as a wave function.

Lecture 1.

The first paper on quantum gravity was written by Rosenfeld in 1930.[1]

A small aside on if gravity needs to be quantized. "Well everything else is [quantized]." True, but gravity is slightly different. The proper answer is: *we don't know for certain but it seems likely.* Lets consider a few thought experiments.

We will try to cover the collapse of the wave function without getting into what it really means. Let us ask two questions:

(1) Does gravity collapse the wave function?
(2) Do other measurements collapse the wave function?

There are four possible answers.

Answer 1: No, No. This is the Everett interpretation of quantum mechanics. This is fine if everything is quantum mechanical, but what if gravity is not quantum mechanical? A classical gravitational field coupled to the quantum mechanical matter results in observable inconsistencies.

- Don N. Page and C.D. Geilker, "Indirect Evidence for Quantum Gravity". *Phys. Rev. Lett.* **47** (1981) pp.979 *et seq.* doi:10.1103/PhysRevLett.47.979

Page and Geilker experiment testing if gravity is classical and matter is quantum mechanical.

Answer 2: No, Yes. The paper for this perspective:

- Kenneth Eppley and Eric Hannah, "The necessity of quantizing the gravitational field". *Foundations of Physics* **7** (1977) pp.51–68 doi:10.1007/BF00715241

Eppley and Hannah argue if this were the case, we could send information faster than light. Their argument is a tad elaborate.

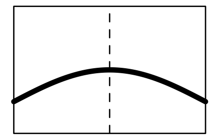
*This is a page from https://pqnelson.github.io/notebk/
Compiled: November 9, 2022 at 3:20pm (PST)

[1] I believe this is, in fact, two papers by Leon Rosenfeld:

(1) "Zur Quantelung der Wellenfelder", *Ann.Phys.* **397** (1930) 113–152. An English translation may be found thanks to D. Salisbury, Max Planck Institute for the History of Science, Preprint 381 (2009) https://pure.mpg.de/rest/items/item_2274368_1/component/file_2274366/content.
(2) "Über die Gravitationswirkungen des Lichtes". *Z. Phys.* **65** (1930) 589–599.

The curious reader may peruse Peruzzi and Rocci's "Tales from the prehistory of Quantum Gravity. Léon Rosenfeld's earliest contribution" arXiv:1802.08878 for a summary of Rosenfeld's contributions to quantum gravity.

Consider a particle in a box symmetric in the middle. We lower some barrier in the middle (the dashed line to the right), split the box in two. Send one to Pluto, the other remains here. Measure the gravitational field. The measured field shouldn't be that of a whole electron since that violates the conservation of energy, and such a violation is bad. We are assuming that gravity is classical, so both observers should measure the gravitational field for half of an electron. Open the box [on Earth]. If the electron is present, the wave function collapses, and information instantaneously changes — the gravitational field of Pluto's box *instantaneously* changes. That's bad.



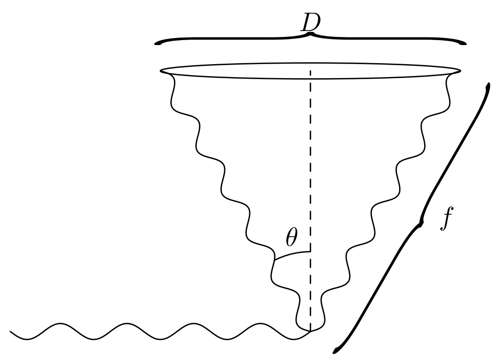
What if we try to weaken causality? Well, causality is either there or not, it's like pregnancy.

One may be able to weasel out of it by supposing that measurements may be generalized a bit.

Answer 3: Yes, No. This is Roger Penrose's idea. We modify Schrodinger's equation to include some "weak nonlinearities" from gravity.

Answer 4: Yes, Yes. Gravity — albeit classical — causes collapse of the wave function and measurement does as well. This leads to violation of uncertainty, or the conservation of energy(?).

Example (Heisenberg microscope)**.** Consider a microscope and an electron some distance f from the lense. We shine some photon to see the electron. We ignore factors and use small-angle approximations. Also we set $c = 1$.



Lets look at the uncertainty in momentum. The electron receives momentum from the collision of the photon with it. Suppose the energy of the electron is E. We have

$$\Delta p_x \sim E\sin(\theta). \tag{1.1}$$

What about the uncertainty in position? This comes from the diffraction limit, we can approximate

$$\theta_c \sim \lambda/D, \tag{1.2}$$

we can find the exact calculations from Jackson [*Classical Electrodynamics*]. We have

$$\Delta x \sim f\theta_c \sim (f\lambda/D) \sim \lambda/\theta. \tag{1.3}$$

So we find

$$\Delta x\,\Delta p_x \sim E\lambda, \tag{1.4a}$$

then using the de Broglie relation $E\lambda \sim h$ gives us

$$\Delta x\,\Delta p_x \sim h. \tag{1.4b}$$

Classically, for a gravitational wave, we can have E as low as we want, and λ as large as we want. This violates the uncertainty principle.

If we violate the uncertainty principle, presumably all of quantum mechanics is undermined. On the other hand, momentum conservation is violated if the uncertainty principle is preserved.

There are limits to how accurately we can measure low energy gravitational waves. The apparatus has to be smaller and more massive, but that may collapse into a black hole. This may be a loophole to the aforementioned [Heisenberg microscope] argument.

Although none of these are conclusive, they seem to *imply* that gravity is quantized.

1.1 Semiclassical Gravity

Suppose we have classical gravity and quantum fields. The Einstein field equations become

$$\widehat{T}_{\mu\nu} \mid \psi\rangle = \frac{1}{8\pi} G_{\mu\nu} \mid \psi\rangle, \tag{1.5}$$

which may be a bit too restrictive since $\widehat{T}_{\mu\nu}$ may have noncommuting elements. On the other hand, we could make it

$$\langle \widehat{T}_{\mu\nu} \rangle = \frac{1}{8\pi} G_{\mu\nu}. \tag{1.6}$$

The metric now depends on the matter field, and the matter field depends on the metric. This becomes nonlinear, albeit a "weak" nonlinearity.

We can look at the Newtonian version of this:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \mid \psi\rangle &= \left(\frac{-\hbar^2}{2m} \nabla^2 + V \right) \mid \psi\rangle, \\ \nabla^2 V = 4\pi G m \rho &= 4\pi G m \sum_j m_j |\psi_j|^2. \end{aligned} \tag{1.7}$$

There is a paper on this:

- P.J. Salzman, S. Carlip, "A possible experimental test of quantized gravity". arXiv:gr-qc/0606120, 9 pages.

Suppose we start with a single particle with a Gaussian wave function. For small mass, it behaves like a free particle. For a large mass, the width narrows since gravitational collapse "wins out". For somewhere in between, there is nonlinear wiggling.

If we neglect the self-gravitating part, we recover the Hartree approximation.

There is another potential problem that the covariant divergence of the quantum stress-energy tensor is not conserved. We need to include in the stress-energy tensor the contribution of the measurement apparatus.

1.2 Positive Aspects of Quantizing Gravity

There are some positive aspects of the quantization of gravity!

(1) There are singularities in general relativity which need to be dealt with. This is similar to back when quantum mechanics was starting and we were answering questions like, "Why doesn't the electron fall into the nucleus?"

(2) Quantum gravity may deal with the problem of infinities in quantum field theory. Consider the renormalization of mass,

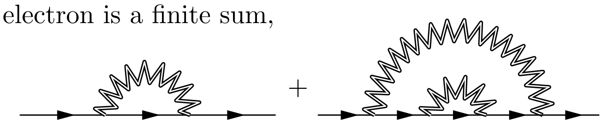
$$m(\varepsilon) = m_0 + \frac{e^2}{\varepsilon}, \tag{1.8}$$

where we include the electric self-energy (which looks like e^2/ε). If we include the classical self-energy to this, we have,

$$m(\varepsilon) = m_0 + \frac{e^2}{\varepsilon} - \frac{Gm(\varepsilon)^2}{\varepsilon}. \tag{1.9}$$

We can solve for $m(\varepsilon)$ to find that this is finite, it is something like the Planck mass times 137 or 1/137.

We can also see the sum of Feynman diagrams of the gravitational self-interaction of the electron is a finite sum,



$+ \cdots = \text{finite}, \quad (1.10)$

despite each term being divergent! (People are finding sets of finite sums of Feynman diagrams in supergravity. There is no proof yet.)

(3) There are a few physical systems we would like to understand that only quantum gravity can answer. The very early universe when quantum effects were present as well as gravity being the dominant force. Black holes also may be better understood with the quantization of gravity.

1.3 Why not Quantum Gravity?

Well, why not quantize gravity? In ordinary quantum theory, the basic observables are local. Consider a scalar field $\widehat{\varphi}(x)$, the value of the field at point x, the axiomatic formulations of quantum field theory these are observables. This does not make sense, since x does not make sense. There is no background. The symmetry of general relativity is diffeomorphism invariance, i.e., invariance under change of coordinates. If $x \to x+a$, then $\widehat{\varphi}(x) \to \widehat{\varphi}(x+a)$ which does not make sense.

This is already an issue in classical general relativity. We need to be careful not to write "the position x of *blah*", but instead "the time an atomic clock reads for a laser to reach some location". This is nonlocal, but what about this treatment in quantum theory? It's fine in classical general relativity, but we have problems in quantum mechanics with nonlocal stuff.

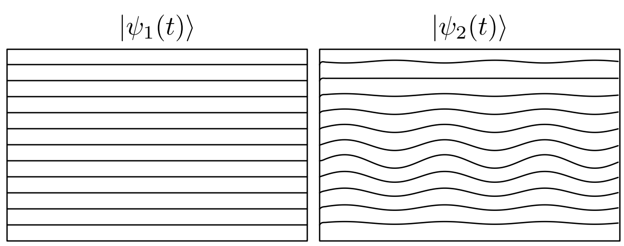
Lecture 2.

In classical general relativity, there are no local observables, so we do not know what the right operators should be. For a proof of the absence of local observables, see:

- C.G. Torre "Gravitational Observables and Local Symmetries". *Phys. Rev.* **D48** (1993) R2373–R2376(R); arXiv:gr-qc/9306030.
  doi:10.1103/PhysRevD.48.R2373

A particular example of this is the "problem of time".

Consider a free scalar field in flat Minkowski spacetime, pick an initial time slice and a final time slice to be the same in two different foliations. Is the time evolution in one foliation equivalent to the time evolution in the other?



We *should* be able to ask if we have

$$|\psi_1(t)\rangle = \mathcal{U}|\psi_2(t)\rangle, \tag{2.1}$$

where $\mathcal{U}$ is a unitary matrix indicating a change of bases.

Torre and Varadarajan [17, 18] show, in general, these are not related by a unitary matrix. But we *can* relate two operators by orderings, which hold in the classical limit.

Determining time by spatial hypersurfaces requires using the metric. Perhaps we can use the expectation value of the metric while demanding it to be spatial but this depends on the wave function which we're trying to find.

A lot of these problems come from thinking in the Schrodinger picture, perhaps using the Heisenberg picture fixes it. There are indications from lower dimensional approaches that this may be correct.

2.1 Quantization

If we want to quantize general relativity, we need to talk about what it means to quantize something. This is—for physicsts—the wrong question. There may be more than one way to go to the classical limit, but we work with the ones that are experimentally correct.

We start with some classical phase space with coordinates (q, p) and some Poisson bracket

$$\{p, q\} = 1. \tag{2.2}$$

We want to put hats on everything

$$[\widehat{p}, \widehat{q}] = \mathrm{i}\hbar. \tag{2.3}$$

We look for unitary irreducible representations on a Hilbert space, and so on. That's the quantum theory. We have the rule

$$\{q, p\} \mapsto \frac{1}{\mathrm{i}\hbar}[\widehat{q}, \widehat{p}], \tag{2.4}$$

and for any observables A and B we have:

$$\{A, B\} \mapsto \frac{1}{\mathrm{i}\hbar}[\widehat{A}, \widehat{B}]. \tag{2.5}$$

In general, this is impossible. There's a "no go theorem" from van Hove proving there's no consistent way to do this.

We have to choose some subset of functions on the phase space, some set of preferred phase space functions, that is "small enough" that this mapping from Poisson brackets to commutators is consistent. But it must be "large enough" so that any other function can be expressed in terms of the preferred set. This is what we do when we quantize the Hydrogen atom.

Suppose we have a phase space with a symmetry group G which relates any point with any other point. So we have the Poisson bracket be preserved

$$\{gA, gB\} = \{A, B\}. \tag{2.6}$$

If H is the stabilizer of x_0 — so $h \in H$ implies $hx_0 = x_0$ — then G/H is the phase space. In this case, we choose the generators of the action of the group on the symmetric space for the preferred functions to quantize.

The Stone–von Neumann theorem ensures the representation of translations is unique up to unitary equivalence. But this theorem does not hold in infinite-dimensions [i.e., for field theories].

There is another approach to quantization called "deformation quantization". We have a quantization map,

$$\mathcal{Q}\colon \text{phase space} \to \text{operators} \tag{2.7}$$

such that

(1) Linearity: $\mathcal{Q}(c_1 f_1 + c_2 f_2) = c_1\mathcal{Q}(f_1) + c_2\mathcal{Q}(f_2)$
(2) Preserves identity: $\mathcal{Q}(1) = \mathbf{1}$
(3) $\mathcal{Q}(x)$, $\mathcal{Q}(p)$ are represented irreducibly
(4) $\mathcal{Q}(\{f, g\}) = \frac{\mathrm{i}}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)] + \mathcal{O}(1)$

See:

- P. Tillman, "Deformation Quantization, Quantization, and the Klein-Gordon Equation". *J.Phys. A* **40** (2007) 7017–7024; arXiv:gr-qc/0610141. doi:10.1088/1751-8113/40/25/S55
- P. Tillman, "Deformation Quantization: From Quantum Mechanics to Quantum Field Theory". arXiv:gr-qc/0610159
- S. Twareque Ali, Miroslav Engliš, "Quantization Methods: A Guide for Physicists and Analysts". *Rev.Math.Phys.* **17** (2005) pp.391–490; arXiv:math-ph/0405065. doi:10.1142/S0129055X05002376

There is the path integral, which is just the continuous sum over the paths, we write this formally as:

$$\int [\mathrm{d}q]\mathrm{e}^{\mathrm{i}S}. \tag{2.8}$$

We can get different answers depending on how we define the derivative, and we get extra terms of order $\hbar$. We think of these ambiguities as normalization.

Lecture 3.

We spoke about what it means to quantize a system. This time we will discuss naive quantization of a system with constraints.

An addendum from last time: Take a one-dimensional particle moving along a line. We have q, p be the canonical coordinates and we make the Poisson bracket into commutators

$$\{q, p\} \mapsto \frac{\mathrm{i}}{\hbar}[\widehat{q}, \widehat{p}]. \tag{3.1}$$

The operator $\exp(\mathrm{i}a\widehat{p}/\hbar)$ generates translations in position, so:

$$\mathrm{e}^{\mathrm{i}a\widehat{p}/\hbar}\widehat{q}\mathrm{e}^{-\mathrm{i}a\widehat{p}/\hbar} = \widehat{q} \pm a. \tag{3.2}$$

Hence $\widehat{q}$ could take on any value.

Suppose we move on the positive real line, not the entire line. We can use the affine commutation relations. We use $\widehat{q}$ and

$$\widehat{D} = \widehat{qp}. \tag{3.3}$$

Classically we have

$$\{q, D\} = q, \tag{3.4}$$

yet quantum mechanically,

$$[\widehat{q}, \widehat{D}] = \mathrm{i}\hbar\widehat{q}. \tag{3.5}$$

This is a different representation than the first set of commutators.

We have

$$\mathrm{e}^{\mathrm{i}a\widehat{D}/\hbar}\widehat{q}\mathrm{e}^{-\mathrm{i}a\widehat{D}/\hbar} = \mathrm{e}^{a}\widehat{q}. \tag{3.6}$$

So this $\widehat{D}$ operation is just dilation.

This ought to be important since the "position" [in general relativity] is the metric on a spatial hypersurface, it should be positive definite. In the naive way, we can get timelike directions or nondefinite values, etc. We probably ought to use affine commutators.

(The Poisson bracket is unique. If we used the Heisenberg brackets, there would be an ordering problem, though not a serious one since we could use a symmetrized product.)

3.1 Quantization of Constrained Systems

In general, we have some action (we use I for the action since, in Euclidean quantum gravity, the action is minus the entropy and S is used for entropy)

$$I = \int L(q, \dot{q})\,\mathrm{d}t. \tag{3.7}$$

In general, higher-order derivatives in the Lagrangian generically leads to unbounded energies. For a review paper on this, see:

- R.P. Woodard, "Avoiding Dark Energy with $1/R$ Modifications of Gravity". *Lect. Notes Phys.* **720** (2007) pp.403–433; arXiv:astro-ph/0601672. doi:10.1007/978-3-540-71013-4_14

There are some exceptions, for example, when there are an infinite number of derivatives (but this is a sort of nonlocality, and there have been some papers recently on a nonlocal generalization of the Einstein–Hilbert action), or when we can integrate by parts to get to first-order. This is what happens with Einstein–Hilbert action.

The action of a system with constraints looks like

$$I = \int (L(q, \dot{q}) - \lambda C(q, \dot{q}))\,\mathrm{d}t, \tag{3.8}$$

where λ is the Lagrange multiplier and $C(q, \dot{q})$ is the constraint. By varying the action with respect to λ, we obtain a constraint on the initial data

$$C(q, \dot{q}) = 0. \tag{3.9}$$

We can rewrite the action to take into account constraints; if there are no second-order time derivatives of certain variables, then there will be constraints.

For general relativity, from the conservation laws, we have

$$\nabla_\mu G^{\mu\nu} = \partial_\mu G^{\mu\nu} + \dots \tag{3.10a}$$
$$= \partial_t G^{t\nu} + \partial_i G^{i\nu} + \dots. \tag{3.10b}$$

This means that each term has to have one time derivative and one spatial derivative (in the Einstein tensor).

We can look at it one way and say the space of initial data is smaller than we thought. Usually constraints "generate" gauge transformations, meaning we can look at it as:

$$\delta q = \{\varepsilon C, q\} \tag{3.11a}$$
$$\delta p = \{\varepsilon C, p\}, \tag{3.11b}$$

where ε is an arbitrary function of time. This is a generator of canonical transformations. In general, they're gauge transformations.

Here's the sketch of the basic idea (see Henneaux and Teitelboim's *Quantization of Gauge Systems* for further details). We want to consider the variation of the action δI. Let's consider the action in Hamiltonian form:

$$I = \int (p\dot{q} - H - \lambda C)\,\mathrm{d}t. \tag{3.12}$$

We often write

$$H^* := H + \lambda C, \tag{3.13}$$

and refer to it as the "Extended Hamiltonian". Let's consider the variation of the kinetic term:

$$\{\varepsilon C, p\dot{q}\} = \{\varepsilon C, p\}\dot{q} + p\frac{\mathrm{d}}{\mathrm{d}t}\{\varepsilon C, q\} \tag{3.14a}$$
$$= \left(\varepsilon\frac{\partial C}{\partial q}\right)\dot{q} + p\frac{\mathrm{d}}{\mathrm{d}t}\left(-\varepsilon\frac{\partial C}{\partial p}\right) \tag{3.14b}$$
$$= \varepsilon\frac{\partial C}{\partial q}\dot{q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon p\frac{\partial C}{\partial p}\right) + \varepsilon\frac{\partial C}{\partial p}\dot{p} \tag{3.14c}$$
$$= \varepsilon\left(\frac{\partial C}{\partial q}\dot{q} + \frac{\partial C}{\partial p}\dot{p}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon p\frac{\partial C}{\partial p}\right) \tag{3.14d}$$
$$= \varepsilon\frac{\mathrm{d}C}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon p\frac{\partial C}{\partial p}\right) \tag{3.14e}$$
$$= -\dot{\varepsilon}C + \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon C - \varepsilon p\frac{\partial C}{\partial p}\right). \tag{3.14f}$$

The next term we need to examine is $\{\varepsilon C, H\}$ which, in general, could be anything. If C remains a constraint under time translations, then the bracket with the Hamiltonian H is also a constraint. In general this is true, the commutator between the Hamiltonian and a constraint is another constraint. Let

$$\{H, C\} = vC \tag{3.15}$$

where v is some function, then

$$\{\varepsilon C, H\} = -\varepsilon vC. \tag{3.16}$$

Putting all of this together, we find

$$\{\varepsilon C, p\dot{q} - H - \lambda C\} = -\dot{\varepsilon}C + \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon C - \varepsilon p\frac{\partial C}{\partial p}\right) + \varepsilon vC - \{\varepsilon C, \lambda C\}. \tag{3.17}$$

If

$$\delta\lambda = -(\dot{\varepsilon} - \varepsilon v), \tag{3.18}$$

then the variation of the action is zero. This is because $\delta C = \{\varepsilon C, C\} = 0$, so

$$\{\varepsilon C, \lambda C\} = \{\varepsilon C, \lambda\}C = (\delta\lambda)C. \tag{3.19}$$

Plugging this back into Eq (3.17) makes $\{\varepsilon C, p\dot{q} - H - \lambda C\}$ into a total derivative, which contributes nothing to the action.

The moral of the story is that constraints generate gauge transformations and, in general (with the exception of some pathological counterexamples), the converse holds too. Note: if $\{C, C\} \propto C$, then the results still hold.

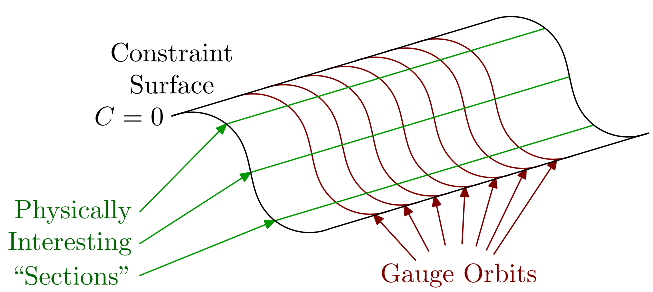
Now we use the results, and generalize to multiple constraints. We need

$$\{C_i, C_j\} = f_{ij}{}^{k}C_k \tag{3.20}$$

where $f_{ij}{}^{k}$ are "structure constants" and these are called "first-class constraints". (If we change Poisson brackets to commutators, these are the generators of the gauge algebra — or, at least, the structure constants are those from the Lie algebra of the gauge group.)

There are also "second-class constraints" which do not generate gauge transformations.

There are various ways to handle constraints in the quantization process. There is a constraint surface in the phase space when the constraints are satisfied. We have this gauge invariance which takes a physical state on this constraint surface and produces another distinct point in the phase space, but is physically indistinguishable from the original physical state.



The space of orbits is what is interesting. We take the physical degrees of freedom by taking some subsurface which cuts through the phase space orbits only once for each orbit.

There are times when a section may not contain an orbit (or some other unpleasant problem), which is the Gribov ambiguity.

For electromagnetism, we have $A_\mu \to A_\mu + \partial_\mu\Lambda$.

The approaches to quantizing ("canonically") systems with constraints:

Approach 1: Reduced phase space quantization, the recipe is:
(1) Clasically solve the constraints.
(2) Choose a section ("gauge fix").
(3) Insert into the action I and quantize.

Part of the problem is that, well, sometimes solving the constraints is fairly hard. For example, for classical general relativity, we do not know the general solution for Einstein's field equations.

A second problem is when fixing a gauge, when we jump to the third step, the resulting field is typically nonlocal. Consider electromagnetism. We have A^μ which is ambiguous, we could have

$$A^\mu = \widetilde{A}^\mu + \partial^\mu \Lambda. \tag{3.21}$$

We can gauge fix using the Lorenz gauge $\partial_\mu \overline{A}^\mu = 0$ for some gauge fixed potential $\overline{A}^\mu$. We can expand this to be:

$$\partial_\mu \overline{A}^\mu = \partial_\mu (A^\mu + \partial^\mu \Lambda) \tag{3.22a}$$

$$= \partial_\mu A^\mu + \Box \Lambda = 0. \tag{3.22b}$$

Then we have

$$\Lambda = -\Box^{-1} \partial_\mu A^\mu, \tag{3.23}$$

and then

$$\overline{A}^\mu = A^\mu - \partial^\mu \Box^{-1} \partial_\nu A^\nu, \tag{3.24}$$

or

$$A^\mu = \overline{A}^\mu - \partial^\mu \Box^{-1} \partial_\nu A^\nu. \tag{3.25}$$

The second term on the right-hand side is horribly nonlocal. BRST says that sticking a differential gauge transformation back into the system when solved is illegal. There are particular cases when this works; but the more complicated the theory, the harder this approach becomes.

Approach 2: Dirac quantization. The basic recipe is:
(1) Quantize the whole system.
(2) Impose constraints as operator conditions. That is, we define the physical states as

$$\widehat{C} \mid \text{physical}\rangle = 0, \tag{3.26}$$

the kernel of a "constraint operator" (or the intersection of kernels of constraint operators). The states are automatically gauge invariant this way.
(3) Define the inner product on physical states (intuitively: "gauge fixing the inner product"). How to do this is less obvious and usually hard.
(4) Find physical operators $\widehat{\mathcal{O}}_{\text{phys}}$ that take physical states to physical states (so if one realizes this, then it's equivalent to the operators which commutes with the constraints). That is, we need to find

$$[\widehat{\mathcal{O}}_{\text{phys}}, \widehat{C}] = 0. \tag{3.27}$$

(For general relativity, these physical operators are in general nonlocal and we don't know what they are.)

Example. The parametrized particle. For a one-dimensional particle subjected to some potential, then

$$I = \int (p\dot{q} - H)\,\mathrm{d}t. \tag{3.28}$$

If we change t to some monotonic function of time, then $p\dot{q}\,\mathrm{d}t$ remains invariant *but* the Hamiltonian contribution $H\,\mathrm{d}t$ doesn't quite remain invariant. Let's define a parameter τ such that,

$$I = \int \left(p\frac{\mathrm{d}q}{\mathrm{d}\tau} - H\frac{\mathrm{d}t}{\mathrm{d}\tau} \right) \mathrm{d}t. \tag{3.29}$$

Let us write $q^0 = t$ and $p_0 = H$, so we can write the action as:

$$I = \int \left(p_\mu \frac{\mathrm{d}q^\mu}{\mathrm{d}\tau} \right) \mathrm{d}\tau. \tag{3.30}$$

But to do this, we observe $H = H(p, q)$, so we need to introduce a constraint:

$$I = \int \left(p_\mu \frac{\mathrm{d}q^\mu}{\mathrm{d}\tau} - \lambda(p_0 + H(p, q)) \right) \mathrm{d}\tau. \tag{3.31}$$

The constraint generates parametrization invariance under

$$\tau \to \tau + \delta\tau. \tag{3.32}$$

So far, so good.

The reduced phase space approach solves the constraint, which is trivially $p_0 = -H$. We plug this back in:

$$p_\mu \frac{\mathrm{d}q^\mu}{\mathrm{d}\tau} = p_1 \frac{\mathrm{d}q^1}{\mathrm{d}\tau} + p_0 \frac{\mathrm{d}q^0}{\mathrm{d}\tau} \tag{3.33a}$$

$$= p_1 \frac{\mathrm{d}q^1}{\mathrm{d}\tau} + (-H) \frac{\mathrm{d}q^0}{\mathrm{d}\tau}. \tag{3.33b}$$

We plug in our gauge-fixing $q^0 = t$, then

$$p_\mu \frac{\mathrm{d}q^\mu}{\mathrm{d}\tau} = p_1 \frac{\mathrm{d}q^1}{\mathrm{d}\tau} + (-H) \frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{3.33c}$$

The Dirac approach where the wave function $\Psi[q^\mu]$, the commutators $[p_\mu, q^\nu] = \mathrm{i}\hbar {\delta_\mu}^\nu$, the constraint operator is

$$(\widehat{p}_0 + \widehat{H}_{\mathrm{phys}})\psi_{\mathrm{phys}} = 0 \tag{3.34a}$$

$$= \left(-\mathrm{i}\hbar \frac{\partial}{\partial q^0} + \widehat{H}_{\mathrm{phys}} \right) \psi_{\mathrm{phys}}. \tag{3.34b}$$

We need to gauge fix the inner product

$$\begin{aligned} \int \Psi_1^*(q^\mu)\Psi_2(q^\mu)\,\mathrm{d}q^i\mathrm{d}q^0 &= \int \langle \Psi_1 \mid \Psi_2 \rangle_{\mathrm{phys}}\,\mathrm{d}t \\ &= \infty \end{aligned} \tag{3.35}$$

where $\langle \Psi_1 \mid \Psi_2 \rangle_{\mathrm{phys}}$ is the usual old-school inner product of quantum mechanics. This integral over time generates infinities, we use a rigged Hilbert space to define the inner product—roughly speaking, we "divide out by infinity".

Part I
Canonical Quantum Gravity

Lecture 4.

I was too sick to attend, but I have been told: Professor Carlip argued the gauge symmetries of general relativity are isometries described by Killing equation, derived ADM coordinates including lapse and shift functions, described extrinsic curvature in terms of lapse and shift, rewrote the Einstein–Hilbert action in ADM coordinates, derived canonically conjugate momentum to metric, wrote first-order form of the action, argued lapse and shift functions are Lagrange multipliers.

In second-order formalism positions x and velocities $\dot{x}$ are treated as independent variables, but first-order formalism treats positions x and momenta p as independent variables. Also, Professor Waldron refers to the ADM action's terms as:

$$I_{ADM} = \int \Big(\underbrace{\pi^{ij}\dot{q}_{ij}}_{\substack{\text{symplectic}\\ \text{term}}} - \underbrace{N_i\mathcal{H}^i - N\mathcal{H}}_{\text{constraints}}\Big)\,\mathrm{d}^4x. \tag{4.1}$$

Also notation: D_i determined using the spatial metric q_{ij} such that $D_i q_{jk} = 0$.

Caveat: these are notes I've written, not based on Dr Carlip's lectures, but from what I've learned over the years.

We start by choosing some coordinate t and foliate spacetime with spacelike hypersurfaces Σ_t indexed by t. We have the unit normal n^μ on each hypersurface Σ_t, as well as the induced 3-metric q_{ij}. For spacelike hypersurfaces

$$n^\mu n_\mu = +1. \tag{4.2}$$

We then have a projection of tensors onto their spatial components

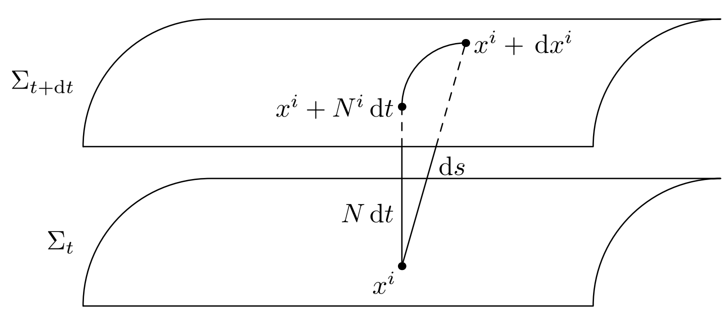
$$h^\mu{}_\nu = \delta^\mu{}_\nu - n^\mu n_\nu. \tag{4.3}$$

We define the extrinsic curvature as the spatial projection of the covariant derivative for the unit normal,

$$K_{\mu\nu} = h_\mu{}^\rho \nabla_\rho n_\nu. \tag{4.4}$$

It's not hard to see $n^\mu K_{\mu\nu} = 0$ and $n^\nu K_{\mu\nu} = 0$ (since it's a spatial tensor).

Now, we have the ADM decomposition of the metric. We begin with the line element, using the Lorentzian analog of the Pythagorean theorem. Intuitively, we should imagine something like the picture:



We start off with a point on a hypersurface Σ_t. If we translate along the time dimension from $t \to t + \mathrm{d}t$, then we end up at $x^i + N\,\mathrm{d}t$ — this is because the coordinates are arbitrary, but $N\,\mathrm{d}t$ should be the "infinitesimal proper time" not the "infinitesimal coordinate time".

Similarly, we could have some rotational effect, which we would account for by adding a translation on $\Sigma_{t+\mathrm{d}t}$ by $-N^i\,\mathrm{d}t$. This gives us the line element

$$\mathrm{d}s^2 = N^2\,\mathrm{d}t^2 - q_{ij}(\mathrm{d}x^i + N^i\,\mathrm{d}t)(\mathrm{d}x^j + N^j\,\mathrm{d}t). \tag{4.5}$$

Here N is called the "Lapse function", the N^i are called the "Shift vector".

It's not too hard (I think it's an exercise in homework 2) to show that

$$K_{ij} = \frac{1}{2N}(\partial_t q_{ij} - D_i N_j - D_j N_i) \tag{4.6}$$

where D_j is the spatial covariant derivative (compatible with q_{ij}, i.e., $D_i q_{jk} = 0$). We also find the inverse 4-metric decomposes like

$$g^{ab} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & -q^{ij} + \frac{N^i N^j}{N^2} \end{pmatrix}, \tag{4.7}$$

where $N^i = q^{ij}N_j$, and q^{ij} is the inverse of q_{ij} (i.e., $q^{ij}q_{jk} = \delta^i_k$).

We rewrite the Lagrangian using the Gauss–Codazzi equations

$${}^{(4)}R = {}^{(3)}R + K_{ij}K^{ij} - K^2 - 2\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\mu n^\mu). \tag{4.8}$$

Then the action

$$I_{EH} = \frac{1}{16\pi G}\int {}^{(4)}R\,\sqrt{-g}\,\mathrm{d}^4x \tag{4.9a}$$

$$= \frac{1}{16\pi G}\iint [{}^{(3)}R + K_{ij}K^{ij} - K^2](N\sqrt{q})\,\mathrm{d}^3x\,\mathrm{d}t + (\text{boundary terms}). \tag{4.9b}$$

We find the conjugate momenta to the 3-metric q_{ij} are

$$\pi^{ij} = \frac{\partial L}{\partial(\partial_t q_{ij})} = \frac{1}{16\pi G}(K^{ij} - q^{ij}K). \tag{4.10}$$

Using this, we can write the canonical action

$$I = \frac{1}{16\pi G}\iint (\pi^{ij}\partial_t q_{ij} - \mathcal{H}_{\text{can}})\,\mathrm{d}^3x\,\mathrm{d}t, \tag{4.11}$$

where $\mathcal{H}_{\text{can}} = \pi^{ij}\partial_t q_{ij} - \mathcal{L}$.

We should expect there to be constraints, since the components of the metric N and N_i do not enter the action with any time derivatives. In fact, it turns out we have

$$\mathcal{H} = \frac{16\pi G}{\sqrt{q}}(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2) - \frac{1}{16\pi G}\sqrt{q}\,{}^{(3)}R \tag{4.12a}$$

and

$$\mathcal{H}^i = -2D_j\pi^{ij} \tag{4.12b}$$

are the two constraints, called the Diffeomorphism constraint (or Hamiltonian constraint) and the Momentum constraints, respectively.

The Poisson brackets would be defined on a spatial hypersurface (so $t = \text{constant}$) as

$$\{q_{ij}(\boldsymbol{x}), \pi^{k\ell}(\boldsymbol{x}')\} = \frac{1}{2}(\delta^k_i\delta^\ell_j + \delta^k_j\delta^\ell_i)\widetilde{\delta}^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \tag{4.13}$$

where we use the densitized delta $\delta^{(3)}(\boldsymbol{x})\sqrt{q} = \widetilde{\delta}^{(3)}(\boldsymbol{x})$ since it satisfies:

$$\int \widetilde{\delta}^{(3)}(\boldsymbol{x})\,\mathrm{d}^3x = 1. \tag{4.14}$$

A number of exercises concerning the Poisson bracket may be found in Homework 3. In particular, the Poisson bracket of the constraints generate diffeomorphisms (morally speaking).

Lecture 5.

The action of general relativity in Hamiltonian form,

$$I_{ADM} = \int [\pi^{ij}\dot{q}_{ij} - N\mathcal{H} - N_i\mathcal{H}^i]\,\mathrm{d}^3x\,\mathrm{d}t. \tag{5.1}$$

The sign conventions varies, but the Hamiltonian is:

$$\mathcal{H} = \frac{16\pi G}{\sqrt{q}}(\pi^{ij}\pi_{ij} - \pi^2) - \frac{1}{16\pi G}\sqrt{q}\,{}^{(3)}R, \tag{5.2a}$$

and the momentum constraints,

$$\mathcal{H}^i = -2D_j\pi^{ij}. \tag{5.2b}$$

This is a Hamiltonian for general relativity based on a certain set of variables: the metric for a spatial hypersurfaces as the position variable and its time derivative for its conjugate momenta.

We can consider the Poisson brackets for this field:

$$\{q_{ij}(x), \pi^{k\ell}(x')\} = \frac{1}{2}(\delta^k_i\delta^\ell_j + \delta^k_j\delta^\ell_i)\widetilde{\delta}^{(3)}(x - x'), \tag{5.3}$$

where the tilde indicates a densitized delta function, so

$$\int \widetilde{\delta}^{(3)}(x)\,\mathrm{d}^3x = 1. \tag{5.4}$$

In particular, we do not need to explicitly write out $\sqrt{q}$. Using a densitized delta should make intuitive sense, since $\pi^{k\ell}$ is a tensor density.

This is a completely constrained system, with the momentum constraints generating spatial change of coordinates. Consider:

$$\{\int \xi^i\mathcal{H}_i(x)\,\mathrm{d}^3x, q_{k\ell}(x')\} = \{-2\int \xi^i D^j\pi_{ij}(x)\,\mathrm{d}^3x, q_{k\ell}(x')\} \tag{5.5a}$$

$$= \{\int (\xi_i D_j + \xi_j D_i)\pi^{ij}(x)\,\mathrm{d}^3x, q_{k\ell}(x')\} \tag{5.5b}$$

$$= -(D_k\xi_\ell + D_\ell\xi_k) \tag{5.5c}$$

$$= -\mathcal{L}_\xi q_{k\ell}. \tag{5.5d}$$

This means that $\mathcal{H}_i$ are generators of spatial coordinate transformations. The Poisson bracket for the momentum constraints and the π^{ij} are a bit more complicated.

We are working on spatial hypersurfaces, so there is a question of what "$\mathcal{H}$ generates time translations" even means. The easy bracket is with q_{ij}, technically what these yield are "surface deformations". (On the horizon of a black hole, surface deformations are not equivalent to changes of coordinates which could be bad…)

5.1 Reduced Phase Space Quantization

We have two ways to quantize this system: reduced phase space approach, and the Dirac approach. Lets begin with the reduced phase space approach.

Recall the basic idea with the reduced phase space quantization is to solve the constraints, find new variables, then quantize. The problem with this is solving the constraints, which is roughly the same as solving Einstein's field equations. We don't have it, or anything near it. We need to assume some sort of symmetry (e.g., cylindrical symmetry). This leads to "minisuperspace" or "midisuperspace". In some sense, this is the wrong thing to do because assuming symmetry at this level assumes that quantum states have this symmetry too.

An alternative approach is to change variables that change 4 nightmarish PDEs into 4 simpler equations.

This is the York time-slicing, the work on this was done predominantly by Fischer and Moncrief. We start with

$$q_{ij} = \phi^4 \widetilde{q}_{ij} \tag{5.6}$$

where ϕ is the conformal factor, $\widetilde{q}_{ij}$ is such that the Ricci scalar is

$${}^{(3)}R[\widetilde{q}] \in \{0, \pm 1\}. \tag{5.7}$$

This is the Yamabe condition. We can always do this for any Riemannian manifold. (The reason why ${}^{(3)}R[\widetilde{q}] = 0$ or ± 1 is due to the topological properties of the spatial hypersurface; it is some deep result in topology that is not immediately obvious.) This is for spatially compact universes (or asymptotically flat ones).

We need the decomposition of the canonical momentum:

$$\pi^{ij} = \frac{1}{16\pi G}[\phi^{-4}p^{ij} - \underbrace{\frac{2}{3}K\phi^2\widetilde{q}^{ij}\sqrt{\widetilde{q}}}_{\text{trace part}} + (\underbrace{\widetilde{D}^i Y^j + \widetilde{D}^j Y^i}_{\substack{\text{symmetrized covariant}\\ \text{derivative of a vector}}} - \frac{2}{3}\widetilde{q}^{ij} D_k Y^k)] \tag{5.8}$$

where Y^i is a density, p^{ij} is a density, and $\widetilde{D}_i \widetilde{q}_{jk} = 0$. (Locally any vector in 3 dimensions can be written as $\boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{A}$.) We have

$$\widetilde{D}_i p^{ij} = 0 \tag{5.9a}$$

and

$$\widetilde{q}_{ij} p^{ij} = 0. \tag{5.9b}$$

The momentum constraint $D_i \pi^{ij} = 0$ can be translated into a covariant derivative with respect to $\widetilde{q}$, we have

$$D_i \pi^{ij} = (\dots)\widetilde{D}_i p^{ij} + (\dots)\partial^j K + (\dots)\widetilde{D}_i(\widetilde{D}^i Y^j + \dots) = 0. \tag{5.10}$$

The crucial step: we choose — Crucial Step

$$t = -K, \tag{5.11}$$

constant mean extrinsic curvature. This is not an obvious choice, but there are proofs that this is neat, nice, and consistent. For a black hole, the hypersurfaces curve around the singularity.

For a large class of solutions, Anderson and Moncrief have recent proofs this is kosher.

We have

$$K = \frac{1}{N}\partial_t(\ln\sqrt{q}), \tag{5.12}$$

some signs vary.

This choice tremendously simplifies things, we are left with

$$D_i \pi^{ij} = (\dots)\widetilde{D}_i(\widetilde{D}^i Y^j + \dots) = 0. \tag{5.13}$$

If we assume spatial compactness or Y falls off at infinity, we get $Y^i = 0$. So we have simplified the conjugate momenta to be:

$$\pi^{ij} = \frac{1}{16\pi G}[\phi^{-4}p^{ij} - \frac{2}{3}K\phi^2\widetilde{q}^{ij}\sqrt{\widetilde{q}}]. \tag{5.14}$$

Here K is the proportional time rate of change of the local volume.

We solved the momentum constraints, p is freely specified provided it satisfies certain conditions. Now, the Hamiltonian constraint, which is hard. We are left with really 2 independent components in $\widetilde{q}$ and in p. We are left with the conformal factor ϕ to determine.

The Hamiltonian constraint determines it! The Hamiltonian constraint puts the condition on ϕ:

$$\widetilde{\Delta}\phi - \frac{1}{8}\phi + \frac{1}{12}t^2\phi^5 - \frac{1}{8}\left(\frac{\widetilde{q}_{ij}\widetilde{q}_{k\ell}p^{ik}p^{j\ell}}{\widetilde{q}^2}\right)\phi^{-7} = 0. \tag{5.15}$$

This is a second-order elliptic PDE.

Plugging this back into the action, we get (combining *everything* back together):

$$I = \left(\frac{1}{16\pi G}\right)^2 \int [p^{ij}\dot{\widetilde{q}}_{ij} - \frac{4}{3}\sqrt{\widetilde{q}}\phi^6]\mathrm{d}^3x\,\mathrm{d}t. \tag{5.16}$$

With the Hamiltonian constraint implying we can write the conformal factor as a function of p and $\widetilde{q}$, $\phi = \phi(p, \widetilde{q})$. In our notion of time, that Hamiltonian is very nonlocal. It is effectively

$$\mathcal{H} = \frac{4}{3}\sqrt{\widetilde{q}}\phi^6. \tag{5.17}$$

Due to this nightmarish nonlocality, we don't know how to put hats on stuff.

(We have been working with a zero cosmological constant $\Lambda = 0$, there should be some contribution from it in ϕ for nonzero Λ.)

In $2+1$ dimensional gravity, the $q^2p^2\phi^{-7}$ term goes away, and we have a local Hamiltonian, and everything's nice.

This was based on a particular decomposition, we'd like to keep something similar to the decomposition of π.

5.2 Dirac Quantization

Let's begin Dirac quantization of the system. We basically impose the constraints at the quantum level. We have our wave function $\Psi[q]$, so we have

$$\widehat{\mathcal{H}}^i\Psi[q] = 0 \tag{5.18a}$$

and

$$\widehat{\mathcal{H}}\Psi[q] = 0. \tag{5.18b}$$

We use the Schrodinger picture to have

$$\widehat{\pi}^{ij} = -\mathrm{i}\frac{\delta}{\delta q_{ij}}. \tag{5.19}$$

The momentum constraint smeared by some vector ζ^i is

$$\int \zeta_j D_i \frac{\delta}{\delta q_{ij}}\Psi[q]\,\mathrm{d}^3x = 0. \tag{5.20}$$

Integration by parts gives us,

$$\int (D_i\zeta_j + D_j\zeta_i)\frac{\delta}{\delta q_{ij}}\Psi[q]\,\mathrm{d}^3x = 0. \tag{5.21}$$

By functional Taylor expansion, we have

$$\Psi[q_{ij} + D_i\zeta_j + D_j\zeta_i] - \Psi[q_{ij}] = 0. \tag{5.22}$$

So Ψ is invariant under such coordinate transformations. This is not as easy as it seems.

Lecture 6.

Spacelike hypersurfaces defined by metric, but in general we don't know the metric in quantum gravity, so we're out of luck. (We are assuming that $\mathcal{M}=\mathbb{R}\times\Sigma$ where $\mathbb{R}$ is time, Σ is a spatial 3-manifold, at least in the canonical approach.) Anyways, back to the Dirac approach.

We're imposing the constraints as operators on the wave function. We interpret the momentum constraint

$$\widehat{\mathcal{H}}^{i}\Psi[q]=0 \tag{6.1}$$

as telling us the wave function is invariant under spatial diffeomorphisms. We should be able to, at least have the urge to assume $\widetilde{H}$ is telling us the wave function is invariant under temporal diffeomorphism but realize that this is meaningless. We're on a spatial hypersurfaces, after all!

The DeWitt supermetric

$$G_{ijk\ell}=\frac{1}{2}\frac{1}{\sqrt{q}}(q_{ik}q_{j\ell}+q_{i\ell}q_{jk}-q_{ij}q_{k\ell}). \tag{6.2}$$

This is like a metric of metrics. We have the deformation of a metric δq^{ij} have the length

$$\|\delta q^{ij}\|^{2}=\int G_{ijk\ell}\delta q^{ij}\delta q^{k\ell}\,\mathrm{d}^{3}x. \tag{6.3}$$

This defines the distance on the space of metrics. (The signature of the supermetric is $(-+++++)$, we take each pair of indices as a single index resulting in a 6-by-6 matrix.)

We introduce

$$\widehat{\pi}^{ij}=-\mathrm{i}\frac{\delta}{\delta q_{ij}}. \tag{6.4}$$

We plug it into the Hamiltonian constraint, and write:

$$\widehat{\mathcal{H}}=16\pi G G_{ijk\ell}\frac{\delta}{\delta q_{ij}}\frac{\delta}{\delta q_{k\ell}}+\frac{1}{16\pi G}\sqrt{q}\,{}^{(3)}R. \tag{6.5}$$

Resist the urge to make the first term a Laplacian. The Ricci 3-scalar ${}^{(3)}R$ is intuitively a sort of potential term, when viewed as a function of q_{ij}. So then we plug it back into

$$\widehat{\mathcal{H}}\Psi[q]=0. \tag{6.6}$$

This is the Wheeler–DeWitt Equation.

We need an inner product, wave functions alone do not suffice for a quantum theory. There are 2 obvious thing to try to do.

The first thing, the ordinary Schrodinger picture using the 3-metric

$$\int\Psi^{*}\Phi\,[\mathrm{d}q]=\infty \tag{6.7}$$

always since the Hamiltonian constraint, we need to gauge fix the inner product, like a path integral with some extra symmetry.

- R. P. Woodard, "Enforcing the Wheeler-de Witt Constraint the Easy Way". *Class. Quant. Grav.* **10** (1993), 483–496.
  doi:10.1088/0264-9381/10/3/008

We can think of $\widehat{\mathcal{H}}\Psi=0$ as a sort of Klein–Gordon equation, and the correct inner product there is:

$$\int\Psi^{*}\frac{\overleftrightarrow{\delta}}{\delta q}\Phi\,[\mathrm{d}q]. \tag{6.8}$$

There is some ambiguity here, we have a number of inner products since $\delta/\delta q$ is nonunique.

We could quantize the Wheeler-DeWitt equation, which is a third quantization. This creates and annihilates metrics (which correspond to universes), which we do not really observe.

There is another approach to finding an inner product which Woodard proposes, which for simple models looks like the Klein-Godon inner product. The wave function encodes some information about the placement of the spatial hypersurface in the universe, which has some information about time.

Another technical problem is the first term of $\widehat{\mathcal{H}}$ has two functional derivatives, which is problematic. We could try to put in some regulator, so the first term looks like:

$$\widehat{\mathcal{H}} = \lim_{x\to x'} \widetilde{G}_{ijk\ell} \frac{\delta}{\delta q_{ij}(x)} \frac{\delta}{\delta q_{k\ell}(x')} + \dots. \tag{6.9}$$

We need to show the result is independent of regularization. We also need to be conscious of the Poisson bracket $\{\mathcal{H}, \mathcal{H}^i\}$ must be recovered from the commutator $[\widehat{\mathcal{H}}, \widehat{\mathcal{H}}^i]$ with our own regularization.

6.1 Perturbative Expansion

The other thing we could try is a perturbative expansion, which is natural if we cannot get an exact solution.[2] We assume the wave functional Ψ satisfies the momentum constraints.

We can do what is roughly the Born–Openheimer approximation, wherein we couple gravity and matter. (Basic idea of the Born–Openheimer approximation is we have 2 independent processes, e.g., there is some background on which matter moves slowly, but there is some backreaction.)

Let us write:

$$\left(16\pi G\hbar G_{ijk\ell} \frac{\delta}{\delta q_{ij}} \frac{\delta}{\delta q_{k\ell}} + \frac{1}{16\pi G\hbar}\sqrt{q}\,{}^{(3)}R + \mathcal{H}_m\right)\Psi = 0, \tag{6.10}$$

where we have the matter Hamiltonian $\mathcal{H}_m \approx T_{00}$. Let us do a sort of WKB approximation:

$$\Psi = A\exp\left(\frac{\mathrm{i}}{32\pi G\hbar} S_0\right). \tag{6.11}$$

We can expand in powers of the Planck length. The lowest order expansion is just

$$\frac{-1}{4} G_{ijk\ell} \frac{\delta S_0}{\delta q_{ij}} \frac{\delta S_0}{\delta k\ell} + \sqrt{q}\,{}^{(3)}R = 0. \tag{6.12}$$

(This is the Hamilton–Jacobi equation for gravity uncoupled to matter.) So at this level we have some background that's fixed and looks classical.

The next order:

$$\mathrm{i}G_{ijk\ell} \frac{\delta S_0}{\delta q_{ij}} \frac{\delta A}{\delta q_{k\ell}} + \frac{\mathrm{i}}{2} G_{ijk\ell} \left(\frac{\delta^2 S_0}{\delta q_{ij}\,\delta q_{k\ell}}\right) A + \mathcal{H}_m A = 0. \tag{6.13}$$

If we are clever, we can choose the functional derivative of A to *look like*:

$$\frac{\delta A}{\delta q_{k\ell}} \sim \text{“}\frac{\mathrm{d}}{\mathrm{d}t} A\text{”}. \tag{6.14a}$$

Remember A is the coefficient for our wave functional Ψ. Explicitly,

$$A = D[q]\widetilde{\Psi}, \tag{6.14b}$$

[2] A good reference for this subsection is Claus Kiefer [6], *Quantum Gravity*, third edition, section 5.4.

choose D such that

$$\frac{\mathrm{i}}{2}G_{ijk\ell}\left(\frac{\delta^2 S_0}{\delta q_{ij}\,\delta q_{k\ell}}\right)D + \mathrm{i}G_{ijk\ell}\frac{\delta D}{\delta q_{ij}}\frac{\delta A}{\delta q_{k\ell}} = 0. \tag{6.14c}$$

Then,

$$\mathrm{i}G_{ijk\ell}\frac{\delta S_0}{\delta q_{ij}}\frac{\delta\widetilde{\Psi}}{\delta q_{k\ell}} + \mathcal{H}_m\widetilde{\Psi} = 0. \tag{6.15}$$

This looks a lot like the Schrodinger equation. We have

$$G_{ijk\ell}\frac{\delta S_0}{\delta q_{ij}} \text{ ``}\sim\text{''} \frac{\delta q_{k\ell}}{\delta T} \tag{6.16}$$

where T is "time" (we don't know exactly what this is in quantum gravity). We can be far more rigorous in certain midisuperspace models.

We can go to higher orders, where we get backreaction, where S_0 gets corrections from S_2 (the effects of gravity self-gravitating). Barvinsky has worked out a systematic formalism using doodles that look like Feynman diagrams.[3] It's not known if the approximation is renormalizable.

The zeroeth order Eq (6.12) describes how spacetime curves, the first-order corrections in Eq (6.13) tells matter how to move, the second-order correction tells spacetime curves due to matter, then the third-order correction tells matter how to react to third-order corrections, and so on.

We can do cosmology in this formalism. (Halliwell(?) did some old work here.)[4] Time has sort of emerged, which is nice, but this tells us that time emerges when the universe is approximately classical. (What about in other universes?)

6.2 Strong Coupling Limit

There's another approximation that has appeal. That is to take $\hbar G$ as large (the so-called **"Strong Coupling Approximation"**). This might be good to tell us about the small scale structure of spacetime.[5] The leading order contribution in the Wheeler-DeWitt equation is the first term. This tells u s that the metric "decouples" at each point. To lowest order we have "almost independent" metrics at each point.

Classically, at each point, the general solution is the Kasner universe

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2p_1}\mathrm{d}x^2 + \mathrm{e}^{2p_2}\mathrm{d}y^2 + \mathrm{e}^{2p_3}\mathrm{d}z^2, \tag{6.17}$$

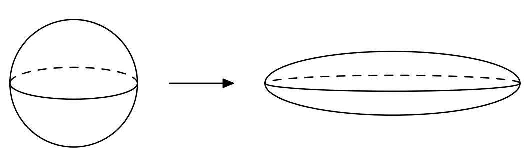
where, at lowest order, the terms p_j are constants satisfying:

$$p_1 + p_2 + p_3 = 1, \tag{6.18a}$$

and

$$p_1^2 + p_2^2 + p_3^2 = 1. \tag{6.18b}$$

The next order correction treats the p_j as slowly-varying terms.



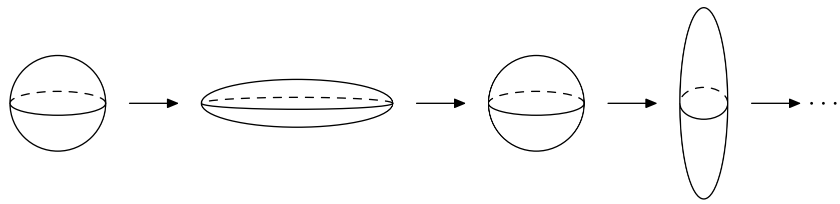
[3] Although a reference was not given, I believe it is Barvinsky and Kiefer [4].

[4] I think this refers to Jonathan Halliwell arXiv:gr-qc/9208001, possibly other papers.

[5] Professor Carlip wrote a review paper with a good discussion of this approximation in §2 of arXiv:1009.1136.

Misner called this the Mixmaster universe. There's a huge literature on this (lookup the Belinskii-Khalatnikov-Lifshitz [BKL] model). It is conjectured that near the Big Bang, the universe behaved this way.

At higher-order corrections, the oscillations look like:



The constraints imply $p_i > 0$, $p_j > 0$ and $p_k < 0$ where $i \neq j \neq k$ and $i, j, k = 1, 2, 3$.

Part II
Loop Quantum Gravity

Lecture 7.

For reduced phase space quantization, we are left with one horrible equation—as opposed to many horrible equations in the Dirac approach.

There's also the problem that we chose $t = -K$. It's sometimes not the obvious choice for certain problems, for example the Schwarzschild solution is completely scary. Do we get different quantum theories with different time slicings? We don't know, this is kind of an anomaly problem—is the quantum theory generally covariant in the reduced phase space approach?

The Dirac approach has a few problems. We need to gauge fix the inner product, but in practice we don't know how to do this. Another problem is that the Wheeler-DeWitt equation has a piece that looks like the product of two functional derivatives at a point, and this results in a $\delta(0)$ contribution. This is a standard problem in quantum field theory, regularization is needed. We could regulate it in theory as

$$\frac{\delta}{\delta g(x)}\frac{\delta}{\delta g(x')} \to \frac{\delta}{\delta g(x)} K_\varepsilon(x, x') \frac{\delta}{\delta g(x')} \tag{7.1}$$

where $K_\varepsilon(x, x')$ is some regulator invariant under spatial diffeomorphisms, preserves the Poisson brackets, and becomes a δ function. No one has a proof that results are independent of how we regulate. It could be possible it makes sense, we just don't know enough about functional differential equations.

Even if this all worked out, the problem remains how to make sense of basic variables. We have physical states be annihilated by the constraints

$$\widehat{\mathcal{H}}\Psi_{\text{phy}} = 0. \tag{7.2}$$

We want a physical operator $\widehat{\mathcal{O}}$ to map physical states to physical states

$$\widehat{\mathcal{O}}\Psi_{\text{phy}} = \Psi'_{\text{phy}}. \tag{7.3}$$

This requires

$$[\widehat{\mathcal{H}}, \widehat{\mathcal{O}}] \approx 0. \tag{7.4}$$

We know no operators that do this. There have been proofs that such operators are necessarily nonlocal, which we don't know how to deal with. There's been work by some to make the Hamiltonian constraint "almost local".

This is where things stood roughly in the 1980s. There are some simplified models where the Wheeler-DeWitt equation simplifies, just freeze out degrees of freedom, very simplified settings. The Wheeler-DeWitt equation becomes an ordinary differential equation.

In the early 1980s, two new approaches emerged:

(1) Loop Quantum Gravity (which sought to simplify the Wheeler-DeWitt equation), and

(2) String Theory (possibly contains quantum gravity).

Then in the 1990s there was a new approach called dynamical triangulations. We'll cover these three for the rest of the quarter.

7.1 Loop Quantum Gravity

We'll begin with gravity in the first-order formulation; i.e., a tetrad/vierbein/frame field ${e^I}_\mu$. The capital Latin indices track the basis vector, the Greek indices track the components of the vector. We have

$$g^{\mu\nu}{e^I}_\mu {e^J}_\nu = \eta^{IJ}. \tag{7.5}$$

It follows that

$$\eta_{IJ}{e^I}_\mu {e^J}_\nu = g_{\mu\nu}. \tag{7.6}$$

We have an additional symmetry: local Lorentz symmetry.

Given such a tetrad, we can introduce the covariant derivative

$$\nabla_\mu A^I = \partial_\mu A^I + {\omega_\mu}^I{}_J A^J, \tag{7.7}$$

where ${\omega_\mu}^I{}_J$ is the spin connection. Spin connections came about when people tried to introduce the spinor to general relativity. We could demand metric compatibility to specify the spin connection. The notation gets difficult, but let $\widetilde{\nabla}_\mu$ be th eordinary covariant derivative for tensors. The demand is that

$$\widetilde{\nabla}_\mu {e_\nu}^J + {\omega_\mu}^I{}_J {e_\nu}^J = 0 \tag{7.8}$$

determines the spin connection ω in terms of the frame e and Christoffel connection.

We can now do ordinary general relativity with this. So

$$[\nabla_\mu, \nabla_\nu]A^I = {R_{\mu\nu}}^I{}_J A^J, \tag{7.9}$$

where

$${R_{\mu\nu}}^\alpha{}_\beta {e^I}_\alpha {e_J}^\beta = {R_{\mu\nu}}^I{}_J. \tag{7.10}$$

We write

$$A^I = {e_\mu}^I A^\mu, \tag{7.11}$$

and by our specification of the covariant derivative (specifically, the spin connection) permits us to write the commutator.

The Einstein field equations are derived from the action:

$$I_{EH} = \frac{1}{16\pi G}\int |e| e^{\mu I} e^{\nu J} R_{\mu\nu IJ}\, \mathrm{d}^4 x, \tag{7.12}$$

where $|e| = \det |{e_\mu}^I| = \sqrt{-g}$ is the determinant of the tetrad. We can express R in terms of the spin connection, computed directly from the commutator, as:

$${R_{\mu\nu}}^I{}_J = \partial_\mu {\omega_\nu}^I{}_J + {\omega_\mu}^I{}_K {\omega_\nu}^K{}_J - (\mu \leftrightarrow \nu). \tag{7.13}$$

We can also treat the tetrad and connection as independent variables. This isn't new: Palatini showed this holds for the metric and Γ back in the 1930s.

The variation of the action, when treating tetrad and connection as independent variables, gives us:

$$\begin{aligned} \delta e: &\quad e^{\nu I} R_{\mu\nu IJ} = 0 = R_{\mu J} \\ \delta\omega: &\quad \nabla_\mu(e(e^{\mu I}e^{\nu J} - e^{\mu J}e^{\nu I})) = 0. \end{aligned} \tag{7.14}$$

(The second variation is just the same as $\nabla^{\text{total}}_{\mu} e_{\nu}{}^{J} = 0$.) The Wheeler-DeWitt equation isn't more interesting, difficult, or simple. But we can do interesting stuff!

We can write this in terms of forms

$$e^I = e_\mu{}^I \, \mathrm{d}x^\mu, \tag{7.15a}$$

$$\omega^I{}_J = \omega_\mu{}^I{}_J \, \mathrm{d}x^\mu. \tag{7.15b}$$

We can write down the curvature 2-form

$$\mathcal{R}^I{}_J = \mathrm{d}\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J. \tag{7.16}$$

The action becomes (up to some sign error):

$$I = \pm\frac{1}{64\pi G}\int \epsilon_{IJKL} e^I \wedge e^J \wedge \mathcal{R}^{KL}. \tag{7.17}$$

This makes it *look* neater.

Let us call

$$\mathcal{B}^{IJ} := e^I \wedge e^J. \tag{7.18}$$

Then the action looks like

$$I = \int \epsilon_{IJKL} \mathcal{B}^{IJ} \wedge \mathcal{R}^{KL}. \tag{7.19}$$

We impose the condition $\mathcal{B}^{IJ} = e^I \wedge e^J$ (e.g., $\mathcal{B}^{IJ} \wedge \mathcal{B}^{KL} = e\varepsilon^{IJKL}$). The converse (having $\mathcal{B}^{IJ}$ defined by the condition $\mathcal{B}^{IJ} \wedge \mathcal{B}^{KL} = e\varepsilon^{IJKL}$) is *almost* true. We then have:

$$I = \frac{1}{64\pi G}\int \left(\underbrace{\epsilon_{IJKL}\mathcal{B}^{IJ} \wedge \mathcal{R}^{KL}}_{\text{a ``BF'' theory}} + \phi_{IJKL} \underbrace{(\mathcal{B}^{IJ} \wedge \mathcal{B}^{KL} - e\varepsilon^{IJKL})}_{\text{constraint}} \right), \tag{7.20}$$

and the ϕ_{IJKL} are Lagrange multipliers. If we didn't have tje constraint, we'd have a flat spacetime with a sort of gauge theory living on it.

So writing things in new variables suggests new approaches. Let us try some new variables.

7.2 Self-Dual 2-Forms

First, we define a $*$ operator on a 2-form:

$$F^*_{IJ} = \frac{-\mathrm{i}}{2}\epsilon_{IJKL}F^{KL}, \tag{7.21a}$$

$$F^{**} = F, \tag{7.21b}$$

where $F_{[IJ]} = 0$ (i.e., F is antisymmetric). So this is a dual of F (there are many notions of "duality"). We say F is **"Self-Dual"** if

$$F^* = F, \tag{7.22}$$

and F is **"Anti-Self-Dual"** if

$$F^* = -F. \tag{7.23}$$

We can write, for an arbitrary 2-form F,

$$F^{\pm IJ} = \frac{1}{2}\left(F^{IJ} \pm F^{*IJ}\right). \tag{7.24}$$

We can define a self-dual connection,

$$A_\mu{}^{IJ} = \frac{1}{2}\left(\omega_\mu{}^{IJ} - \frac{\mathrm{i}}{2}\epsilon^{IJ}{}_{KL}\omega_\mu{}^{KL}\right). \tag{7.25}$$

We define the self-dual curvature as:

$$F_{\mu\nu}{}^{IJ} = \partial_\mu A_\nu{}^{IJ} + A_\mu{}^I{}_K A_\nu{}^{KL} - (\mu \leftrightarrow \nu) \tag{7.26a}$$

$$= \frac{1}{2}(R_{\mu\nu}{}^{IJ} + R_{\mu\nu}{}^{IJ*}). \tag{7.26b}$$

We complexified, doubling the degrees of freedom, roughly speaking the self-dual and antiself-dual splits the degrees of freedom.

The Ashtekar–Sen Action is then:

$$I_{AS} = \frac{1}{8\pi G}\int e e^{\mu I} e^{\nu J} F_{\mu\nu IJ}\,\mathrm{d}^4x. \tag{7.27}$$

By treating the self-dual connection as separate [independent] from the tetrad, we get the Einstein field equations. (There is actually an extra term like $\sim e^{\mu I}e^{\nu J}R_{\mu\nu}{}^{KL} = \epsilon_{IJKL}R^{IJKL} = 0$.) The constraints simplify *dramatically.*

We need to have **"Reality Conditions"** so we don't have anything imaginary. Classically, they are:

$$\omega_\mu{}^{IJ} = A_\mu{}^{IJ} + A_\mu{}^{IJ*} \tag{7.28a}$$

$$\frac{-\mathrm{i}}{2}\epsilon^{IJ}{}_{KL}\omega_\mu{}^{KL} = A_\mu{}^{IJ} - A_\mu{}^{IJ*} \tag{7.28b}$$

$$= \frac{-\mathrm{i}}{2}\epsilon^{IJ}{}_{KL}(A_\mu{}^{IJ} + A_\mu{}^{IJ*}). \tag{7.28c}$$

The statement is that

$$A_\mu{}^{IJ} - A_\mu{}^{IJ*} = \frac{-\mathrm{i}}{2}\epsilon^{IJ}{}_{KL}(A_\mu{}^{IJ} + A_\mu{}^{IJ*}). \tag{7.29}$$

This is a second-class condition, which relates the real and imaginary parts of the connection.

Given the change of variables to the Ashtekar–Sen action, we can do a $3+1$ dimensional split. We will introduce new indices ($\widehat{I}$, $\widehat{J}$, $\ldots = 1, 2, 3$) for tetrad indices and (i, j, $\ldots =$ 1, 2, 3) for coordinate indices. Let's look at the components:

$$A_\mu{}^{0\widehat{L}} = \frac{1}{2\mathrm{i}}\epsilon^{0\widehat{L}}{}_{\widehat{I}\widehat{J}}A_\mu{}^{\widehat{I}\widehat{J}} = \frac{1}{2\mathrm{i}}A_\mu{}^{\widehat{L}}, \tag{7.30a}$$

$$A_\mu{}^{\widehat{I}\widehat{J}} = \frac{1}{2}\epsilon_0{}^{\widehat{I}\widehat{J}\widehat{K}}A_{\mu\widehat{L}}. \tag{7.30b}$$

If we went back to the original spin connection, we find it is related to the extrinsic curvature

$$\omega_i{}^{0\widehat{I}} = K_i{}^{\widehat{I}}. \tag{7.31}$$

We can define

$$\Gamma_i{}^{\widehat{I}} = \frac{1}{2}\epsilon_0{}^{\widehat{I}\widehat{J}\widehat{K}}\omega_{i\widehat{J}\widehat{K}}, \tag{7.32}$$

which is basically the connection on the spatial hypersurface ignoring the embedding. We do this so we can write the self-dual connection as

$$\begin{aligned} A_i{}^{\widehat{I}} &= \Gamma_i{}^{\widehat{I}} + \mathrm{i}K_i{}^{\widehat{I}} \\ &= \begin{pmatrix}\text{Ordinary Connection}\\ \text{On the Slice}\end{pmatrix} + \mathrm{i}\begin{pmatrix}\text{Extrinsic Curvature}\\ \text{On the Slice}\end{pmatrix} \end{aligned} \tag{7.33}$$

We can generalize, letting γ be "some parameter"

$$A_i^{(\gamma)\widehat{I}} = \Gamma_i{}^{\widehat{I}} + \gamma K_i{}^{\widehat{I}} \tag{7.34}$$

where γ is the **"Immirzi–Barbero Parameter"**. The self-dual connection is really just a canonical transformation.

Lecture 8.

(Remark: If we can write the constraints in two independent groups, then we can do a mixture of Dirac quantization and reduced phase-space quantization.)

Last time we ended up with a kind of gauge-like field,

$$A_i^{(\gamma)\widehat{I}} = \Gamma_i{}^{\widehat{I}} + \gamma K_i{}^{\widehat{I}}. \tag{8.1}$$

We can write this gauge-like field in terms of the spin connection as:

$$A_i^{(\gamma)\widehat{I}} = \frac{1}{2}\epsilon^{0\widehat{I}\widehat{J}\widehat{K}}\omega_{i\widehat{J}\widehat{K}} + \gamma\omega_i{}^{0\widehat{I}}. \tag{8.2}$$

We can think of this as a canonical transformation. In the ADM formalism, $K_i{}^{\widehat{I}}$ is more or less the canonical conjugate momentum, and we're adding some terms involving derivatives of the tetrad to it.

The next step is slightly dodgy, but makes the math easier. We gauge fix Lorentz-boosts:

$$e^t{}_{\widehat{I}} = 0. \tag{8.3}$$

If we don't do this, then we get second-class constraints. This may give a different representation (there is some evidence of it yielding a different representation[6]). We can now define

$$e^t{}_{\widehat{0}} = 1/N \tag{8.4a}$$

$$e^i{}_{\widehat{0}} = -N^i/N, \tag{8.4b}$$

where N is the Lapse function and N^i is the shift function both from the ADM formalism.We have

$$q^{ij} = e^i{}_{\widehat{I}}e^{j\widehat{I}}, \tag{8.5}$$

so we can write

$$g^{ij} = q^{ij} - \frac{N^iN^j}{N^2} \tag{8.6a}$$

$$= e^i{}_{\widehat{0}}e^{j\widehat{0}} + e^i{}_{\widehat{I}}e^{j\widehat{I}}. \tag{8.6b}$$

With this gauge fixing, we recover the ADM decomposition of the metric.

Notation 1. *Let's define*

$$\widetilde{E}^i_{\widehat{I}} := \sqrt{q}e^i{}_{\widehat{I}}. \tag{8.7}$$

It is a tensor density, and it is a triad on a spatial hypersurface.

[6] Unfortunately, I didn't ask for references on this.

Given all of this, we can go back to the Einstein–Hilbert action, do all the computations, we find:

$$I = \frac{1}{8\pi G}\int \left(\frac{1}{\gamma}A_i{}^{\widehat{I}}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}^i{}_{\widehat{I}} - \underbrace{\mathrm{i}A_{0\widehat{I}}G^{\widehat{I}} + \mathrm{i}N^iV_i - \frac{1}{2}\frac{N}{\sqrt{q}}S}_{\text{constraints}} \right) \mathrm{d}^3x\,\mathrm{d}t. \tag{8.8}$$

If we didn't impose our gauge-fixing condition, we'd have a more complicated constraint algebra and one more constraint. Now, let us examine these constraints:

(1) We have $G^{\widehat{I}} = D_i\widetilde{E}^{i\widehat{I}}$ where D_i is the gauge covariant derivative treating this as a gauge theory.

(2) $V_j = \widetilde{E}^i{}_{\widehat{I}}F_{ij}{}^{\widehat{I}}$ where $F_{ij}{}^{\widehat{I}} = \partial_iA_j{}^{\widehat{I}} - \partial_jA_i{}^{\widehat{I}} + \epsilon^{\widehat{I}\widehat{J}\widehat{K}}A_{i\widehat{J}}A_{j\widehat{K}}$. (This should ring a bell as the field strength tensor for a nonabelian gauge theory.) Observe these two do not involve the Immirzi parameter γ directlry.

(3) The remaining constraint is a monster:

$$S = \epsilon^{\widehat{I}\widehat{J}\widehat{K}}\widetilde{E}^i_{\widehat{I}}\widetilde{E}^j_{\widehat{J}}F_{ij\widehat{K}} - 2\left(\frac{1+\gamma^2}{\gamma^2}\right)\widetilde{E}^i_{[\widehat{I}}\widetilde{E}^j_{\widehat{J}]}(A_i^{(\gamma)\widehat{I}} - \Gamma_i{}^{\widehat{I}})(A_j^{(\gamma)\widehat{J}} - \Gamma_j{}^{\widehat{J}}). \tag{8.9}$$

The factor of $A_i^{(\gamma)\widehat{I}} - \Gamma_i{}^{\widehat{I}}$ should remind us of the extrinsic curvature.

Let us consider the Poisson brackets of quantities.

$$\{\widetilde{E}^i_{\widehat{I}}(x), A_j^{(\gamma)\widehat{J}}(x')\} = -8\pi G\gamma\delta_{\widehat{I}}{}^{\widehat{J}}\widetilde{\delta}^{(3)}(x-x'). \tag{8.10}$$

If we look at this as a nonabelian gauge theory, the Poisson bracket looks like an electric field and potential. The constraint $D_i\widetilde{E}^{i\widehat{I}} = G^{\widehat{I}}$ looks like Gauss's law.

It looks like the physical phase space of an SU(2) gauge theory. We're then imposing two additional constraints, and calling the result quantum gravity. The natural thing to do is treat the $A_j^{(\gamma)\widehat{J}}$ as positions and the $\widetilde{E}^i_{\widehat{I}}$ as momenta.

If we work at $G^{\widehat{I}}$, it tells us the wave functions are gauge invariant, the V^i constraints generate spatial diffeomorphisms, and the S is the Hamiltonian constraint. We have:

$$S = \underbrace{\epsilon^{\widehat{I}\widehat{J}\widehat{K}}\widetilde{E}^i_{\widehat{I}}\widetilde{E}^j_{\widehat{J}}F_{ij\widehat{K}}}_{\text{scalar curvature term}} \underbrace{- 2\left(\frac{1+\gamma^2}{\gamma^2}\right)\widetilde{E}^i_{[\widehat{I}}\widetilde{E}^j_{\widehat{J}]}(A_i^{(\gamma)\widehat{I}} - \Gamma_i{}^{\widehat{I}})(A_j^{(\gamma)\widehat{J}} - \Gamma_j{}^{\widehat{J}})}_{\text{the }\pi^2\text{ term}}. \tag{8.11}$$

The π^2 term is ugly since the Γ term depend on E, so we have a constraint with quadratic terms in E.

There is a trick here, discovered originally by Thiemann, called the Thiemann trick, where we represent the ugly term in terms of nested Poisson brackets. So it is possible to make it really pretty.

Let's forget the Hamiltonian constraint for the time being. Let's try to solve the other constraints, beginning with the Gauss's Law constraint. The constraints

$$G^{\widehat{I}} = 0 \tag{8.12}$$

implies the wave functionals Ψ are gauge-invariant. We will write

$$A = A^{\widehat{I}}{}_i\,\mathrm{d}x^i\,\tau_{\widehat{I}} \tag{8.13}$$

where $\tau_{\widehat{I}}$ are the generators of SU(2) or SO(3) depending on gauge, let $g \in \mathrm{SU}(2)$ then the A field transforms like:

$$A \to g^{-1}\,\mathrm{d}g + g^{-1}Ag. \tag{8.14}$$

(For electromagnetism, $g^{-1}Ag = g^{-1}gA = A$ and $g^{-1}\,\mathrm{d}g = \mathrm{d}\Lambda$.) Recall the field strength 2-form is then,

$$F = \mathrm{d}A + A\wedge A. \tag{8.15}$$

We see, from $\mathrm{d}(gg^{-1}) = 0$ we have,

$$\mathrm{d}(g^{-1}) = -g^{-1}\,(\mathrm{d}g)g^{-1}. \tag{8.16}$$

In particular,

$$\mathrm{d}(g^{-1}\,\mathrm{d}g + g^{-1}Ag) = -g^{-1}\,\mathrm{d}g\,g^{-1}\,\mathrm{d}g - g^{-1}\,\mathrm{d}g\,g^{-1}Ag + g^{-1}\,\mathrm{d}A\,g - g^{-1}A\,\mathrm{d}g. \tag{8.17}$$

Then applying this and Eq (8.14) to the field strength 2-form gives us,

$$\begin{aligned} F \to F' &= \mathrm{d}(g^{-1}\,\mathrm{d}g + g^{-1}Ag) + (g^{-1}\,\mathrm{d}g + g^{-1}Ag)\wedge(g^{-1}\,\mathrm{d}g + g^{-1}Ag) && (8.18a)\\ &= g^{-1}\,\mathrm{d}A\,g + (g^{-1}Ag)\wedge(g^{-1}Ag) && (8.18b)\\ &= g^{-1}(\mathrm{d}A + A\wedge A)g && (8.18c)\\ &= g^{-1}Fg. && (8.18d)\end{aligned}$$

This isn't terribly surprising, it's basic differential geometry.

The kinetic term is

$$\mathrm{Tr}(F^2) = F^{\widehat{I}}_{\mu\nu}F^{\mu\nu}_{\widehat{I}} \tag{8.19}$$

for the gauge field. It's invariant under gauge transformations.

Now we would like to construct a basis of gauge invariant quantities (easier said than done). But we can consider the parallel transport of a gauge field on a cloed curve on a surface, the holonomy is gauge invariant! We have the parallel transport,

$$\frac{\mathrm{d}v^{\widehat{I}}}{\mathrm{d}s} + \frac{\mathrm{d}x^i}{\mathrm{d}s}\left(A_i{}^{\widehat{I}}\epsilon_{\widehat{K}}{}^{\widehat{I}}{}_{\widehat{J}}\right)v^{\widehat{J}} = 0. \tag{8.20}$$

This is the equation for parallel transport, it's basic differential geometry.[7] The result is that

$$v^{\widehat{I}}(s) = \mathcal{U}^{\widehat{I}}{}_{\widehat{J}}(s,s_0)v^{\widehat{J}}(s_0), \tag{8.21}$$

where we have the path-ordering exponential,

$$\mathcal{U}^{\widehat{I}}{}_{\widehat{J}}(s,s_0) = \mathcal{P}\exp\left(-\int_{s_0}^{s} A_i{}^{\widehat{K}}\underbrace{\epsilon^{\widehat{I}}{}_{\widehat{J}\widehat{K}}}_{\text{generators}}\,\mathrm{d}x^i\right). \tag{8.22}$$

We can generalize to any representation of SU(2), just replace the $\epsilon^{\widehat{I}}{}_{\widehat{J}\widehat{K}}$ "generators" factor with $\tau_{\widehat{K}}$.

This is more general than curvature, we're not doomed to infinitesimal nightmares.

We see, taking care with ordering due to noncommutativity, that:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{U}(s,s_0) = -A(s)\mathcal{U}, \tag{8.23}$$

and similarly,

$$\frac{\mathrm{d}}{\mathrm{d}s_0}\mathcal{U}(s,s_0) = -\mathcal{U}A(s_0). \tag{8.24}$$

[7] See, e.g., §13 of my notes on general relativity http://pqnelson.github.io/assets/notebk/GR.pdf.

Let $\widetilde{\mathcal{U}} = g(s)^{-1}\mathcal{U}g(s_0)$, so we have:

$$\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathcal{U}} = -g^{-1}(s)\frac{\mathrm{d}g(s)}{\mathrm{d}s}\widetilde{\mathcal{U}} - g^{-1}(s)A\mathcal{U}g(s_0) \tag{8.25a}$$

$$= \left(-g^{-1}Ag - g^{-1}\frac{\mathrm{d}g}{\mathrm{d}s}g\right)\widetilde{\mathcal{U}} \tag{8.25b}$$

$$= -\widetilde{A}\widetilde{\mathcal{U}}. \tag{8.25c}$$

We read this backwards, when

$$A \to g^{-1}\,\mathrm{d}g + g^{-1}Ag, \tag{8.26}$$

we simply have

$$\mathcal{U}(s, s_0) \to g^{-1}(s)\mathcal{U}(s, s_0)g(s_0). \tag{8.27}$$

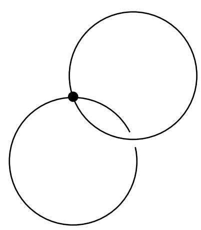
Note: redo the computations starting from Eq (8.23) for enlightening insight.

In particular, for a closed curve C, we find $\mathrm{Tr}(\mathcal{U}(s, s_0))$ is gauge invariant. This is the Wilson loop, and it gives us a complete set of gauge invariant variables for a gauge theory.

There is one problem with this, the Wilson loops give an overcomplete set of variables (i.e., they're not all linearly independent of each other, due to the Mandelstam identities):

$$\mathcal{U}_{C_1}\mathcal{U}_{C_2} = \mathcal{U}_{C_1\circ C_2} + \mathcal{U}_{C_1\circ C_2^{-1}}, \tag{8.28}$$

where C_1, C_2 are closed curves sharing a point, as doodled below:



There's a nice basis called the **"Spin Network Basis"**, and we can claim to solved 3 of the constraints of quantum gravity.

Lecture 9.

Consider a wave functional $\Psi[\varphi(x)]$ in a Schrodinger type picture in quantum field theory. Consider infinitesimal deformations of the field

$$\Psi[\varphi(x) + \varepsilon(x)] = \Psi[\varphi(x)] + \int \frac{\delta\Psi}{\delta\varphi}(x_1)\varepsilon(x_1)\,\mathrm{d}^n x_1 + \mathcal{O}(\varepsilon^2). \tag{9.1}$$

If the field is invariant under $\varphi \to \varphi + \widetilde{\varepsilon}$, then

$$\Psi[\varphi + \widetilde{\varepsilon}] = \Psi[\varphi], \tag{9.2}$$

and moreover

$$\int \frac{\delta\Psi}{\delta\varphi}(x_1)\widetilde{\varepsilon}(x_1)\,\mathrm{d}^n x_1 = 0. \tag{9.3}$$

This is useful for computing vacuum expectation values.

Suppose we have a constraint. For us, we have the Gauss Law constraint

$$D_i\widetilde{E}^{i\widehat{I}} = \partial_i\widetilde{E}^{i\widehat{I}} + \epsilon^{\widehat{I}\widehat{J}\widehat{K}}A_{i\widehat{J}}\widetilde{E}^{i}{}_{\widehat{K}} = 0, \tag{9.4}$$

where in the Schrodinger picture we have,

$$\widetilde{E}^{i\widehat{I}} = -8\pi\gamma G_N \hbar \frac{\delta}{\delta A_{i\widehat{I}}}. \tag{9.5}$$

The constraint is linear in functional derivatives. What we can do is look at the integral of the constraint against a test function,

$$\int \lambda_{\widehat{I}} D_i \widetilde{E}^{i\widehat{I}} \,\mathrm{d}^n x = 0. \tag{9.6}$$

If this is true for arbitrary $\lambda_{\widehat{I}}$, then integration by parts

$$\int \lambda_{\widehat{I}} D_i \widetilde{E}^{i\widehat{I}} \,\mathrm{d}^n x = 8\pi G_N \hbar\gamma \int D_i \lambda_{\widehat{I}} \frac{\delta}{\delta A_{i\widehat{I}}} \,\mathrm{d}^n x. \tag{9.7}$$

Our constraint is then, when applied to a wave functional,

$$8\pi G_N \hbar\gamma \int D_i \lambda_{\widehat{I}} \frac{\delta}{\delta A_{i\widehat{I}}} \,\mathrm{d}^n x \,\Psi[A] = 0. \tag{9.8}$$

This is the first term in a Taylor expansion

$$\Psi[{A_i}^{\widehat{I}} + D_i\lambda^{\widehat{I}}] = \Psi[{A_i}^{\widehat{I}}]. \tag{9.9}$$

We can also have gauge transformations not built up from infinitesimal transformations (e.g., time reversal) called **"Large Gauge Transformations"**.

We get to the Wilson line (a.k.a., the parallel propagator), we have the holonomy

$$\mathcal{U}^{\widehat{J}}{}_{\widehat{K}} = \mathcal{P}\exp\left[-\int_C {A_i}^{\widehat{J}} \epsilon_{\widehat{I}}{}^{\widehat{J}}{}_{\widehat{K}} \,\mathrm{d}x^i\right], \tag{9.10}$$

or suppressing indices and letting $\tau_{\widehat{I}}$ be the generators of the gauge algebra,

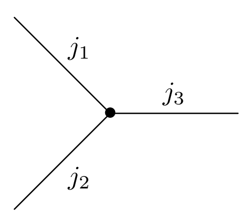
$$\mathcal{U} = \mathcal{P}\exp\left[-\int_C {A_i}^{\widehat{I}} \tau_{\widehat{I}} \,\mathrm{d}x^i\right]. \tag{9.11}$$

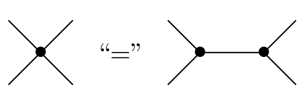
Observe this transforms under change of "coordinates" as

$$\mathcal{U} \to g^{-1}(s_2)\,\mathcal{U}\,g(s_1). \tag{9.12}$$

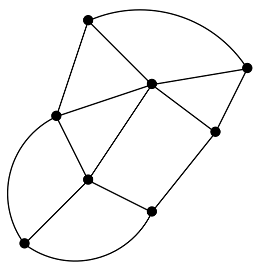
We wish to construct invariants, so we construct closed loops then take the trace of the holonomy $\mathcal{U}$ over the loop. This is an overcomplete set of variables.

The way out is to consider the intersection, as doodled to the right. We wish to consider this in detail. Give each edge of the graph depicting the intersection a representation of SU(2). We assign the vertex a Clebsch-Gordon coefficient. If we generalize this to n-edges meeting at a vertex, we can use an intertwiner instead of a Clebsch–Gordon coefficient. For SU(2), we have a neatway to combine things as vertices:





This works for SU(2), it may not necessarily work for an arbitrary gauge group. So in short:
(1) At each node, we have an intertwiner;
(2) At each edge, we have a representation of SU(2).



The spin network (generically doodled to the left) gives a complete (but not overcomplete) basis of states. Loop quantum gravity theorists like to say a spin network is a state. What they mean is: the spin network is a function of the connection A, which is all a state *is* in quantum gravity. A spin network eats in a value of A and spits out a complex number. One could use this for computation in, e.g., QCD (this is group field theory).

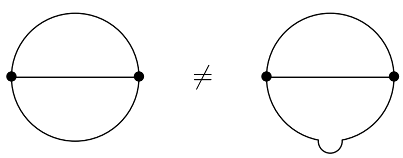
We don't need a smooth connection, we can *generalize* the connection so it gives Wilson lines along finite parts of space (it could be only where the edges are in fact). If we take the space of all connections modulo gauge transformations, and complete it (so it's a Hilbert space), then that's the Hilbert space we use.

Since a spin network is a state, we should probably define an inner product between two spin networks. We should consider the usual way to define the inner product on the Hilbert space just described as something like

$$\langle\Psi \mid \Phi\rangle \sim \int_{\mathcal{A}/\mathcal{G}} \Psi^*[A]\Phi[A]\,[\mathrm{d}A]. \tag{9.13}$$

This is the only gauge-invariant inner product. We can get close to a delta function, specifying geomtries down to the Planck length, using weave states.

Note: the spin networks doodled below,

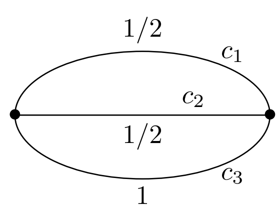


are distinct, since the lines are Wilson lines, the integral *changes*.

Lecture 10.

We need the spatial topology to not change, otherwise we can end up with a number of baby universes ("polymer topology").

Let us consider the simplest spin network, we need 2 nodes and 3 edges. (If we had 2 edges, then we would obtain the identity spin network.) We don't have spin 0, as the propagator would be the identity.



For each of these lines we have the Wilson line

$$\mathcal{U} = \mathcal{P}\exp[-\int A]. \tag{10.1}$$

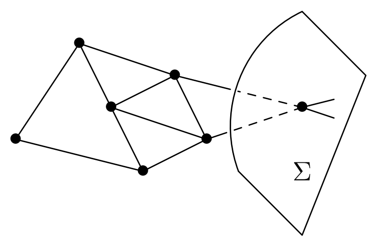
We have three Wilson lines $\mathcal{U}_1{}^{m_1}{}_{n_1}$, $\mathcal{U}_2{}^{m_2}{}_{n_2}$, and $\mathcal{U}_3{}^{m_3}{}_{n_3}$. Conceptually, the $\mathcal{U}$'s tell spin-1/2 objects rotate in spin space. We have $m_1 = 1/2, -1/2$ for spin up and spin down (respectively). We see m_2 also describes a spin-1/2 object, but m_3 describes a spin-1 object with possible values $m_3 = -1, 0, +1$. We can now use the Clebsch–Gordon coefficients $\langle j\, m \mid j_1\, m_1\, , j_2\, m_2\rangle$ and find

$$\sum_{\substack{m_1,m_2,m_3\\ n_1,n_2,n_3}} \mathcal{U}_1^{m_1}{}_{n_1}\mathcal{U}_2^{m_2}{}_{n_2}\mathcal{U}_3^{m_3}{}_{n_3}\langle 1\, m_3 \mid \frac{1}{2}\, m_1\, \frac{1}{2} m_2\rangle\langle 1\, n_3 \mid \frac{1}{2}\, n_1\, \frac{1}{2} n_2\rangle, \tag{10.2}$$

which is a function of A.

10.1 Area Operator

We take a surface Σ, we can ask "What is the area of the surface?" Suppose we have some spin network that "goes through" our surface Σ:



We won't consider an edge of the spin network "grazing" the surface, or lies inside the surface: we care about puncturing edges.

We will only really consider a simple example choosing a surface where $x^3 = 0$, the area of the surface would be classically

$$A = \int \sqrt{{}^{(2)}g}\,\mathrm{d}^2x. \tag{10.3}$$

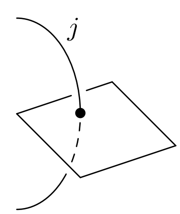
We see

$${}^{(2)}g = g_{11}g_{22} - 2(g_{12})^2 = \widetilde{E}^3_{\widehat{I}}\widetilde{E}^{3\widehat{I}}. \tag{10.4}$$

So the area is

$$A = \int \sqrt{\widetilde{E}^3_{\widehat{I}}\widetilde{E}^{3\widehat{I}}}\,\mathrm{d}^2x. \tag{10.5}$$

Consider a more general surface with coordinates σ^1, σ^2. Then our considerations change by

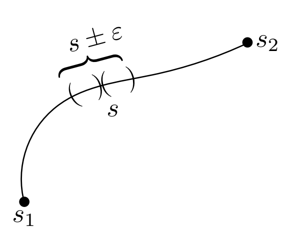


$$\widetilde{E}^3{}_{\widehat{I}} \to \epsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\widetilde{E}^k{}_{\widehat{I}}. \tag{10.6}$$

In the classical arena, the criteria for a "small region" is not really well-defined; in the quantum arena, we just require a single piercing of a spin network with the surface (as doodled in the margin). We can turn this now into an operator

$$\widetilde{E}_{\widehat{I}} = -8\pi G_N\gamma\int \epsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\frac{\delta}{\delta A^{\widehat{I}}_k}\,\mathrm{d}\sigma^1\mathrm{d}\sigma^2. \tag{10.7}$$

We need to consider $\delta\mathcal{U}/\delta A^{\widehat{I}}_k$. If we didn't have path-ordering, then this would be trivial. But we must be careful, since things do not commute. Consider the path doodled below:



We have

$$\mathcal{U}(s_2, s_1) = \mathcal{U}(s_2, s)\mathcal{U}(s, s_1) \tag{10.8a}$$

$$= \mathcal{U}(s_2, s+\varepsilon)\mathcal{U}(s+\varepsilon, s-\varepsilon)\mathcal{U}(s-\varepsilon, s_1). \tag{10.8b}$$

If ε is small enough, we have

$$\frac{\delta}{\delta A_i^{\widehat{I}}(s)}\mathcal{U}(s_2, s_1) = \mathcal{U}(s_2, s)\left(-\tau_{\widehat{I}}\frac{\mathrm{d}x^i}{\mathrm{d}s}\right)\mathcal{U}(s, s_1). \tag{10.9}$$

More generally,

$$\frac{\delta}{\delta A_i^{\widehat{I}}(x)}\mathcal{U}(s_2, s_1) = \int \delta^{(3)}(C(s)-x)\,\mathcal{U}(s_2, s)\left(-\tau_{\widehat{I}}\frac{\mathrm{d}x^i}{\mathrm{d}s}\right)\mathcal{U}(s, s_1)\,\mathrm{d}s, \tag{10.10}$$

and is zero if x does not lie on the curve C. We now see that

$$E_{\widehat{I}}\mathcal{U}(s_2, s_1) = 8\pi\gamma G_N \int \epsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\frac{\partial x^k}{\partial s}\delta^{(3)}(C(s)-x)\,\mathcal{U}(s_2, s)\tau_{\widehat{I}}\mathcal{U}(s, s_1)\,\mathrm{d}\sigma^1\mathrm{d}\sigma^2\mathrm{d}s. \tag{10.11}$$

We see that

$$\int \epsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\frac{\partial x^k}{\partial s}\delta^{(3)}(C(s)-x)\,\mathrm{d}\sigma^1\mathrm{d}\sigma^2\mathrm{d}s$$

is called the "oriented intersection number" (it's ± 1 if $C(s)$ intersects Σ, and 0 otherwise).

The moral of the story is that the oriented intersection number $I(C, \Sigma)$ is used to find

$$E_{\widehat{I}}\mathcal{U}(s_2, s_1) = 8\pi G_N\gamma I(C, \Sigma)\mathcal{U}(s_2, s)\tau_{\widehat{I}}\mathcal{U}(s, s_1), \tag{10.12}$$

where $C(s)$ is the point of intersection.

Lets consider

$$E_{\widehat{I}}E^{\widehat{I}}\mathcal{U}(s_2, s_1) = (8\pi G_N\gamma)^2\mathcal{U}(s_2, s)\tau_{\widehat{I}}\tau^{\widehat{I}}\,\mathcal{U}(s, s_1). \tag{10.13}$$

We see for SU(2), $\tau_{\widehat{I}}\tau^{\widehat{I}}$ is the quadratic Casimir (it's J^2 in quantum angular momentum), so we can plug in its eigenvalue $j(j+1)$ giving us

$$E_{\widehat{I}}E^{\widehat{I}}\mathcal{U}(s_2, s_1) = (8\pi G_N\gamma)^2 j(j+1)\mathcal{U}(s_2, s_1), \tag{10.14}$$

assuming there is an intersection. We can write a spin network state $\mid s\rangle$, so

$$E_{\widehat{I}}E^{\widehat{I}} \mid s\rangle = \sum_{\text{intersections}} (8\pi G_N\gamma)^2 j(j+1) \mid s\rangle. \tag{10.15}$$

We can now define the area operator

$$\widehat{A} = \sum_{\substack{\text{small regions}\\ \text{of } \Sigma}} \sqrt{E_{\widehat{I}}E^{\widehat{I}}}. \tag{10.16}$$

Classically we had

$$A = \int (E^3_{\widehat{I}}E^{3,\widehat{I}})^{1/2}, \tag{10.17}$$

so we see a direct connection really, it's a sensible definition. Given this area operator, we see that when it acts on a spin network that

$$\widehat{A}_\Sigma \mid s\rangle = \sum_{\text{intersections}} 8\pi G_N\gamma\sqrt{j(j+1)} \mid s\rangle. \tag{10.18}$$

The spetrum of the area operator is discrete. The spacing between high j's is smaller than the spacing for low j's. There are some attempts at number theoretic explanations.

The volume operator can be defined similarly using the product of 3 $\widetilde{E}$ factors instead of 2. Its construction is horrible. The area operator has contributions from edges, but the volume operator has contributions from vertices "with enough edges" (4 edges at a node should be viewed as dual to a tetrahedron, 3 edges has area but no volume). At this point, the volume spectrum is not well understood.

If we have a cubic network, if we have too many edges, then the metric "looks flat". There are angle operators with fairly odd properties. One fairly old reference but a beautiful introduction to these geometric operators:

- C. Rovelli and P. Upadhya, "Loop quantum gravity and quanta of space: A Primer". arXiv:gr-qc/9806079.

Lecture 11.

The Hamiltonian constraint gives surface deformations. So far we have not looked at the momentum or Hamiltonian constraints. It's like Einstein's field equations are missing. We have

$$V_i = F^{\widehat{I}}_{ij}\widetilde{E}^j_{\widehat{I}} = 0. \tag{11.1}$$

Let us first consider a trick useful in many circumstances. Consider

$$\xi^i F^{\widehat{I}}_{ij} = \xi^i(\partial_i A^{\widehat{I}}_j - \partial_j A^{\widehat{I}}_i + \epsilon^{\widehat{I}\widehat{M}\widehat{N}} A_{i\widehat{M}} A_{j\widehat{N}}) \tag{11.2a}$$

$$= \xi^i\partial_i A^{\widehat{I}}_j - \partial_j(\xi^i A^{\widehat{I}}_i) + (\partial_j\xi^i)A^{\widehat{I}}_i + \epsilon^{\widehat{I}\widehat{M}\widehat{N}}\xi^i A_{i\widehat{M}} A_{j\widehat{N}} \tag{11.2b}$$

$$= \underbrace{\xi^i\partial_i A^{\widehat{I}}_j + (\partial_j\xi^i)A^{\widehat{I}}_i}_{\substack{\text{change of } A \text{ under an}\\ \text{infinitesimal change of}\\ \text{coordinates}}} - D_j(\xi^i A^{\widehat{I}}_i) \tag{11.2c}$$

$$= \underbrace{\delta_\xi A^{\widehat{I}}_i}_{\text{Lie derivative}} - D_j(\xi^i A^{\widehat{I}}_i). \tag{11.2d}$$

The moral is that a vector contracted with the field strength tensor looks liek a covariant derivative and a diffeomorphism (gauge transformation). So we have

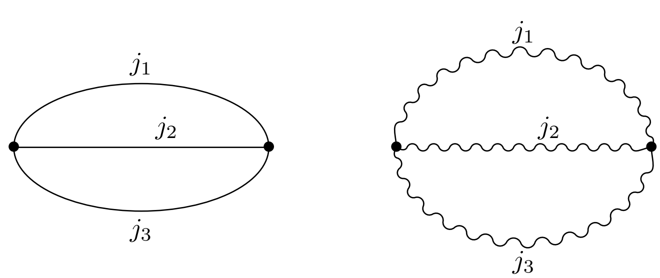
$$\int \xi^i V_i \,\mathrm{d}^3x \sim \int (-D_i(\xi^i A^{\widehat{I}}_i) + \delta_\xi A^{\widehat{I}}_i)\frac{\delta}{\delta A^{\widehat{I}}_i}\mathrm{d}^3x, \tag{11.3a}$$

which acts on fields like:

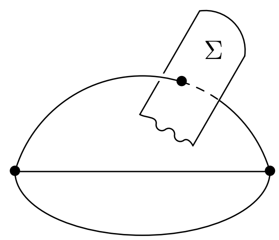
$$A \to A - D\varepsilon + \delta_\xi A, \tag{11.3b}$$

where $\varepsilon^{\widehat{I}} = \xi^i A^{\widehat{I}}_i$. This is exactly analogous to the statement in quantum mechanics that $\widehat{p}$ generates translations in position.

We have these spin network states. We would like them to be invariant under gauge transformations. Forget about position and think about a spin network in the graph theoretic notion of a network. For instance consider the two spin networks doodled below:



We now treat them as the same spin network (despite in our earlier treatment, we would have treated them as distinct spin networks). That is to say, *before* we would have found the inner product of these two spin states would vanish, *but now* we will find their inner product is unity (i.e., 1). There is an obvious problem here, though: spin networks are not "functions" of connections since thre is no longer any background space. This is fine on the one hand, but it's harder to find how to get physics from this. The trick is to define a diffeomorphism invariant notion of what a surface is, so whn we perform some diffeomorphism of a surface, and the surface Σ is punctured by a spin network, dragging the surface Σ would cause a diffeomorphism in the spin network; i.e., everything is dragged along.



Here we need some extra parameters, e.g., if we had a scalar field, there would be some extra information that needs to be preserved under diffeomorphisms. (This is what we do in practice.)

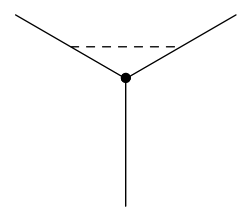
Thus far we have left out the Hamiltonian constraint, but for agood reason: no one knows how to deal with it. Recall it looks like

$$S \sim FEE + (1+\gamma^2)(\text{big mess}). \tag{11.4}$$

Here the Hamiltonian is quadratic in functional derivatives. There are some tricks that may possibly work; e.g., we can see the volume operator looks like $V \sim E^3$, so we have the Poisson bracket $\{V, A\} \sim E^2$, which permits us to rewrite the first term as

$$FEE \sim F\{V, A\}. \tag{11.5}$$

Recall, to get F we found the path ordered exponential integral (a.k.a., the holonomy) $\mathcal{P}\mathrm{e}^{-\int A} \sim 1 + \int F + \cdots$ But to do that for a spin network, we work around a node and take the dashed line (doodled to the right) to go to the node (its length vanishing). But diffeomorphism invariance allows us towiggle the line, so its vanishing no longer really matters. The regulation becomes independent of the regulator.

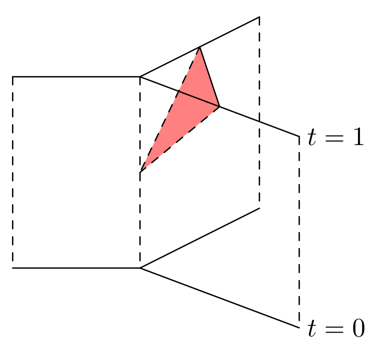


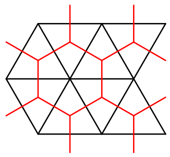
As far as the "big mess" term in the Hamiltonian constraint, Thomas Thiemann has done a lot of work trying to simplify it.

Another problem is that the Hamiltonian acts node by node, so it may be too local (resulting in nothing propagating).

It may be these are not problems, we just don't know. We'd need to know information about the solution to the Hamiltonian constraint.

Spin foams may give an alternative point of view to time evolution of spin networks. If we want to consider how a spin network changes, we can change the spin label or the intertwiners (which is hard) or we can add edges (or remove edges). We have several ingredients: the node of a spin network is promoted to an edge in the spin foam, the edge becomes a plane, and we need some way to deal with the vertex in the spin foam.





Recall the notion of duals to simplices. In two dimensions, the dual is doodled to the left in red, and in three dimensions it is generalized accordingly. The trick is for the spin foam, we have a sort of dictionary identifying various things to arious other parts of the dual 4-simplex. The use of a spin foam is to give transition amplitudes to evolving spin networks.

As a closing remark, there are field theories which use group elements on the edges of its Feynman diagrams, but these appear to be "dual" to spin foams—so in a sense, gravity "emerges".[8]

[8] I believe this remark refers to group field theory.

Part III
String Theory

Lecture 12.

References for loop quantum gravity.

- About Spin Foams
  - J. C. Baez, "An Introduction to Spin Foam Models of *BF* Theory and Quantum Gravity". *Lect. Notes Phys.* **543** (2000), 25-93; arXiv:gr-qc/9905087 doi:10.1007/3-540-46552-9_2
  - A. Perez, "Spin foam models for quantum gravity". *Class. Quant. Grav.* **20** (2003) R43; arXiv:gr-qc/0301113 doi:10.1088/0264-9381/20/6/202
- Group Field Theory
  - D. Oriti, "The Group field theory approach to quantum gravity". arXiv:gr-qc/0607032.
- Critique of Loop Quantum Gravity:
  - H. Nicolai and K. Peeters, "Loop and spin foam quantum gravity: A Brief guide for beginners". *Lect. Notes Phys.* **721** (2007) 151-184; arXiv:hep-th/0601129. doi:10.1007/978-3-540-71117-9_9
- Covariant Canonical Quantization
  - A. Ashtekar, L. Bombelli and O. Reula, "The Covariant Phase Space of Asymptotically Flat Gravitational Fields". In *Mechanics, Analysis and Geometry: 200 Years After Lagrange*, pp.417–450, Elsevier, 1991. doi:10.1016/B978-0-444-88958-4.50021-5.

12.1 String Theory

A very brief introduction to string theory, but we'll focus on its relevance to gravity. There are several points to consider

(1) Closed loops have a massless spin-2 excitation ("graviton")
(2) Strings propagate only in a spacetime satisfying the Einstein field equations (plus some negligible corrections)

(Any theory with self-interacting spin-2 massless excitations is a hint that gravity is in the game.)

(3) Background spacetime of the second point is equivalent to a coherent state of excitations of the first point.

Review of classical spin-2 "gravitons"

Let us examine the first point. We have the basic field be some tensor with two indices $h_{\mu\nu}$ and the field equations look like:

$$\Box h_{\mu\nu} + (\text{terms involving } \partial_\rho h^{\rho\sigma}) = T_{\mu\nu}. \tag{12.1}$$

The most general result is that we end up with a spin-2 part, a vector (spin-1) part and a scalar (spin-0) part. We need these extra (vector and scalar) parts vanish, which is a gauge choice (analogous to a spin-1 field $\Box A_\mu + k\partial_\mu\partial_\nu A^\nu = J_\mu$ requires $\partial^\mu J_\mu$, otherwise we do not have electromagnetism). The gauge invariance for us is

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{12.2}$$

which demands

$$\partial_\mu T^{\mu\nu} = 0 \tag{12.3}$$

for consistency. We can choose gauge $\partial_\rho h^{\rho\sigma} = 0$ (Lorenz gauge, Harmonic gauge, de Donder gauge, Fock gauge, Harmonic gauge, Feynman gauge, Lorenz gauge, etc.). The reason harmonic is sometimes we write $\Box X^\mu$ where X^μ is some parametrized version of the coordinates which individually transform as scalars (and $\Box$ uses derivatives with respect to the parameters).

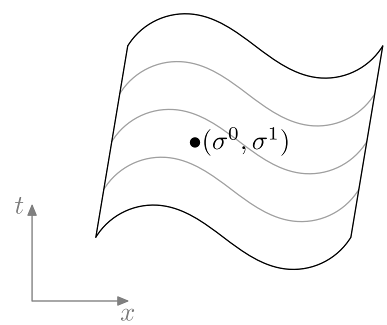
For consistency, after choosing the gauge, we add contributions of order h^2 to the stress-energy tensor to the right-hand side

$$\Box h_{\mu\nu} = T_{\mu\nu} + T^{(h)}_{\mu\nu} \tag{12.4}$$

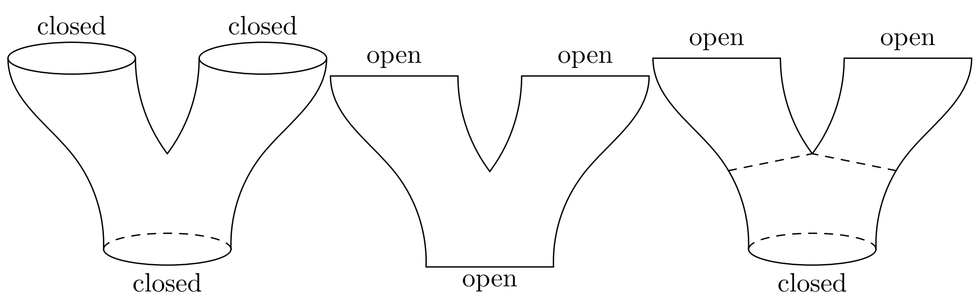
but then we will need to add cubic interactions, and then quartic interactions, etc. Deser showed (reprinted as arXiv:gr-qc/0411023), using an incredibly clever choice of variables, the series terminates. Damour and Henneaux (arXiv:hep-th/0007220 and arXiv:hep-th/0009109) used clever arguments to show there are some extra terms using cohomological techniques. This work done by Deser, Damour and Henneaux, are entirely classical, but there are some soft graviton theorems.

Strings, Worldsheet

Let's start with string theory, we will start by talking about strings. Open strings trace out a 2-dimensional surface with intrinsic coordinates $(\sigma, \tau) = (\sigma^0, \sigma^1)$. We can write the 4-coordinates of the surface in terms of σ and τ.



We can model interactions using pant diagrams, for example:



These are the only possible interactions. These are, of course, in a fixed background. The common statistic is that there are 10^{500} possible backgrounds. We write the metric for this background as $G_{\mu\nu}$ and the induced metric on the worldsheet is,

$$h_{ab} = G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}. \tag{12.5}$$

Nambu–Goto Action
Polyakov Action

The generalization of the action for a world line is the area of the worldsheet,

$$I = \frac{-1}{2\pi\alpha'} \int \sqrt{-h}\, \mathrm{d}^2\sigma. \tag{12.6}$$

This is the Nambu–Goto action.

Quantizing the Nambu–Goto action turns out to be quite difficult, which we should expect with any Lagrangian involving the squareroot of quadratic terms. This leads us to consider the generalization of this action, the Polyakov action, which requires us to use a new induced metric γ_{ab},

$$I = \frac{-1}{4\pi\alpha'}\int \gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}\sqrt{-\gamma}\,\mathrm{d}^2\sigma. \tag{12.7}$$

Classically this is equivalent to the Nambu–Goto action, where the metric γ behaves as a Lagrange multiplier (since its derivatives do not appear in the action). We see varying the action with respect to γ yields

$$\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = 0. \tag{12.8}$$

We can solve this equation for γ_{ab} to write the induced metric as

$$\gamma_{ab} = 2f(\sigma^0,\sigma^1)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \tag{12.9}$$

where

$$\frac{1}{f(\sigma^0,\sigma^1)} = \gamma^{cd}\partial_c X^\mu \partial_d X^\nu G_{\mu\nu}. \tag{12.10}$$

This action has Weyl symmetry,

$$\gamma_{ab} \to \Omega(\sigma^c)\gamma_{ab}, \tag{12.11}$$

where Ω is everywhere positive. We see that the Polyakov action may be rewritten as,

$$I = \frac{-1}{4\pi\alpha'}\int \frac{\sqrt{-\gamma}}{f(\sigma^0,\sigma^1)}\,\mathrm{d}^2\sigma. \tag{12.12}$$

Taking advantage of the fact f is everywhere positive, we see we can classically recover the Nambu–Goto action by the Weyl symmetry transformation $\gamma_{ab} \to \gamma_{ab}/\sqrt{f}$. Polchinski proved the two actions are the same quantum mechanically. We can look at the string worldsheet as fundamental, then view the X^μ living on the worldsheet, and four-dimensional spacetime "emerges".[9]

Lecture 13.

Today we will discuss only a crude quantization of strings, just to see how a picture of quantum gravity might look like.

We will first consider an open string in flat Minkowski spacetime. Its action,

$$I = \frac{-1}{4\pi\alpha'}\int \gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\sqrt{-\gamma}\,\mathrm{d}^2\sigma. \tag{13.1}$$

The equations of motion are roughly

$$\partial^a \partial_a X^\mu \sim 0. \tag{13.2}$$

We need to impose boundary conditions; the two obvious ones are Neumann $\boldsymbol{n}^a\partial_a X^\mu = 0$ at the endpoints (where $\boldsymbol{n}^a$ is the normal vector at the endpoints), and the Dirichlet $X^\mu = \text{fixed}$ at the endpoints.

Observe the graviton has no such boundary conditions, and open strings cannot describe massless particles of spin greater than 1, so we must use a closed string for the graviton.[10] We have 3 gauge invariances (2 diffeomorphisms, 1 Weyl invariance). This is simple enough that we can choose 3 gauge conditions (technically we should verify the consistency with the constraints, but it doesn't get to the interesting point):

[9] Ed Witten's "Reflections on the Fate of Spacetime" (*Physics Today*, April 1996, pp.24–30) argues this heuristically; Nick Huggett and Christian Wüthrich's "Out of Nowhere: The 'emergence' of spacetime in string theory" (arXiv:2005.10943) reviews this general subject.

[10] For more on open strings, see Carlo Angelantonj and Augusto Sagnotti's review article arXiv:hep-th/0204089.

(0) Define $X^{\pm} = (X^0 \pm X^1)/\sqrt{2}$ (observe: this breaks Lorentz invariance),
(1) Choose $X^+ = \tau$ "light-front coordinates" (or "light-cone coordinates" or "light-cone gauge"),
(2) Choose $\partial_\sigma \gamma_{\sigma\sigma} = 0$, and
(3) Choose $\det(\gamma_{ab}) = -1$.

These choices have no deep physical meaning, they just simplify the mathematics. The first two gauge conditions ($X^+ = \tau$ and $\partial_\sigma\gamma_{\sigma\sigma} = 0$) deal with the diffeomorphisms, whereas the last gauge condition ($\det(\gamma_{ab}) = -1$) deals with Weyl invariance. Before going further, we use the notation

$$\bar{X}^- = \frac{1}{\ell}\int_0^\ell X^- \,\mathrm{d}\sigma \tag{13.3}$$

for the center-of-mass for the X^- coordinate.

Plugging these choices into the action gives us,

$$\begin{aligned} I = \frac{-1}{4\pi\alpha'}\iint \Big[\gamma_{\sigma\sigma}(2\partial_\tau\bar{X}^- - \partial_\tau X^{'}\partial_\tau X^{'}) \\ - 2\gamma_{\sigma\tau}(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) \\ + \frac{1}{\gamma_{\sigma\sigma}}(1-\gamma_{\sigma\tau}^2)\partial_\sigma X^i\partial_\sigma X^i\Big]\,\mathrm{d}\sigma\mathrm{d}\tau \end{aligned} \tag{13.4}$$

where $i = 2, \dots, D$ and $X^- = \bar{X}^- + Y^-$.

By varying the action with respect to Y^- gives us the equations of motion

$$\partial_\sigma\gamma_{\sigma\tau} = 0. \tag{13.5}$$

Observe, since Y^- doesn't appear in the action with a time derivative, it acts like a Lagrange multiplier. Since $\gamma_{\sigma\tau} = 0$ at the boundary, and $\partial_\sigma\gamma_{\sigma\tau} = 0$ at the boundary, it follows that $\gamma_{\sigma\tau} = 0$ everywhere. This simplifies the action to two pieces

$$I = \frac{-1}{4\pi\alpha'}\iint\left[\gamma_{\sigma\sigma}(2\partial_\tau\bar{X}^- - \partial_\tau X^{'}\partial_\tau X^{'}) + \frac{1}{\gamma_{\sigma\sigma}}\partial_\sigma X^i\partial_\sigma X^i\right]\mathrm{d}\sigma\mathrm{d}\tau. \tag{13.6}$$

We've eliminated X^+ and Y^-, so roughly speaking we have $D-2$ components. The transverse fluctuations are described by the second term, the motion of the center-of-mass is described by the first term. We write for the first term's momentum,

$$p_- = \frac{-\ell}{2\pi\alpha'}\gamma_{\sigma\sigma} = -p^+, \tag{13.7a}$$

and the second term's momentum,

$$\pi^i = \frac{p^+}{\ell}\partial_\tau X^i. \tag{13.7b}$$

We see the first term in the action Eq (13.6) is just the relativistic particle, and the second term is just a harmonic oscillator. We can now write,

$$H = \frac{\ell}{4\pi\alpha' p^+}\int\left[2\pi\alpha'(\pi^i)^2 + \frac{1}{2\pi\alpha'}(\partial_\sigma X^i)^2\right]\mathrm{d}\sigma \tag{13.8}$$

We see

$$\begin{aligned} \partial_\tau p^+ &= \frac{\partial H}{\partial \bar{X}^-} \quad \text{by Hamilton's equations} && (13.9\text{a}) \\ &= 0 && (13.9\text{b}) \end{aligned}$$

hence p^+ is a constant of motion. We can choose units $\ell/(2\pi\alpha' p^+) = 1$.

We now want to quantize it, we see that $\bar{X}^-$ and p^+ are conjugate variables, so

$$[\bar{X}^-, p^+] = \mathrm{i}. \tag{13.10}$$

We can now write

$$X^i(\sigma,\tau) = \bar{X}^i + \underbrace{\left(\frac{p^i}{p^+}\right)}_{\substack{\text{arbitrary choice}\\ \text{makes life}\\ \text{easier}}}\tau + \underbrace{\mathrm{i}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i \mathrm{e}^{-\mathrm{i}\pi n\tau/\ell}\cos\left(\frac{\pi n\sigma}{\ell}\right)}_{\text{just a Fourier series expansion for } X} \tag{13.11}$$

If we plug this in and find the conjugate momentum, we find the commutation relations

$$[\bar{X}^i, p^j] = \mathrm{i}\delta^{ij}, \tag{13.12}$$

and similarly

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{0,m+n}. \tag{13.13}$$

This shouldn't be too surprising, $\alpha_{-n}^i = (\alpha_n^i)^\dagger$ for X to be real. (This should be familiar: it *is* a simple harmonic oscillator.)

Let's look at the states of the string, there is a vacuum $\mid 0, k\rangle$ with

$$p^+ \mid 0,k\rangle = k^+ \mid 0,k\rangle \tag{13.14a}$$

$$p^i \mid 0,k\rangle = k^i \mid 0,k\rangle \tag{13.14b}$$

$$\alpha_m^i \mid 0,k\rangle = 0 \quad \text{for } m \geq 0. \tag{13.14c}$$

We basically have a bunch of harmonic oscillators.

Now, we can work out the Hamiltonian, and we will find,

$$H = \frac{1}{2p^+}(p^i)^2 + \frac{1}{2p^+\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i + \underbrace{A}_{\text{constant, zero-point energy}} \tag{13.15}$$

This is just the harmonic oscillator Hamiltonian. The constant term is thus

$$A = (D-2)\sum_{n=1}^{\infty}\frac{1}{2}n \tag{13.16}$$

As everyone knows

$$\sum_{n=1}^{\infty} n = \frac{-1}{12}. \tag{13.17}$$

There are two ways to see this. The first way is to use the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. We analytically continue it, and its value at $s=-1$ is $\zeta(-1) = -1/12$.

The second way, the physicist's way, is to consider

$$\sum_{n=1}^{\infty} n\mathrm{e}^{-\varepsilon n} = -\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\sum_{n=1}^{\infty}\mathrm{e}^{-\varepsilon n} = -\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{\mathrm{e}^{-\varepsilon}}{1-\mathrm{e}^{-\varepsilon}}\right) = \frac{1}{\varepsilon^2} - \frac{1}{12} + \mathcal{O}(\varepsilon). \tag{13.18}$$

Being physicists, we throw away the divergent part, then take $\varepsilon \to 0$.

Either way, we plug this into our Hamiltonian, we find:

$$\begin{aligned} H &= \frac{1}{2p^+}(p^i)^2 + \frac{1}{2p^+\alpha'}\sum_{n=1}^{\infty} N_n - \frac{D-2}{24} \\ &= p^- \end{aligned} \tag{13.19}$$

since p^- generates time translations and we chose $X^+ = \tau$. We can use creation operators on the vacuum which gives excited states on the string. We can ask what is the value for the mass squared (since mass squared is a Lorentz invariant quantity),

$$\begin{aligned} m^2 &= p_\mu p^\mu = 2p^+p^- - (p^i)^2 \\ &= \frac{1}{\alpha'}\left(N - \frac{D-2}{24}\right) \end{aligned} \tag{13.20}$$

This means for the states of the string, $N = 0$ (vacuum) has $m^2 < 0$, so it's Tachyonic (which is bad!). For $N = 1$,

$$m^2 = \frac{1}{\alpha'}\left(1 - \frac{D-2}{24}\right) \tag{13.21}$$

and the states are just $\alpha^i_{-1} \mid 0, k\rangle$.

Lorentz invariance requires $m^2 = 0$, which requires

$$\frac{D-2}{24} = 1 \implies D = 24 + 2 = 26. \tag{13.22}$$

We can use the Chan–Paton generators to do some fancy tricks.

Lecture 14.

Now the massless state for the open string $\alpha^i_{-1} \mid 0, k\rangle$ acting on the vacuum, this corresponds to A^i $(i = 2, \dots, D)$. This either is a massless field, or we've broken Lorentz invariance.

For the closed string, the procedure is pretty much the same. We have the Fourier expansion be,

$$X^i = \bar{X}^i + \left(\frac{p^i}{p^+}\right)\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}^{\infty}\left[\frac{\alpha^i_n}{n}\mathrm{e}^{-\mathrm{i}2\pi n(\sigma+\tau)/\ell} + \frac{\widetilde{\alpha}^i_n}{n}\mathrm{e}^{-\mathrm{i}2\pi n(\sigma-\tau)/\ell}\right]. \tag{14.1}$$

If we're fixing the endpoints, the modes obey this boundary condition. If we've got a closed string, then the waves can propagate.

Now, we find the mass squared as,

$$m^2 = \frac{2}{\alpha'}\left(N + \widetilde{N} + 2\left(\frac{2-D}{24}\right)\right). \tag{14.2}$$

There are two number operators now. There is an extra symmetry, the zero-point for σ is arbitrary. Working through the constraints we find that,

$$P = -\int \pi^i \partial_\sigma X^i \,\mathrm{d}\sigma = -\frac{2\pi}{\ell}(N - \widetilde{N}) = 0. \tag{14.3}$$

For the excited state above the vacuum,

$$\alpha^i_{-1}\widetilde{\alpha}^j_{-1} \mid 0, 0, k\rangle \qquad m^2 = \frac{2}{\alpha'}\left(2 + 2\left(\frac{2-D}{24}\right)\right) \tag{14.4}$$

we see $m^2 = 0$ if $D = 26$.

Since α, $\widetilde{\alpha}$ are not symmetric, we can break their product up into a symmetric traceless part (which transforms as a spin-2 particle) G^{ij}, the trace which is just a scalar Φ called the dilaton, and there is also the antisymmetric part B^{ij} (sometimes referred to as the axion). The axion acts like a sort of gauge potential,

String spectrum:
G^{ij} graviton
Φ dilaton
B^{ij} axion

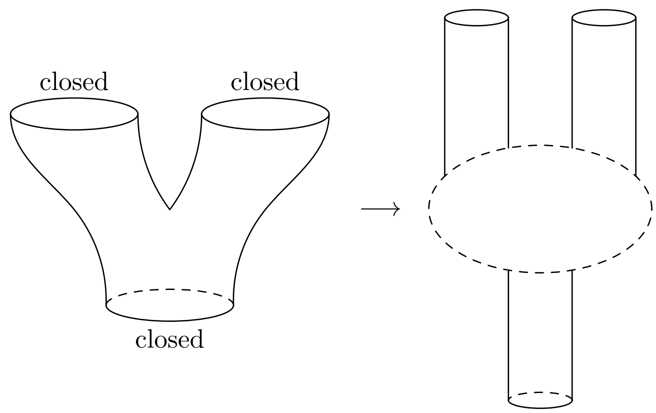
$$B_{ij} \to B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i. \tag{14.5a}$$

The analog of the field-strength tensor would be,

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}. \tag{14.5b}$$

This is the string spectrum (or particles present) in our action.

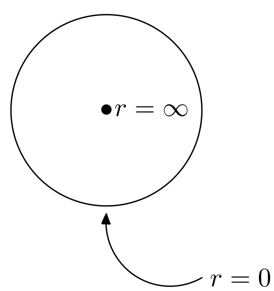
Now, let's look at a string interaction. We will examing the asymptotic behaviour of incoming and outgoing strings, something like:



Consider the metric of a 2-dimensional cylinder, labeling the height with the r coordinate,

$$\mathrm{d}s^2 = \mathrm{e}^{2\sigma}(\mathrm{d}r^2 + \mathrm{d}\theta^2) \tag{14.6}$$

Let $z = \exp(-r + \mathrm{i}\theta)$ and $\bar{z} = \exp(-r - \mathrm{i}\theta)$, then as $r \to \infty$ we have $z \to 0$ and $\bar{z} \to 0$. We similarly find, as $r \to 0$ that $z \to \exp(+\mathrm{i}\theta)$ (that is, z and $\bar{z}$ go to the unit circle) This gives us a way to describe interactions using the unit disk.



The state-operator correspondence specifies the outcome of a string interaction by examining an operator near the center of the disk. We have

$$\begin{aligned} \alpha^{\mu}_{-m} &= \sqrt{\frac{2}{\alpha'}} \int z^{-m} \partial_z X^m \frac{\mathrm{d}z}{2\pi} \\ &\sim (\partial_z^m X^{\mu})(0). \end{aligned} \tag{14.7}$$

This is a vertex operator. In practice we rarely work with string diagrams, we usually have vertex operators acting on some [Riemann] sphere. We see that

$$|\,0, k\rangle = {:}\mathrm{e}^{\mathrm{i}kx}{:}\,|\,0, 0\rangle. \tag{14.8}$$

Then we have a **"Vertex Operator"**,

$$V = (\text{const.}) \int \left[(\gamma^{ab} S_{\mu\nu} + \mathrm{i}\varepsilon^{ab} a_{\mu\nu}) \partial_a X^{\mu} \partial_b X^{\nu} + \alpha' \phi\, {}^{(2)}R \mathrm{e}^{\mathrm{i}kx} \right] \sqrt{-\gamma}\, \mathrm{d}^2\sigma \tag{14.9}$$

where $S_{\mu\nu}$ is symmetric, $a_{\mu\nu}$ is antisymmetric which corresponds to the graviton and axion vertex operators (respectively). The last term corresponds to the dilaton. We can work backwords starting from

$$G_{\mu\nu} = \eta_{\mu\nu} + S_{\mu\nu}, \tag{14.10}$$

then the action

$$I = \int \gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}\sqrt{-\gamma}\,\mathrm{d}^2\sigma \tag{14.11}$$

has its path-integral's integrand expands in powers of $S_{\mu\nu}$ like

$$\mathrm{e}^{\mathrm{i}I} = \mathrm{e}^{\mathrm{i}I(G=\eta)}\left(1 + \int \gamma^{ab}\partial_a X^\mu \partial_b X^\nu S_{\mu\nu}\sqrt{-\gamma}\,\mathrm{d}^2\sigma + \dots\right). \tag{14.12}$$

Lecture 15.

Remember, if we look at the massless states for open and closed strings we get $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, and A_μ.

We can also add fermions in two ways. We can look at spacetime fermions (described not just by x but also by fermionic coordinates). We end up with the Green–Schwartz model.

The other way is to add world sheet fermions. We have $X^\mu(\sigma,\tau)$ be our coordinates, $\psi^\mu(\sigma,\tau)$ be our fermion. To do this, it's easier to use 2-spinors:

$$\psi^\mu = \begin{pmatrix}\psi^\mu\\ \widetilde{\psi}^\mu\end{pmatrix} \tag{15.1}$$

There is a nice supersymmetry of the form

$$\delta X^\mu \sim \eta\psi^\mu - \eta^*\widetilde{\psi}^\mu \tag{15.2a}$$
$$\delta\psi^\mu \sim \eta\partial_z X^\mu \tag{15.2b}$$
$$\delta\widetilde{\psi}^\mu \sim \eta^*\partial_{\bar{z}} X^\mu, \tag{15.2c}$$

where $z = \tau + \mathrm{i}\sigma$. The Lagrangian looks like

$$\mathcal{L} \sim \partial_z X^\mu \partial_{\bar{z}} X_\mu + \psi^\mu\partial_{\bar{z}}\psi_\mu + \widetilde{\psi}^\mu\partial_z\widetilde{\psi}_\mu. \tag{15.3}$$

For a closed string, we can have boundary conditions $\psi(\sigma+\ell) = \psi(\sigma)$ or $\psi(\sigma+\ell) = -\psi(\sigma)$. We can make either choice consistently, the former is known as Ramond boundary conditions, the latter is Neveu–Schwartz boundary condition. We can consider a mode expansion

$$\begin{aligned} R\colon &\qquad \psi^\mu = \sum_{m\in\mathbb{Z}} d^\mu_m \mathrm{e}^{\mathrm{i}2\pi m\sigma/\ell}\\ NS\colon &\qquad \psi^\mu = \sum_{m\in\mathbb{Z}+\frac{1}{2}} b^\mu_m \mathrm{e}^{\mathrm{i}2\pi m\sigma/\ell}\end{aligned} \tag{15.4}$$

We can look at the coefficients as creation/annihilation operators

Vacuum states

$$\begin{aligned} R\colon &\qquad d^\mu_r \mid 0\rangle = 0 \text{ for } r \geq 1\\ NS\colon &\qquad b^\mu_r \mid 0\rangle = 0 \text{ for } r \geq \frac{1}{2}\end{aligned} \tag{15.5}$$

We can work out the commutators and anticommutators

$$\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}, \tag{15.6}$$

so the d_0 are Γ matrices. The Ramond boundary conditions yields spacetime spinors.

We can now ask what are the consistency conditions. There are five types:

type	fields	strings allowed	boundary conditions imposed
IIA	ψ_+, $\widetilde{\psi}_-$	closed	ψ has Ramond, $\widetilde{\psi}$ has NS
IIB	ψ_+, $\widetilde{\psi}_+$	closed	both have Ramond
I	SO(32)	open and closed	
heterotic	SO(32)	half closed, half open and closed	
heterotic	$\mathtt{E}_8$	half closed, half open and closed	

A few more string theory miracles. Suppose $X^9 \sim X^9 + 2\pi R$ (so we have a cylinder). There are two implications: T-duality

(1) The momentum is quantized $p^9 = n/R$,
(2) If we fix time, $X^9(\sigma+\ell) = X^9(\sigma) + 2\pi R\omega$ where $\omega \in \mathbb{Z}$ (and it is, in fact, the winding number).

We Fourier expand it,

$$X^9 = \frac{2\pi R\omega}{\ell}\sigma + \frac{n}{R}\tau + (\text{oscillators}). \tag{15.7}$$

We can work out the mass spectrum

$$m^2 = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{(\alpha')^2} + (\text{oscillator contributions}). \tag{15.8}$$

Observe as $R \to \alpha'/R$ and $n \leftrightarrow \omega$, we end up with exactly the same mass. The winding modes and momentum modes get switched.

This switching of n and ω is equivalent to switching τ and σ. The left moving momentum $\underset{\sim}{p_L} \to \underset{\sim}{p_L}$ but the right moving momentum $p_R \to -p_R$. The fermions also change sign $\underset{\sim}{\psi} \to -\psi$. This means IIA $\leftrightarrow$ IIB. This is the gist of T-duality.

For open strings, $R \to 0$ corresponds to the center-of-mass motion as $m \to \infty$. If we consider $\sigma \leftrightarrow \tau$, then Neumann boundary conditions $\partial_\sigma X^9 = 0$ become Dirichlet boundary conditions $\partial_\tau X^9 = 0$. The endpoints of the string sees $D-1$ dimensions, the rest of the string sees all D dimensions.

Lecture 16. AdS/CFT Correspondence, Causal Dynamical Triangulations.

The AdS/CFT [anti de Sitter Space, Conformal Field Theory] correspondence is a really interesting, nonperturbative string theory requiring a negative cosmological constant. Although unphysical, it could contain interesting analogies for our purposes. Recall Anti-De Sitter space is a flat solution to the Einstein field equations with a negative cosmological constant. We usually write the cosmological constant as

$$\Lambda = \frac{-1}{\ell^2} \tag{16.1}$$

where ℓ is the radius of the universe. The metric is

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1+(r/\ell)^2} + r^2\,\mathrm{d}\Omega^2 - \left(1 + \frac{r^2}{\ell^2}\right)\mathrm{d}t^2. \tag{16.2}$$

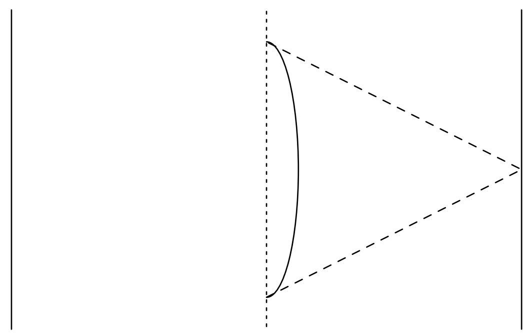
There is another set of coordinates $r = \ell\sinh(\rho)$, so

$$\mathrm{d}s^2 = \ell^2\left(\mathrm{d}\rho^2 + \sinh^2(\rho)\,\mathrm{d}\Omega^2 - \cosh^2(\rho)\,\mathrm{d}t^2\right). \tag{16.3}$$

Observe that ρ is just the proper distance. If we look at the limit as $\rho \to \infty$, the asymptotic limit, we have

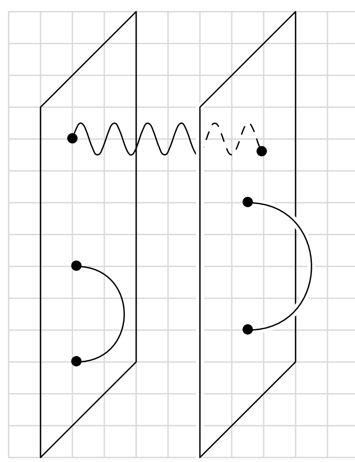
$$\mathrm{d}s^2 \sim \ell^2\,\mathrm{d}\rho^2 + \frac{\ell^2}{4}\mathrm{e}^{2\rho}(\mathrm{d}\Omega^2 - \mathrm{d}t^2). \tag{16.4}$$

Near infinity, AdS space looks like a flat cylinder with a radial coordinate added. If we look at geodesics, we can doodle timelike geodesics [solid line] and lightlike geodesics [dashed line] in the Penrose diagram:



Timelike geodesics which start at the center of the cylinder will try to "move out", but the cosmological constant then acts like an attractive potential. For lightlike geodesics, such geodesics will reflect off of the boundary. In theory, an observer could receive back a pulse of light they sent provided they live long enough. The isometry group for AdS is $\mathrm{SO}(d-1,2)$.

If we have a pair of D-branes, we can have the endpoints of a string be both on one D-brane, or one endpoint on each D-brane; something like sketched below:



On 2 D-branes, open string states have 2 indices varying from 1, 2. For N D-branes we have N indices with values being 1, 2. This may be oriented, not necessarily symmetrized. It turns out these indices have an $\mathrm{SU}(N)$ symmetry. There is an $N\to\infty$ limit where we may take to decouple gravitons, giving us an $\mathrm{SU}(N)$ Yang–Mills theory with a supersymmetry. We can look at the strong coupling limit, we end up with charged black branes.

We can look at things quite simply: these are dual to each other. That is, they each describe the entire picture at various limits. What we find is the near horizon geometry of charged black branes is AdS $\times$ (*compact*). If we look at N coincident 3-branes (3 spatial dimensions + time), this gives $\mathrm{AdS}_5\times S^5$ and its boundary is a four-dimensional flat cylinder.

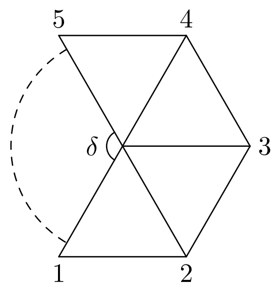
Maldacena AdS/CFT conjecture: string theory in bulk ($\mathrm{AdS}_5\times S^5$) $\iff$ $N=4$ Supersymmetric $\mathrm{SU}(N)$ Yang–Mills theory on a flat 4-dimensional cylinder. That is, "in bulk" means "in spacetime that's asymptotically $\mathrm{AdS}_5\times S^5$".

16.1 Causal Dynamical Triangulations

A lattice approach to quantum gravity. Nonrenormalizability may be a statement about the perturbative approach, we're doing the perturbation wrong—there's nothing wrong with the theory. We have the path integral approach, which has of supreme importance the partition function

$$Z = \int \mathrm{e}^{\mathrm{i}(I[g]+\int g_{\mu\nu}J^{\mu\nu})}\,[\mathrm{d}g] \tag{16.5}$$

where $J^{\mu\nu}$ is a fixed source. We approximate this as a sum over finite geometries, so we havea discrete approach using lattices (e.g., Regge calculus approach). The defecit angle δ, the Ricci scalar is determined by parallel transport $R = \delta\cdot\delta^{(2)}(\boldsymbol{x})$. For three dimensions, we work with 3-simplices, so imagine an edge sticking out of the paper at the vertex doodled below and making each triangle a face on the tetrahedron.



We find that, in three-dimensions,

$$\int R\sqrt{|g|}\,\mathrm{d}^3x = \sum_i \delta_i\ell_i. \tag{16.6}$$

In four-dimensions, we have,

$$\int R\sqrt{|g|}\,\mathrm{d}^4x = \sum_{\text{2-d hinges}} \delta_i A_i. \tag{16.7}$$

Regge figured this out in 1961.

So we end up with a discretization of the path integral

$$Z \approx \sum_{\substack{\text{discrete}\\ \text{geometries}}} \mathrm{e}^{\mathrm{i}(I_{\text{regge}}+\Lambda\sum_n V_n)} \tag{16.8}$$

There are several approaches to summing over geometries.

Regge advocated fixing ttriangulation, summing over lengths (and angles). The other approach is dynamical triangulations fix edge-lengths and sum over triangulations. In principle, either approach approximates smooth surfaces. The dynamical triangulations approach walks over the space of possible states. There are two phases in the computation

(1) The crumpled phase
(2) The branched polymer phase.

Dynamical triangulations is modified to become Causal Dynamical Triangulations, prohibiting polymer phases. We have a sort of timelike foliation, restricting the sort of triangulations. It's not entirely clear how to recover the Newtonian limit.

Lecture 17. Black Hole Thermodynamics.

The reference for today's lecture will be:

- Steven Carlip, "Black Hole Thermodynamics and Statistical Mechanics". arXiv:0807.4520, 35 pages.

Black holes are not black but thermal objects

$$kT = \frac{\hbar\kappa}{2\pi} \tag{17.1}$$

where κ is the "surface gravity" (which is $1/4GM$ for Schwarzschild black holes). They also have entropy

$$S_{BH} = \frac{A_{\text{horizon}}}{4\hbar G}. \tag{17.2}$$

In a sense, this is part of quantum gravity, since it involves $\hbar$, G.

Suppose we have a black hole with mass M, there is a "box of gas" with [characteristic] length L, temperature T, and mass m. We see that the change in entropy is

$$\Delta S = \frac{\Delta E}{T} = -\frac{m}{T}. \tag{17.3}$$

We better have a corresponding change in entropy for the black hole, or else a black hole could be used for a perpetual motion machine. Suppose the black hole is Schwarzschild, so its metric is

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2. \tag{17.4}$$

The proper distance ρ in the Schwarzschild metric for the box to be on the surface of the black hole is

$$\rho = \int_{2GM}^{2GM+\delta r} \frac{\mathrm{d}r}{\sqrt{1 - 2GM/r}} \sim \sqrt{GM\,\delta r} \tag{17.5}$$

We have $\rho = L$ when $\delta r \sim L^2/GM$. The mass m is redshifted when w eare far from the black hole, so the change of mass for the black hole would be,

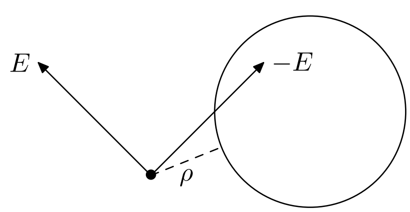
$$\Delta M \sim m\sqrt{1 - \frac{2GM}{2GM + \delta r}} \sim \frac{mL}{GM}. \tag{17.6}$$

To maximize the loss of entropy, we need $L \sim \hbar/T$. We find then

$$\Delta S \sim \frac{GM\Delta M}{\hbar} \sim \frac{\Delta A}{\hbar G}. \tag{17.7}$$

One of the laws of Black Hole Thermodynamics is $\Delta M = \frac{\kappa}{8\pi G}\Delta A$.

By the uncertainty principle, a virtual pair with energy E can exist for time $\sim \hbar/E$. Fort a virtual pair, with the negative energy particle entering the black hole, locally it's as though the black hole swallowed a particle of positive energy. Far away, it looks as though it emitted a particle of energy E.



The proper time for the particle to fall into the black hole would be

$$\tau \sim \sqrt{GM\,\delta r} \sim \frac{\hbar}{E} \tag{17.8}$$

hence

$$E \sim \frac{\hbar}{\sqrt{GM\,\delta r}}. \tag{17.9}$$

So to an observer far away, the energy is redshifted, so as seen from infinity,

$$E_\infty \sim \frac{\hbar}{\sqrt{GM\,\delta r}}\sqrt{1 - \frac{2GM}{2GM + \delta r}} \sim \frac{\hbar}{GM}. \tag{17.10}$$

This is precisely the energy corresponding to the black hole temperature. (Usually temperature is derived first, then standard thermodynamics is used to derive entropy.)

Let us briefly review quantum field theory, using discrete momentum. Let

Review: QFT in curved spacetime

$$u_k = \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega_k t), \tag{17.11}$$

where $\omega_k = +\sqrt{|k|^2 + m^2}$. The Fourier decomposition of the free field,

$$\varphi = \sum_k (a_k u_k + a_k^\dagger u_k^*), \tag{17.12}$$

we observe the commutation relations are such that

$$[a_k^\dagger, a_{k'}] = \delta_{k,k'}, \tag{17.13a}$$

$$a_k \mid 0\rangle = 0. \tag{17.13b}$$

We generalize to curved spacetime,

$$(\Box + m^2)u_k = 0 \tag{17.14}$$

where $\Box = \nabla_\mu\nabla^\mu$ uses covariant derivatives. There's an infinite number of orthogonal solutions (this happens in flat spacetimes, too, e.g., Bessel functions, Fourier decomposition, etc. etc. etc.). We can choose two different modes

$$\varphi = \sum_k (a_k u_k + a_k^\dagger u_k^*) = \sum_k (\bar{a}_k \bar{u}_k + \bar{a}_k^\dagger \bar{u}_k^*) \tag{17.15a}$$

where

$$\bar{u}_k = \sum (\underbrace{\alpha_{ki}\, u_i + \beta_{ki}\, u_i^*}_{\text{Bogoliubov coefficients}}) \tag{17.15b}$$

We have two number operators, N_k and $\bar{N}_k$. Using the orthogonality of mode functions, we can prove,

$$\langle\bar{0} \mid N_k \mid \bar{0}\rangle = \sum_j |\beta_{jk}|^2, \tag{17.16}$$

so how do we choose what vacuum to work in? Hawking argued if we're far away looking at the black hole region prior to collapse, there is a natural vacuum: the Minkowski vacuum.

There is another derivation from Parikh and Wilczek [10]. It's a tunneling approach, where particles tunnel from inside the black hole to the outside. The particle position doesn't move outward, the position of the event horizon moves inward. Then apply the WKB approximation. We have

$$\Gamma = \mathrm{e}^{-2\Im(I/\hbar)} \tag{17.17}$$

where I is the action. Now we just write

$$I = \int_{r_{\text{in}}}^{r_{\text{out}}} p_r\,\mathrm{d}r. \tag{17.18}$$

We use the metric and there is a pole $\sqrt{1 - 2GM/r}$.

In ordinary quantum mechanics, time translation operator is $\langle\text{out} \mid \exp(\mathrm{i}tH) \mid \text{in}\rangle$, for thermodynamics it is $\mathrm{Tr}(\mathrm{e}^{-\beta H})$, if we make time imaginary we get a partition function.

There are at least a dozen different modern ways to derive black hole entropy.

Part IV

Appendices

A Introduction to Quantum Gravity: Homework I[11]

I. **Planck scale**

The planck length is $\ell_{Pl} = \sqrt{\hbar G/c^3}$.

(a) Suppose you wish to probe an area of characteristic size R with a relativistic particle (that is, one for which $E \sim pc$). Consider the following two restrictions
   – uncertainty relation: $\Delta x \Delta p \gtrsim \hbar$
   – no black hole formed by problem: $G\Delta E/Rc^4 \lesssim 1$.

   Find an estimate of the smallest possible value of R. How would this change if you allow a *nonrelativistic* probe (that is, a massive probe with $mc^2 \gg pc$)?

(b) Consider a piece of matter of energy E that is not already a black hole. Its size must be greater than its Compton wavelength (quantum mechanics) and also large enough that it is not a black hole (general relativity). Approximately what is its minimum size?

(c) Recall that for any two quantum mechanical observables $\widehat{A}$ and $\widehat{B}$ an uncertainty principle holds

$$\Delta\widehat{A}\Delta\widehat{B} \geq \frac{1}{2}|\langle[\widehat{A},\widehat{B}]\rangle|$$

   For a free particle (in the Heisenberg picture) with position operator $\widehat{x}(t)$ and momentum operator $\widehat{p}$,

$$\widehat{x}(t) = \widehat{x}(0) + \frac{t}{m}\widehat{p}$$

   and $[\widehat{x}(0),\widehat{p}] = \mathrm{i}\hbar$. Assuming that $\Delta x(t)$ is of the same order as $\Delta x(0)$, find its minimum value as a function of t and m. (This is closely related to what is known as the "standard quantum limit".)

   Now consider measuring a distance L between two points by sending a particle from one to the other and timing its motion. By relativity, we must have $L \leq ct$. If the particle is too massive, the two points we are measuring will be inside a black hole; to avoid this we need $Gm/Lc^2 \lesssim 1$. Find the resulting limit on Δx on the accuracy to which we can measure L.

(d) Find loopholes in these arguments.

II. **Van Hove's theorem**

In the Hamiltonian formalism, a classical dynamical system typically has a phase space that is (at least locally) parametrized by (generalized) positions $\mathbf{q}$ and momenta $\mathbf{p}$. The basic rule in quantization is that "Poisson brackets become commutators." One way to express this is by a quantization map Q from functions of the phase space ($f(\mathbf{q},\mathbf{p})$, $g(\mathbf{q},\mathbf{p})$, etc.) to operators on a Hilbert space. Since we're physicists, we'll denote the action of Q by adding a "hat": $Q(f) = \widehat{f}$. An obvious set of conditions for Q is:

(1) $Q(af + bg) = aQ(f) + bQ(g)$ (linearity)
(2) $Q(1) = 1$
(3) $Q(\mathbf{x})$ and $Q(\mathbf{p})$ are represented irreducibly
(4) $[Q(f), Q(g)] = \mathrm{i}\hbar Q(\{f,g\})$ (where $\{f,g\}$ is the Poisson bracket)

Van Hove's theorem say that this is not possible for a particle moving in one dimension. Prove this.

Hint: In the phase space, one has hat $p^2q^2 = -\frac{1}{9}\{p^3,q^3\}$ and $p^2q^2 = -\frac{1}{3}\{p^2q, q^2p\}$. Show that these give different values for $Q(p^2q^2)$. (To do this mathematically rigorously, you will have to use the irreducibility condition (3), which implies that if $[\widehat{q},\widehat{O}] = 0$ and

[11] This was handed out April 2, 2009.

$[\widehat{p}, \widehat{O}] = 0$ for some operator $\widehat{O}$, then $\widehat{O}$ is proportional to the identity, that is, $\widehat{O}$ is a number. Most "physicists' proofs" don't pay too much attention to this.)

Note: "deformation quantization" replaces condition 4 by

(4′) $[Q(f), Q(g)] = \mathrm{i}\hbar Q(\{f, g\}) + (\text{terms of order } \hbar^2)$

III. **Affine commutators**

(a) Show that if $[\widehat{q}, \widehat{p}] = \mathrm{i}\hbar$, then $\widehat{q}$ generates translations in $\widehat{p}$, that is,

$$\mathrm{e}^{-\mathrm{i}a\widehat{q}/\hbar}\widehat{p}\mathrm{e}^{\mathrm{i}a\widehat{q}/\hbar} = \widehat{p} + a.$$

(b) Suppose that $[\widehat{q}, \widehat{p}] = \mathrm{i}\hbar$. Show that if there is any state that is an eigenfunction (or a generalized eigenfunction—that is, the state need not be normalizable) of $\widehat{p}$, then *all* real numbers appear as eigenvalues of $\widehat{p}$.

(c) Suppose the fundamental operators are instead $\widehat{q}$ and $\widehat{D} = \widehat{qp}$ with $[\widehat{q}, \widehat{D}] = \mathrm{i}\hbar\widehat{q}$. Show that $\widehat{D}$ generates dilatations, that is,

$$\mathrm{e}^{-\mathrm{i}a\widehat{D}/\hbar}\widehat{p}\mathrm{e}^{\mathrm{i}a\widehat{D}/\hbar} = \mathrm{e}^{a}\widehat{p}.$$

(d) With affine commutators, show that if there is any state that is an eigenfunction of $\widehat{p}$ with positive eigenvalue, then all positive real numbers appear as an eigenvalue of $\widehat{p}$.

B Introduction to Quantum Gravity: Homework II[12]

I. Electrodynamics as a constrained system.

Classical electrodynamics is described by a four-vector potential A_μ and an antisymmetric field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Lagrangian is

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu \mathcal{J}^\mu$$

where $\mathcal{J}^\mu$ is the current four-vector, and the ordinary electric and magnetic fields are

$$E^i = F^{0i}, \qquad B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}$$

where i, j, k, ...run from 1 to 3 and μ, ν, ...run from 0 to 3.

The tensor $F_{\mu\nu}$ is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

This suggests the system has constraints (to generate the gauge transformations).

(a) If you are not familiar with this formalism, convince yourself that it really does give you Maxwell's equations.
(b) Show the canonical variables ("generalized positions and momenta") are A_i and E^j, with Poisson brackets $\{A_i(\mathbf{x}), E^j(\mathbf{y})\} = \delta_i^j \delta^3(\mathbf{x} - \mathbf{y})$.
(c) Show that A_0 is a Lagrange multiplier, and the corresponding constraint is the Gauss law $C = \nabla \cdot \mathbf{E} = 0$.
(d) Show the constraint generates the gauge transformations of A_i, that is, $\{\int \Lambda(\mathbf{x}) C(\mathbf{x})\, \mathrm{d}^3 x, A_i(\mathbf{x}')\} \sim (A_i + \partial_i \Lambda)(\mathbf{x}')$.
(e) Correct any mistakes in algebra I have made.

II. **Extrinsic Curvature: Symmetry**

Let n^a be the unit normal to a $t = const.$ hypersurface. (Note this implies $n_a = f \nabla_a t$ for some function f. Why?) Recall the extrinsic curvature tensor is

$$K_{ab} = q_a{}^c \nabla_c n_b \tag{B.1}$$

where $g_{ab} = q_{ab} + n_a n_b$.

(a) Show $K_{ab} = K_{ba}$.
(Hint: use the fact $n_a = f \nabla_a t$ and find $n^c \nabla_c n^b$ in terms of f. The vector $a^b = n^c \nabla_c n^b$ is sometimes called the "acceleration".)
(b) Show $K = \nabla_a n^a$ (where $K = K^a{}_a$).

III. **Gauss–Codazzi**

A "spatial" tensor $T^{ab\ldots}{}_{cd\ldots}$ is one with no normal components, i.e.,

$$n_a T^{ab\ldots}{}_{cd\ldots} = n_b T^{ab\ldots}{}_{cd\ldots} = n^c T^{ab\ldots}{}_{cd\ldots} = \cdots = 0$$

$q^a{}_b$ is a projection operator, that is, it projects any index into a "purely spatial" one. (Why?)

The three-dimensional ("spatial") covariant derivative D_a of a spatial tensor can be defined as

$$D_e T^{ab\ldots}{}_{cd\ldots} = q^a{}_g q^b{}_h \ldots q_c{}^i q_d{}^j \ldots q_e{}^f \nabla_f T^{gh\ldots}{}_{ij\ldots}$$

[12] This was handed out April 10, 2009.

(that is, we take the ordinary four-dimensional covariant derivative and then project all indices onto the $t = const.$ slice). The spatial curvature is defined by the condition

$$[D_a, D_b]v_c = {}^{(3)}R_{abc}{}^{d}v_d$$

for any spatial vector v_d. Using these facts, show that

(a) ${}^{(3)}R_{abcd} = q_a{}^e q_b{}^f q_c{}^g q_d{}^h R_{efgh} + K_{ac}K_{bd} - K_{ad}K_{bc}$
(b) $R_{abcd}n^d = \nabla_a K_{bc} - \nabla_b K_{ac} - n_a(\nabla_b a_c - a_b a_c) + n_b(\nabla_a a_c - a_a a_c)$
(c) $R = {}^{(3)}R - K_{ab}K^{ab} + K^2 + 2\nabla_a(n^b\nabla_b n^a - n^a\nabla_b n^b)$

Note that sign conventions differ reference to reference.

IV. **ADM metric and extrinsic curvature**

Suppose we decompose the metric in ADM form as

$$\mathrm{d}s^2 = N^2\,\mathrm{d}t^2 - q_{ij}(\mathrm{d}x^i + N^i\,\mathrm{d}t)(\mathrm{d}x^j + N^j\,\mathrm{d}t)$$

(a) Confirm that the inverse metric is

$$g^{ab} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & -q^{ij} + \frac{N^i N^j}{N^2} \end{pmatrix}$$

where q^{ij} is the inverse of q_{ij} and I raise and lower indices with the spatial metric tensor q_{ij}.

(b) Show that the extrinsic curvature tensor is

$$K_{ij} = \frac{1}{2N}\left(\partial_t q_{ij} - D_i N_j - D_j N_i\right)$$

Note that sign conventions differ from reference to reference.

(Hint: as noted in problem II, the unit normal is $n_a = f\nabla_a t$. What is this in these coordinates? What is f?)

C Introduction to Quantum Gravity: Homework III[13]

I. **Lie Derivative of the Metric.**

The Lie derivative of a metric along a vector ξ^a is

$$\mathcal{L}_\xi g_{ab} = g_{ac}\partial_b \xi^c + g_{bc}\partial_a \xi^c + \xi^c \partial_c g_{ab}$$

Show this may be rewritten as

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$$

where ∇ is the standard covariant derivative.

II. **Constraints generate diffeomorphism**

Recall that the Hamiltonian and momentum constraints are

$$\mathcal{H} = \frac{16\pi G}{\sqrt{q}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right), \quad \mathcal{H}^i = -2D_j \pi^{ij}$$

and $\pi^{ij} = \frac{1}{16\pi G}\sqrt{q}(K^{ij} - q^{ij}q)$ with $K_{ij} = \frac{1}{2N}(\partial_t q_{ij} - D_i N_j - D_j N_i)$. Let

$$H[\widehat{\xi}] = \int \left[\widehat{\xi}^\perp \mathcal{H} + \widehat{\xi}^i \mathcal{H}_i\right] \mathrm{d}^3 x.$$

Show that $H[\widehat{\xi}]$ generates (spacetime) diffeomorphisms of q_{ij}, that is,

$$\left\{H[\widehat{\xi}], q_{ij}\right\} = (\mathcal{L}_\xi q)_{ij}$$

where $\mathcal{L}_\xi$ is the full spacetime Lie derivative and the spacetime vector field ξ^μ is given by the full spacetime Lie derivative and the spacetime vector field ξ^μ is given by

$$\widehat{\xi}^\perp = N\xi^0, \quad \widehat{\xi}^i = \xi^i + N^i \xi^0$$

The parameters $(\widehat{\xi}^\perp, \widehat{\xi}^i)$ are known as "surface deformation" parameters.

(Hint: use problem 1 and express the Lie derivative of the spacetime metric in terms of the ADM decomposition.)

III. **Surface deformation algebra**

Show that

$$\left\{H[\widehat{\xi}], H[\widehat{\eta}]\right\} = H[\{\widehat{\xi}, \widehat{\eta}\}_{SD}]$$

where the "surface deformation bracket" $\{-,-\}_{SD}$ is

$$\{\widehat{\xi}, \widehat{\eta}\}^\perp_{SD} = \widehat{\xi}^i \partial_i \widehat{\eta}^\perp - \widehat{\eta}^i \partial_i \widehat{\xi}^\perp$$
$$\{\widehat{\xi}, \widehat{\eta}\}^i_{SD} = \widehat{\eta}^j \partial_j \widehat{\xi}^i \widehat{\xi}^j \partial_j \widehat{\eta}^i + q^{ij}\left(\widehat{\xi}^\perp \partial_j \widehat{\eta}^\perp - \widehat{\eta}^\perp \partial_j \widehat{\xi}^\perp\right)$$

Show that for purely spatial deformations ($\xi^0 = \eta^0 = 0$), the surface deformation bracket is equal to the ordinary commutator.

(The surface deformation bracket is a "canonical" bracket, defined at one moment of time. For deformations with ξ^0 or η^0 nonzero, the commutator involves time derivatives; it can be shown that the time derivatives of ξ^μ and η^μ can be chosen so that the surface deformation brackt is again equal to the commutator.)

[13] I do not recall when this was handed out. This was never posted to the course website, unlike the other homework assignments.

D Final Exam[14]

The final exam for "Quantum Gravity" will take place on Tuesday, June 9, from 8–10 am in Roessler 158. The format of the exam will be as follows:

I have listed below twelve topics that have been central themes of the course. On the exam, I will list six of these. You will choose four of the six, and write a short (3–4 paragraph) essay describing each of the four you have chosen.

Your essays do not have to be heavily mathematical, but some mathematics—equations and simple derivations—are appropriate for most of these topics. The goal of the essays is to demonstrate the basic concepts, well enough to (for example) explain the fundamental ideas to another student or read and roughly follow a paper in which they are used. I have included a sample essay on the opposite side of this paper.

Topics:

(1) Ambiguities in quantizing a classical system
(2) Quantization of constrained systems: Dirac and reduced phase space methods
(3) Observables and the "problem of time" in quantum gravity
(4) The ADM form of the metric
(5) The diffeomorphism and Hamiltonian constraints in general relativity
(6) The Wheeler–DeWitt equation
(7) The parallel transport matrix, holonomies, and gauge-invariant observables
(8) Spin networks
(9) The area operator in loop quantum gravity
(10) The Nambu–Goto and Polyakov actions in string theory
(11) How string theory contains gravity
(12) Regge calculus and dynamical triangulations

[14] This was a handout given a few weeks before the final exam. It is transcribed verbatim.

Sample essay on the theme "constraints and symmetries"

(Note that this example is more heavily mathematical than would be appropriate for some of the other topics. Include derivations like this when you can make them simple enough; otherwise, describe them and give a few steps. Of course, this sample is also a bit more polished than I would expect on the exam.)

A constraint is an equation of motion that involves only first time derivatives (in the Lagrangian formalism) or no time derivatives (in the Hamiltonian formalism). It therefore does not describe time evolution, but rather restricts ("constrains") the initial data. In the action, a constraint is most easily described with a Lagrange multiplier. In the Hamiltonian form, for instance,

$$I = \int (p\dot{q} - H - \lambda C)\,\mathrm{d}t \tag{1}$$

where the Lagrange multiplier λ enforces the constraint $C(p,q) = 0$. In order to be preserved under time evolution, the constraint must effectively have a vanishing Poisson bracket with the Hamiltonian:

$$\{C, H\} = VC \implies \frac{\mathrm{d}C}{\mathrm{d}t} = 0 \text{ when } C = 0. \tag{2}$$

With some technical exceptions, a constraint generates a symmetry of the classical action. That is, under the transformation

$$\delta q = \epsilon\{C, q\} = -\epsilon\frac{\mathrm{d}C}{\mathrm{d}p}, \quad \delta p = \epsilon\{C, p\} = \epsilon\frac{\mathrm{d}C}{\mathrm{d}q} \tag{3}$$

the action remains invariant. To see this, note that

$$\begin{aligned}
\delta I &= \int \left[\delta q\left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\mathrm{d}H}{\mathrm{d}q}\right) + \delta p\left(-\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{\mathrm{d}H}{\mathrm{d}p}\right) - \delta\lambda\, C\right]\mathrm{d}t \\
&= \int \left[-\epsilon\frac{\mathrm{d}C}{\mathrm{d}p}\left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\mathrm{d}H}{\mathrm{d}q}\right) + \epsilon\frac{\mathrm{d}C}{\mathrm{d}q}\left(-\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{\mathrm{d}H}{\mathrm{d}p}\right) - \delta\lambda\, C\right]\mathrm{d}t \\
&= \int \left[-\epsilon\frac{\mathrm{d}C}{\mathrm{d}t} - \epsilon\{C, H\} + \delta\lambda\, C\right]\mathrm{d}t = \int \left(\frac{\mathrm{d}\epsilon}{\mathrm{d}t} - \epsilon V - \delta\lambda\right) C\,\mathrm{d}t
\end{aligned}$$

which is zero if we choose $\delta\lambda = (\mathrm{d}\epsilon/\mathrm{d}t) - \epsilon V$.

An example of a constrained system is electromagnetism. For an electromagnetic system, the vector potential $\mathbf{A}$ and the electric field $\mathbf{E}$ are canonically conjugate, and the Gauss law $G = \nabla\cdot\mathbf{E} - \rho = 0$ is a constraint. It is easy to check that G generates gauge transformations of $\mathbf{A}$. General relativity is also a constrained system, in which the constraints generate diffeomorphisms of space and "surface deformations" that are equialent to diffeomorphisms involving time when the equations of motion are satisfied.

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