Newtonian Mechanics in a Nutshell

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Contents

1 Big Picture

1·1. Goal of Mechanics. We model physical bodies and try to predict their motion. This is the entire goal of Newtonian mechanics (and classical mechanics more broadly).

1·2. Physical Body. We model a physical body as a point-particle. That is to say, at any given moment, the location of a physical body is modeled as a point. The motion of a physical body is modeled as a [suitably smooth] curve $\gamma\colon I\to\mathbb{R}^3$ where I is some interval of "time". This works well as an approximation when we use the center-of-mass as the location of the body.

In short, using a point-particle as an idealized approximation of a physical body is "good enough" for most purposes.

1·2.1. *Remark* (Extended bodies, rigidity condition). There are also extended bodies. For classical mechanics, this is a rigid body. We model a rigid body by a set of trajectories such that the distance between any two points in the body is fixed. For example, a coin is modeled as a family of trajectories $\gamma(t;\boldsymbol{x}_0)$ with initial position $\gamma(0;\boldsymbol{x}_0)=\boldsymbol{x}_0$ such that the coin's position at time t is the region

$$\mathcal{C}_t=\{\gamma(t;\boldsymbol{x})\mid\boldsymbol{x}\in\mathcal{C}_0\}. \tag{1·2.1}$$

The "rigidity condition" is: for any $\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{C}_0$, we have $\|\gamma(t;\boldsymbol{x}_1)-\gamma(t;\boldsymbol{x}_2)\|$ be a constant ("rigid") with respect to time. When this condition holds, we call an extended body a "rigid body".

For rigid bodies, we need to model rotational mechanics of the rigid body in addition to the translational mechanics of the rigid body. Euler studied this problem quite extensively. If rotational contributions to a rigid body is negligible, then we may model it as a point-particle located at its center of mass.

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1·2.2. *Remark* (Continuum mechanics). If we remove the "rigidity condition" [i.e., we could allow the distance between points in an extended body to change over time] for an extended body, then we end up with continuum mechanics. This is an extraordinarily difficult field. Fluid mechanics is a subfield of continuum mechanics, and we don't know how to solve the equations of motion in fluid mechanics.

1·3. Equations of Motion. Classical mechanics (and, more generally, physics) describes the motion of bodies by means of the so-called **"Equations of Motion"** which relates the acceleration in terms of velocities and positions. These are second-order ordinary differential equations.

In Newtonian mechanics, the "big insight" is the idea that *forces* determine the "right-hand side" to these differential equations. Undergraduates spend a lot of time setting up and then solving these differential equations. But the difference in formalisms in classical mechanics (Newtonian, Lagrangian, Hamiltonian, etc.) lies in these two steps:

(a) Setting up the equations of motion
(b) Solving the equations of motion

1·4. The real skill in physics is not in solving these differential equations, but in using physical principles [conservation laws or symmetries or whatever] to extract relevant and useful information from them. The different formalisms in classical mechanics allows us to look at a physical system in different lighting.

1·4.1. *Remark* ("ISEE"). If you are learning physics, I would highly encourage you to read Young and Freedman's *University Physics*. Any edition is fine (unless you are taking a course at college or university which requires a specified edition for its homework). The real usefulness lies in its format for examples, which state a problem, the solves it in a template consisting of the following steps:

Identify the relevant concepts:
- Identify the physical conditions stated in the problem to help decide which physical concepts which are relevant
- Identify the target variable, the physical quantity (or quantities) you are trying to find.
- Identify the known quantities (stated or implied in the problem). (Sometimes this is done in the "Setup" step.)

Setup the problem
- Given the concepts, known quantities, and target variables, which we have obtained from the IDENTIFY step, now we choose the equations we'll use to solve the problem (and decide how we'll use them).
- Here you also choose coordinate systems, which is important and often overlooked.
- Draw sketches (if relevant) and free body diagrams (if relevant).

Execute the solution — that is, "do the math", solve the equations you have obtained from the SETUP step. Try to solve these equations *algebraically* ("symbolically"), defer plugging in values as late as possible. This will help us in the next step

Evaluate your answer.
- We can try taking limits of the symbolic solution to recover previous solutions (e.g., taking the "friction goes to zero" limit should recover the frictionless solution) or special cases where we might divide by zero (e.g., if the answer looks like $f(x)/\cos(\theta)$, the $\theta = (n + 1/2)\pi$ solutions lead to a divide-by-zero singularity resulting in an infinity)
- Check the order-of-magnitude of the solution, to see if it makes sense.

This encourages the reader to keep a *répertoire* of worked problems, and nurtures a habit of "successive refinement" to add layer-upon-layer of complexity and realism to a problem.

1·5. Observers and Reference Frames. Before rushing along, we should discuss one subtlety to classical mechanics, namely the idea of observers and reference frames. We can model an observer as a physical body [i.e., a point-particle] equipped with "a ruler and a clock". What does this even mean? It means we have a set of basis spatial vectors for $\mathbb{R}^3$ ["ruler"] and a basis for time $\mathbb{R}$ ["a clock"]. Together these are bundled together and sometimes called a *reference frame*.

The trajectories of physical bodies are expressed in coordinates relative to an observer's reference frame.

2 Newton's Laws of Motion

2·1. Newton laid down three "laws" of motion. The first Law is necessary for the second Law, but the third Law may be viewed as a consequence of the second Law.

2.1 Newton's First Law

2·2. First Law of Motion. This is the famous, "A body at rest tends to stay at rest, a body in motion [in a constant rectlinear velocity] tends to stay in motion, unless acted upon by a force."

Implicitly there are two claims being made:

1. For any trajectory $\gamma\colon I\to\mathbb{R}^3$, we have $\mathrm{d}\gamma(t)/\mathrm{d}t$ be constant unless there is a force acting upon the body;
2. When we apply this to observers, there is a special class of observers whose trajectories are described by $\mathrm{d}\gamma(t)/\mathrm{d}t$ being constant — these are **"Inertial Observers"**. These are special because the laws of physics "look the same" to all inertial observers.

2·3. This second claim strains credulity, it's not immediately obvious *this* is the result of Newton's First Law of Motion. But this is because if an obsever is non-inertial, there are forces acting on the observer which must be "accounted for" when setting up the equations of motion for other bodies. Inertial observers lack such "bonus parts" in equations of motion relative to inertial coordinate systems.

2·4. Galilean Relativity. If we have two different inertial observers K_1 and K_2 with coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively, and if K_2 moves relative to K_1 with a constant velocity $\boldsymbol{v}_{2|1}$, then we have:

$$\boldsymbol{r}_1=\boldsymbol{r}_2+\boldsymbol{v}_{2|1}t. \tag{2·4.1}$$

The time is the same in the two frames, in the sense that when we agree on the units the clocks can differ by a constant displacement $t_2=t_1+(\text{constant})$. This is known as **"Galilean Relativity"** and relates inertial reference frames to each other.

2·5. Non-inertial reference frames. There are two ways to have a non-inertial reference frame (either one disqualifies a reference frame from being inertial):

(1) Rotating reference frames are not inertial,
(2) Accelerating reference frames are not inertial.

2·6. Objection: Status of Inertial Observers. The clever reader may object that inertial observers cannot possibly exist. There are two separate arguments:

(1) We are on a rotating planet, Earth, which also revolves around the Sun — that is, we experience rotation. So we're a rotating reference frame, which is not inertial.
(2) We are told rumour that the expansion of the universe is accelerating. *We* are in the universe. Therefore *we* are an accelerating reference frame.

This is all true, and in Appendix A we modify Newton's laws of motion to facilitate these claims (and quantify the degree to which they affect experiments).

2·7. Puzzle. Consider the trajectory of a body moving in uniform circular motion, something like:

$$\gamma(t)=r_0\cos(\omega t)\boldsymbol{e}_1+r_0\sin(\omega t)\boldsymbol{e}_2, \tag{2·7.1}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are unit vectors for the x, y axes in Cartesian coordinates (relative to some inertial observer); $r_0>0$ is the radius of the circular motion; $\omega\neq0$ is the constant "angular velocity" ($\omega=2\pi/T$ where it takes a time interval T seconds long to orbit the circle once).

But this is not a trajectory with a "uniform rectilinear velocity". We can see this by taking its time derivative:

$$\frac{\gamma(t)}{\mathrm{d}t}=-r_0\omega\sin(\omega t)\boldsymbol{e}_1+r_0\omega\cos(\omega t)\boldsymbol{e}_2. \tag{2·7.2}$$

We see the *speed* is constant $\|\mathrm{d}\gamma/\mathrm{d}t\|=r_0\omega$, so the direction of the velocity changes. But by Newton's first Law, this means there is some force acting on the body. What "force" could this be?

2.2 Newton's Second Law

2·8. Mathematical Statement. This is the big contribution, the main tool, of Newtonian mechanics. It is often presented as: The equations of motion for a body looks like (for every coordinate of the body):

$$\text{Force} = (\text{mass})(\text{acceleration}). \tag{2·8.1}$$

This is a special case of Newton's second Law, it's true when a body's mass is constant. When a body's mass can change over time $m = m(t)$, we instead need the notion of momentum

$$\begin{gathered} \boldsymbol{p}(t) := m(t)\boldsymbol{v}(t) \\ (\text{momentum})(t) := (\text{mass})(t)(\text{velocity})(t) \end{gathered} \tag{2·8.2}$$

where ":=" means "is defined as". Then Newton's second Law is

$$\boxed{\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t}} \tag{2·8.3}$$

where $\boldsymbol{F}$ is the sum of all forces acting on the body. Usually mass m is constant, which is how we obtained the $\boldsymbol{F} = m\boldsymbol{a}$ form of Newton's second Law.

2·8.1. *Remark* (Physical bodies with changing mass). Rockets have variable mass because fuel is expended continuously. We would predict incorrcet motion with $\boldsymbol{F} = m\boldsymbol{a}$: we would be neglecting the entire source of the rocket's motion.

Other systems (like automobiles or airplanes) experience a similar phenomenon, but it is much less pronounced. For an automobile, 10 gallons of gas weighs 61 pounds (or 27.67 kg) compared to about 1500 kg for the rest of the car, i.e., the gas is less than 1.85% the mass of the total car, less than the weight of one adult passenger.

Airplanes, like the Boeing 737, have around 26 000 L fuel capacity, and airplane fuel has a density of 0.8 kg/L for a total mass of 20 828 kg. The maximum weight of a 737 is around 55×10^3 kg, the fuel's consumption is non-negligible in this case (since the fuel is greater than 36% of the initial mass).

2·8.2. *Remark* (Inertial mass and equivalence principle). The mass which appears in Newton's second law is sometimes referred to as a body's *inertial mass*. This is contrasted with a body's *gravitational mass* — analogous to a body's *electric charge* which generates an electric field, the gravitational mass generates a gravitational field. Galileo determined experimentally these two masses are the same, and this is the celebrated *equivalence principle*.

2·9. Equations of Motion. Newton's second Law gives us the equations of motion relative to an inertial reference frame. All we need to do is write down the forces acting on the body, sum them together, then plug the net force vector into the left-hand side of Newton's second Law.

2·10. How to Study This. The way we should study this is by introducing new forces, one at a time, and studying how they act on physical bodies. For a rough roadmap of how most texts study forces:

(1) Uniform gravitational force — on Earth, the gravitational force from the Earth is approximately constant $\boldsymbol{F}_{\text{const. gravity}} = m_{\text{body}} g \boldsymbol{e}_{\text{downwards}}$ where $g \approx 9.8\ \text{m/s}^2$ and m_{body} is the mass of the body experiencing the gravitational force. Newton's second Law becomes $\mathrm{d}^2\gamma(t)/\mathrm{d}t^2 = g\boldsymbol{e}_{\text{downwards}}$. This is studied in the variety of problems:
   - Projectile motion
   - Block on inclined plane, which experiences on uniform gravitational force.
   - Simple pendulum whose "bob" experiences uniform gravitational force, and is connected to the ceiling by a massless rigid rod.

(2) Tension (in a rope or chain), which "pulls" on a body

- Atwoods machines attaches point masses to a [massless] rope which passes through a [massless, frictionless] pulley. The masses experience uniform gravitational force, possibly placed on an inclined plane. This can be "iterated" — we could replace one of the point masses with another Atwood's machine, the rope is holding up the [massless, frictionless] pulley.

(3) Various contact forces — static friction (the "hesitation" of a body from accelerating due to the floor's microscopic "jaggedness"), kinetic friction (the "ongoing" friction once the static friction has been overcome), normal force (the floor "pushing against" the object placed against it, to prevent the object falling through the floor), and fluid resistance (recall, a fluid is either a liquid or a gas, and air is a gas, so a body needs to "push through" the stagnant air which is precisely air resistance).

These are introduced one at a time, which is used to re-examing the previous examples of tension and uniform gravitational force.

(4) Hooke's law for simple harmonic motion. This is usually presented as a way to describe how [massless] springs work, with a force proportional to the distance the spring is dislocated from equilibrium.

More generally, any force proportional to displacement $\boldsymbol{F} = k\boldsymbol{x}$ follows Hooke's law. This is important because every [non-constant] force can be Taylor expanded, and Hooke's law describes the first nontrivial contribution.

This actually guides the "next steps" in a physics textbook, because the other forces of interest are electromagnetism and gravity.

2·11. "Composite" Problems. We can compose these forces together to form more complicated problems. For example, we could attach a massless pulley to the ceiling by a thin massless rod (or, respectively, we could replace the point-mass "bob" of a simple pendulum by an Atwood's machine); we could consider a "double pendulum" (attach a thin massless rod to the ceiling, and attach bob A to the other end; we attach a thin massless rod to bob A and then attach a point-mass B to the other end of this second massless rod); we could replace ropes (or massless rods) with springs; so on and so forth.

As we invent more complicated problems, we should think about how we could "freeze out" complications to recover simpler problems. The solutions to the complicated problems should have some "limiting behaviour" to recover solutions to the simpler problems. This guides one part of the "evaluate" step to writing solutions to these problems.

2.3 Newton's Third Law

2·12. This is the famous, "To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts", law. Sadly, this is misunderstood. We should interpret this as "If body A acts on body B by a force $\boldsymbol{F}_{A\text{ on }B}$, then B acts on body A by an equal but opposite force [i.e., a force equal in magnitude] $\boldsymbol{F}_{B\text{ on }A} = -\boldsymbol{F}_{A\text{ on }B}$".

This is assuming there are no external forces, the only force which body B experiences is $\boldsymbol{F}_{A\text{ on }B}$ and the only force which body A experiences is $\boldsymbol{F}_{B\text{ on }A}$.

In light of Newton's second Law, we see this is precisely the conservation of momentum (i.e., the momentum for the total system $\boldsymbol{p} = \boldsymbol{p}_A + \boldsymbol{p}_B$ [being the sum of momentum for body A and the momentum for body B] is a constant in time):

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{p}_A}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{p}_B}{\mathrm{d}t} = \mathbf{0}. \qquad (2\cdot 12.1)$$

Why believe this? Well, this is because the sum of forces, which Newton's third Law asserts are equal in magnitude but opposite in sign (hence sums to zero).

2·13. Conservation of Momentum. Thus, we identify Newton's third Law with the conservation of momentum for a composite system. Phrased differently: an isolated system experiences no force.

2·13.1. *Remark* (Rockets move by conservation of momentum). The conservation of momentum may be used to *propel* bodies forward. This is how a rocket works.

The same principle applies if you were in a boat with a load of bricks. By throwing one brick out at a time, with most of the initial momentum of the brick being in the direction parallel to the water, you could move your boat foward. Conservation laws are powerful.

2.4 Concluding Remarks about Newton's Laws

2·14. Free Body Diagrams. We have a nice way to obtain the equations of motion by invoking physical principles to obtain the force vector acting on bodies. It is useful to draw a diagram for each body in the system, where each force acting on the body is drawn. This is known as a "free body diagram". [It should be drawn in the "setup" step of writing a solution.]

2·15. Rigid body motion. Advanced Newtonian mechanics just works with more complicated situations, like modeling a coin as a rigid body. Euler pioneered this study of rigid body motion — the analogous propositions to Newton's Laws of motion for rigid body motion is called "Euler's Laws of motion". It turns out the problem may be decomposed as:

(1) Treat the center-of-mass of the system as a point particle subject to Newton's laws of motion, and
(2) Treat the rigid body as rotating about its center of mass; by choosing the center-of-mass as the origin, this problem simplifies considerably.

Euler proved this all works fine, but it's tedious to do the calculations here.

2·16. Differential Equations are Hard. Newton's laws of motion gives us a system of second-order [ordinary] differential equations. In general, second-order ordinary differential equations are hard to solve. This motivates us to find ways to extract information about physical systems which avoids explicitly solving these equations of motion, like the "work–energy theorem" relating the force applied to a body over some distance is equal to the change in the body's kinetic energy.

2·17. Setting up problems is hard. It is surprisingly nontrivial to setup a problem correctly. We could easily forget or overlook certain forces, or translate it as an incorrect vector. For this reason, physicists invented different formalisms for classical mechanics like Lagrangian mechanics and Hamiltonian mechanics.

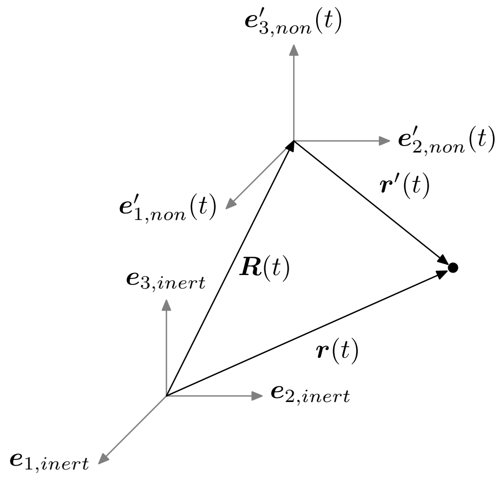
Appendices

A Equations of Motion in Non-inertial Reference Frames

A.1. For non-inertial reference frames, we start by considering the position of the origin of the non-inertial frame relative to an inertial frame

$$\boldsymbol{R}(t) = \begin{pmatrix}\text{Position of the origin for the}\\ \text{non-inertial reference frame}\\ \text{relative to an inertial observer}\end{pmatrix}. \tag{A.1.1}$$

We are trying to describe the trajectory of a body relative to a noninertial reference frame $\boldsymbol{r}'(t)$ (primes indicate the quantity is relative to the noninertial reference frame — this is the physics convention, not my own idiosyncratic notation) in terms of $\boldsymbol{R}(t)$ and the position of the body relative to the inertial observer $\boldsymbol{r}(t)$. The picture we should have in mind (with subscripts "inert" and "non" to emphasize the different reference frames):



We can write this as

$$\boldsymbol{r}(t) = \boldsymbol{R}(t) + \boldsymbol{r}'(t). \tag{A.1.2}$$

Note that $\boldsymbol{R}(t)$ is taken relative to the inertial reference frame. Now we take derivatives of both sides to get velocities

$$\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{r}'(t)}{\mathrm{d}t} \tag{A.1.3}$$

and again for accelerations

$$\frac{\mathrm{d}^2\boldsymbol{r}(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\boldsymbol{R}(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\boldsymbol{r}'(t)}{\mathrm{d}t^2}. \tag{A.1.4}$$

For accelerating reference frames, $\mathrm{d}^2\boldsymbol{R}/\mathrm{d}t^2 \neq 0$, and for rotating reference frames things get more interesting.

A.2. We need to start by writing the velocity of the a body relative to an inertial frame $\boldsymbol{v}_i = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ in terms of the velocity of the body relative to a non-inertial frame $\boldsymbol{v}_n$ (which is just the time derivative of the components of the vector, since the basis vectors $\boldsymbol{e}'_j$ are constant relative to the non-inertial observer),

$$\boldsymbol{v}_i = \frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(r'_1(t)\boldsymbol{e}'_1(t) + r'_2(t)\boldsymbol{e}'_2(t) + r'_3(t)\boldsymbol{e}'_3(t)\right) \tag{A.2.1}$$

$$= \frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{3} r'_j(t)\boldsymbol{e}'_j(t) \tag{A.2.2}$$

$$= \frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t} + \sum_{j=1}^{3} \frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\boldsymbol{e}'_j(t) + r'_j(t)\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t} \tag{A.2.3}$$

$$= \frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t} + \boldsymbol{v}_n + \sum_{j=1}^{3} r'_j(t)\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t} \tag{A.2.4}$$

where $r'_j(t)$ are the coordinates for the position relative to the non-inertial frame, $\boldsymbol{e}'_j(t)$ are the unit vectors in the non-inertial reference frame which vary over time (by, e.g., rotating); the $\boldsymbol{v}_n$ is the velocity of the body as viewed by the non-inertial observer. Acceleration relative to an inertial reference frame is obtained by taking the time derivative with respect to both sides.

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_i = \frac{\mathrm{d}^2\boldsymbol{R}(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{3} \frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\boldsymbol{e}'_j(t) + r'_j(t)\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t} \tag{A.2.5}$$

$$= \frac{\mathrm{d}^2\boldsymbol{R}(t)}{\mathrm{d}t^2} + \sum_{j=1}^{3} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\boldsymbol{e}'_j(t)\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(r'_j(t)\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t}\right) \tag{A.2.6}$$

$$= \frac{\mathrm{d}^2\boldsymbol{R}(t)}{\mathrm{d}t^2} + \sum_{j=1}^{3} \left(\frac{\mathrm{d}^2 r'_j(t)}{\mathrm{d}t^2}\boldsymbol{e}'_j(t) + \frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t} + r'_j(t)\frac{\mathrm{d}^2\boldsymbol{e}'_j(t)}{\mathrm{d}t^2}\right) \tag{A.2.7}$$

$$= \frac{\mathrm{d}^2\boldsymbol{R}(t)}{\mathrm{d}t^2} + \boldsymbol{a}_n + \sum_{j=1}^{3} 2\frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t} + r'_j(t)\frac{\mathrm{d}^2\boldsymbol{e}'_j(t)}{\mathrm{d}t^2} \tag{A.2.8}$$

where $\boldsymbol{a}_n$ is the acceleration relative to the non-inertial frame.

A.3. Rewriting the equations of motion. Newton's second Law for a body with constant mass is then,

$$\boldsymbol{F} = m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_i \tag{A.3.1}$$

For a non-inertial observer, we just rewrite $\mathrm{d}\boldsymbol{v}_i/\mathrm{d}t$ by the complicated mess

$$\boldsymbol{F} = m\boldsymbol{a}_n + m\left(\frac{\mathrm{d}^2\boldsymbol{R}(t)}{\mathrm{d}t^2} + \sum_{j=1}^{3} 2\frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t} + r'_j(t)\frac{\mathrm{d}^2\boldsymbol{e}'_j(t)}{\mathrm{d}t^2}\right). \tag{A.3.2}$$

The "extra terms" on the right may be interpreted as "pseudoforces", and moved to the left-hand side as:

$$\boldsymbol{F}_{\text{pseudo}} = -m\left(\frac{\mathrm{d}^2\boldsymbol{R}(t)}{\mathrm{d}t^2} + \sum_{j=1}^{3} 2\frac{\mathrm{d}r'_j(t)}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{e}'_j(t)}{\mathrm{d}t} + r'_j(t)\frac{\mathrm{d}^2\boldsymbol{e}'_j(t)}{\mathrm{d}t^2}\right), \tag{A.3.3}$$

so Newton's laws of motion for a non-inertial frame is,

$$\boldsymbol{F} + \boldsymbol{F}_{\text{pseudo}} = m\boldsymbol{a}_n. \tag{A.3.4}$$

As a consistency check, we see when we make $\boldsymbol{e}'_j(t)$ constants and $\boldsymbol{R}(t)$ constant, we recover Newton's laws for an inertial observer.

If you want to account for the rotation of the Earth, or the acceleration of the universe, then we can do it. The contributions may be small depending on what we want to do. For example, modeling a planet's atmosphere as a fluid surrounding a rotating sphere requires accounting for pseudoforces from the planet's rotation. But the contribution from the acceleration of the universe is rather negligible.