Lie Groups Notes

Alex Nelson*
Email: pqnelson@gmail.com

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Abstract

Notes taken from Albert Schwarz's class on Lie Groups taught Winter Quarter 2010 at University of California, Davis.

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Lecture 1

Lie algebras and Lie groups, and their representations, will be the subject of the quarter. We will start with an outline for the course.

First of all, an explanation of what is a Lie group. Well, group theory is simply a theory of symmetry. We've studied mostly finite and discrete symmetries, but in the real world symmetries are continuous. For example in 4-dimensional space, everything is invariant to translations labeled by elements of $\mathbb{R}^3$ (this is a continuous symmetry). Everything is also invariant under rotations (symmetry denoted by SO(3)). So really the study of symmetries should be the study of continuous groups. First of all, group of symmetries should be a topological group. (It is a group object in the category of topological spaces.) We have a notion of continuity therefore.

Definition 1.1. A "**Topological Group**" consists of a topological space M equipped with

1. a continuous mapping $\mu\colon M \times M \to M$ called the "Law of composition";
2. a continuous mapping $e\colon \mathbf{1} \to M$ called the "Identity Element";
3. a continuous mapping $\xi\colon M \to M$ called the "Inversion Operation".

We can further demand that M is a manifold. So if we have a group G, t hen we can take a neighborhood of the unit element $e \in U \subseteq G$ where $U \sim \mathbb{R}^n$ is "topologically equivalent" (i.e. we can introduce a "good" coordinate system in U denoted by x^1, ..., x^n which are continuous and have continuous inverses).

A manifold is a topological space where every point $x \in M$ has a neighborhood $U_x \subseteq M$ which is equivalent to $\mathbb{R}^n$. We have a preferred point, namely the unit point (identity element of the group), since we have *shifts*

$$T_g(x) = g \cdot x. \tag{1.1}$$

These shifts are continuous topological transformations with continuous inverses, a continuous identity, etc. We have

$$T_g(e) = g \cdot e = g \qquad \forall g \in G. \tag{1.2}$$

We have coordinates in the neighborhood of any point. A Lie group is a topological group with a coordinizable neighborhood at every point.

Note that this is not a good definition. The coordinates of $x \cdot y$ are coordinates of these two factors $f^j(x^1, \dots, x^n, y^1, \dots, y^n)$ continuous, but that's not enough. We would like these functions to be differentiable (moreover smooth, i.e. C^∞). This is not included in the definition of a Lie group. First could we correct our coordinates to be differentiable (there is a theorem which says we can). Second, we'd like to incorporate smoothness into the definition of a Lie group. A Lie group is then a topological group with a neighborhood at each point with coordinates permitting some smooth structure.

Now what of Lie algebras. If the Lie group is connected, if we know the group in the neighborhood of e (the identity element), then we know it everywhere. The main thing is we can take infinitesimally small neighborhoods[1] of e which is how we get a Lie algebra.

Take the group $\mathrm{GL}(n)$. It is easy to see it is a Lie group. If we take any closed subgroup $G \subseteq \mathrm{GL}(n)$, then we will get a Lie group. It is a kind of submanifold of $\mathrm{GL}(n)$. We can consider the tangent space of e in this submanifold, and it turns out it is a Lie algebra. So it is a vector space closed under a commutator operation. This is the Lie algebra of the group G, denoted $\mathrm{Lie}(G)$.

Definition 1.2. A "**Lie Algebra**" consists of a vector space V equipped with a commutator

$$[\cdot,\cdot]\colon V \times V \to V \tag{1.3}$$

such that, for all $a, b, c \in V$, we have

[1] Do not worry about what this rigorously means, we will use the tools of differential geometry to make such a notion explicit and rigorous.

1. distributivity $[a, b+c] = [a, b] + [a, c]$;
2. anticommutativity $[a, b] = -[b, a]$;
3. Jacobi identity $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

Proposition 1.3. *For every Lie group we can construct a corresponding Lie algebra. Moreover, if $\Phi\colon G \to G'$ is a Lie group morphism, then we can construct a Lie algebra morphism $\phi\colon \mathrm{Lie}(G) \to \mathrm{Lie}(G')$.*

Question. If we have a Lie algebra, does it come from a Lie group? Does it only come from only one Lie group?

The answer is that a finite dimensional Lie algebra gives rise to a simply connected Lie group corresponding to the Lie algebra. Omitting the criteria of finite dimensionality, we don't have an answer — no notion of infinite dimensional Lie groups currently exist!

We have, lastly, a notion of representation. It is very simple, namely a group homomorphism

$$\rho\colon G \to \mathrm{GL}(n). \tag{1.4}$$

By proposition 1.3 this has a corresponding morphism of Lie algebras $\mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{GL}(n))$ which is the representation of Lie algebras. How to restore, how to classify the representations of Lie algebras? It is very easy, contrasted to asking the same question for Lie groups.

We will mostly work with compact Lie groups, classical Lie groups, and their representations. We will also consider infinite dimensional Lie algebras. We will consider applications to physics.

A Kac–Moody algebra is in general infinite dimensional, given by some commutation relations. Every classical Lie Algebra may be considered as a Kac–Moody algebra. Every Lie algebra of a compact Lie group is a Kac–Moody algebra, to be more precise.

Semisimple and reductive Lie algebras are very closely related to the theory of compact Lie algebras of compact Lie groups. This is Hermann Weyl's so-called unitary trick. By default we will consider Lie algebras over $\mathbb{R}$, i.e. consider the underlying vector space to be over $\mathbb{R}$. We can consider over any field (viz. over $\mathbb{C}$). At some moment we may switch to work over $\mathbb{C}$, as we are interested in complex representations.

Lecture 2

The main thing of interest is Lie groups, but Lie algebras are a useful tool to study Lie groups. We will start with Lie algebras. First what is an algebra. Well, more or less, it's obtained from the formula

$$\text{Vector Space} + \text{Ring} = \text{Algebra}, \tag{2.1}$$

with some compatibility conditions. When we speak of vector spaces, we need a field $\mathbb{F}$; a ring has 2 operations: multiplication and addition. (A ring is an Abelian group under addition, and a magma under multiplication.) The only only relation between addition and multiplication is distributivity:

$$a(b+c) = ab + ac \tag{2.2a}$$
$$(b+c)a = ba + ca. \tag{2.2b}$$

In general, a ring doesn't have a multiplicative identity, nor is multiplication an Abelian operation.

The compatibility condition for an algebra is thus

$$\lambda(ab) = (\lambda a)\cdot b \tag{2.3a}$$
$$= a\cdot(\lambda b) \tag{2.3b}$$

where $\lambda \in \mathbb{F}$. This is associativity.

Example 2.1. We have $\mathrm{Mat}_n(\mathbb{F})$ — the collection of all $n \times n$ matrices over a field $\mathbb{F}$ — be a noncommutative, associative algebra.

Example 2.2. If we have some set M, the set of functions on M (denoted by $C(M)$) is an algebra with respect to point-wise addition, multiplication, and $\mathbb{F}$-scalar multiplication. Note that if $\mathbb{F}$ is a field, the algebra is associative. Now $C(M)$ has a lot of subalgebras if M has some additional structure. If M is a topological space, we have the set $C^0(M) \subseteq C(M)$ of continuous functions be a subalgebra. If $M = \mathbb{R}^n$ we may consider $C^\infty(M) \subseteq C(M)$ the subalgebra of smooth functions.

Example 2.3. We can consider $C^\infty(S^n)$ where S^n is the n-sphere. We do this by introducing local coordinates, and define the notion of smoothness in S^n by demanding tit be smooth in every coordinate system on S^n. But it is possible for a function to be smooth in one coordinate system but not another, so we need to use the notion of a transition function.

A smooth manfiold M is covered by smooth local coordinate systems, and the transition function between coordinate systems is smooth. So $C^\infty(M)$, for some smooth manifold M, is a unital, commutative, associative algebra.

We will introduce a construction of an algebra for a group, called the group algebra. We consider all formal linear combinations of group elements with coefficients from a ring:

$$\begin{aligned} \text{Group} &\to \text{Algebra} \\ G &\to \mathbb{F}[G] \end{aligned} \tag{2.4}$$

which has an element resemble $\sum_i^n a_i g_i$ where $g_i \in G$ and $a_i \in \mathbb{F}$ for all i. We have addition be component-wise, and multiplication also be component-wise. So for example

$$(g_1 + g_2) + (g_2 + 3g_3) = g_1 + 2g_2 + 3g_3 \tag{2.5a}$$

$$(g_1 + g_2)(g_2 + 3g_3) = g_1 g_2 + 3g_1 g_3 + g_2^2 + 3g_2 g_3. \tag{2.5b}$$

More generally

$$(\sum_i a_i g_i)(\sum_j b_j g_j) = \sum_{i,j} a_i b_j \cdot (g_i g_j). \tag{2.6}$$

In the language of category theory, this is a functor **Grp** $\to$ **Alg**.

Recall a representation of a group $G \to \mathrm{GL}(V)$ are homomorphisms from G to automorphisms on V. We have very simply for a rep $G \to \mathrm{GL}(n)$ a representation $\mathbb{F}[G] \to \mathcal{L}(V, V) = \mathrm{Mat}_n$ of the algebras. For every $g \in G$ we have its representation $\varphi(g)$, so this induces a representation

$$\sum a_i g_i \mapsto \sum a_i \varphi(g_i), \tag{2.7}$$

where products go to products and sums go to sums. The opposite direction, a representation of $\mathbb{F}[G]$ induces a representation of G, is also true (by the duality principle). Moral: representations of groups induce representations of associative algebras (a representation of associative algebras in general is referred to as "**Modules**").

Consider $\mathscr{A}$ an associative algebra. We will define a new operation on $\mathscr{A}$, namely the bracket as a commutator

$$[a, b] = ab - ba \tag{2.8}$$

for all $a, b \in \mathscr{A}$. So with respect to the bracket, $\mathscr{A}$ is an algebra (distributivity remains, but associativity is broken). But observe

1. $[a, b] = -[b, a]$ i.e. we have antisymmetry of the bracket;
2. the Jacobi identity holds.

This newly constructed algebra is in fact a Lie algebra! So for every associative algebra $\mathscr{A}$, we may construct a Lie algebra on $\mathscr{A}$; this is described by a natural functor, so algebra morphisms are mapping to Lie algebra morphisms.

Remark 2.4. There are other ways to construct Lie algebras.
N.B.: subalgebras of Lie algebras are again Lie algebras.

Example 2.5. The Lie Algebra $\mathrm{Mat}_n(\mathbb{F}) = \mathfrak{gl}_n(\mathbb{F})$ the Lie algebra for $\mathrm{GL}(n, \mathbb{F})$.

Example 2.6. Consider $\mathfrak{sl}_n(\mathbb{F}) = \{A \in \mathfrak{gl}_n(\mathbb{F}) \mid \mathrm{Tr}(A) = 0\}$. This is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$, since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ so $\mathrm{Tr}([A, B]) = 0$ for all $A, B \in \mathfrak{gl}_n(\mathbb{F})$.

We can introduce the notion of an "**Ideal**" in an algebra, especially a Lie algebra! If $I \subseteq R$ where R is a ring, then $IR = I$ is a left ideal, and $RI = I$ is a right ideal. This notion may be generalized to algebras (especially Lie algebras!). Fir Lie algebras, *every ideal is a two-sided ideal.*

Proposition 2.7. $\mathfrak{sl}_n(\mathbb{F})$ *is an ideal in* $\mathfrak{gl}_n(\mathbb{F})$.

If we have a ring morphism $\varphi\colon R \to R'$, its kernel is a two-sided ideal. Moreover we may factorize

$$\mathrm{Im}(\varphi) \cong R/\operatorname{Ker}(\varphi). \tag{2.9}$$

This construction generalizes to algebras.

2.1 Exercises

▸ **EXERCISE 1**

Check that the vector space $\mathbb{R}^3$ is a Lie algebra with respect to cross-product of vectors. Check that this Lie algebra is simple (does not have any non-trivial ideals). Check that all derivations of this Lie algebra are inner derivations.

▸ **EXERCISE 2**

Check that the Lie algebra of Problem 1 is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of real antisymmetric 3×3 matrices and to the Lie algebra $\mathfrak{su}(2)$ of complex anti-Hermitian traceless 2×2 matrices.

Lecture 3: Examples of Lie Algebras

Let us consider n-dimensional space $\mathbb{R}^n$, with coordinates $(x^1, \dots, x^n)$, and we can consider either functions or polynomials of these coordinates $\mathbb{C}[x^1, \dots, x^n]$ and we will consider the differential operators on $\mathbb{C}[x^1, ..., x^n]$. It is an associative algebra, but also a Lie algebra (when the Lie bracket is the commutator). We can consider Lie subalgebras, e.g. first order differential operators[2] $\widehat{A} = A^i\partial_i$. But this is *NOT* a subalgebra of derivations, the product $\widehat{A}\widehat{B}$ is a second order differential operator; however note that

$$\begin{aligned}
[\widehat{A}, \widehat{B}](f) &= A^i\partial_i(B^j\partial_j f) - B^j\partial_j(A^i\partial_i f) && (3.1a)\\
&= A^i(\partial_i B^j)\partial_j f + A^i B^j \partial_i\partial_j f - B^j(\partial_j A^i)\partial_i f - A^i B^j \partial_i \partial_j f && (3.1b)\\
&= A^i(\partial_i B^j)\partial_j f - B^j(\partial_j A^i)\partial_i f && (3.1c)
\end{aligned}$$

So we write

$$\widehat{C} = C^k\partial_k = A^i(\partial_i B^j)\partial_j - B^j(\partial_j A^i)\partial_i \tag{3.2}$$

This $\widehat{C}$ is a derivation on $\mathbb{C}[x^1, ..., x^n]$. We can write $C^k = A^j(\partial_j B^k) - B^j(\partial_j A^k)$. The commutator of first order differential operators is again a first order differential operator.

We would like to express this operator in two different ways. First what are the coefficients A^i? They are the components of a vector field. So this is really the algebra

[2]Note that we are using Einstein summation convention; when one index is upstairs and another is downstairs, we sum over it as a dummy index.

of vector fields, the commutator of vector fields yield a Lie Algebra. Second, we want to introduce the notion of derivation of algebra. It is something satisfying the Leibniz rule. Suppose we have an $\mathscr{A}$-algebra, and a linear map

$$\alpha\colon \mathscr{A} \to \mathscr{A} \tag{3.3}$$

such that

$$\alpha(ab) = \alpha(a)b + a\alpha(b). \tag{3.4}$$

First, these first order differential operators are derivations, and moreover all derivations are first order differential operators.

This is a bit ambiguous, the algebra considered are left unspecified (smooth functions or polynomials!). We will prove it for polynomials, but not for smooth functions. If we know how a derivation behaves on the generators of the polynomial, then we know everything. Let $\widehat{A}(x^i) = A^i(x)$ where $A^i(x)$ is a polynomial.

Remark 3.1. All this stuff works on smooth manifolds despite never specifying what a "smooth manifold" is!

Theorem 3.2. *Given an algebra $\mathscr{A}$, then we may consider* $\mathrm{Der}(\mathscr{A})$ *of derivations of $\mathscr{A}$ which form a Lie algebra.*

Proof. We should prove it is a vector space, but it is obvious; we should prove the commutator of derivations $\alpha, \beta \in \mathrm{Der}(\mathscr{A})$ is a derivation $[\alpha, \beta] \in \mathrm{Der}(\mathscr{A})$. We consider

$$\begin{aligned}
(\alpha \circ \beta)(ab) &= \alpha(\beta(ab)) && (3.5a)\\
&= \alpha(\beta(a) \cdot b + a \cdot \beta(b)) && (3.5b)\\
&= \alpha(\beta(a) \cdot b) + \alpha(a \cdot \beta(b)) \qquad \text{by linearity} && (3.5c)\\
&= (\alpha \circ \beta)(a) \cdot b + \beta(a)\alpha(b) + \alpha(a)\beta(b) + a \cdot (\alpha \circ \beta)(b) && (3.5d)
\end{aligned}$$

Now we can consider the commutator expression of α with β, which amounts to

$$\begin{aligned}
[\alpha, \beta](ab) &= \Big((\alpha \circ \beta)(a) \cdot b + \beta(a)\alpha(b) + \alpha(a)\beta(b) + a \cdot (\alpha \circ \beta)(b)\Big)\\
&\qquad - \Big((\beta \circ \alpha)(a) \cdot b + \beta(a)\alpha(b) + \alpha(a)\beta(b) + a \cdot (\beta \circ \alpha)(b)\Big) && (3.6a)\\
&= (\alpha \circ \beta)(a)b + a \cdot (\alpha \circ \beta)(b) - (\beta \circ \alpha)(a)b - a \cdot (\beta \circ \alpha)(b) && (3.6b)\\
&= [\alpha, \beta](a) \cdot b + a \cdot [\alpha, \beta](b). && (3.6c)
\end{aligned}$$

This concludes our proof. □

One last example of derivations. Consider an algebra $\mathscr{A}$ (either associative or Lie), take $a, x \in \mathscr{A}$ where a is fixed. Consider the derivation

$$\alpha_a(x) = [a, x]. \tag{3.7}$$

For Lie algebras it is absolutely trivial:

$$\begin{aligned}
\alpha_a([x, y]) &= [\alpha_a(x), y] + [x, \alpha_a(y)] && (3.8a)\\
\iff [a, [x, y]] &= [[a, x], y] + [x, [a, y]] && (3.8b)\\
&= -[x, [y, a]] - [y, [a, x]] \qquad \text{Jacobi Identity!} && (3.8c)
\end{aligned}$$

Remark 3.3. Such derivations are called "**Inner Derivations**".

Lets compute the commutator of two inner derivations, the answer is the

$$[\alpha_a, \alpha_b] = \alpha_{[a,b]} \tag{3.9}$$

the result is an inner derivation. We have a homomorphism, so we have a $\mathscr{G}$-Lie algebra so we get a map $\mathscr{G} \to \mathrm{Der}(\mathscr{G})$ which, for all $a \in \mathscr{G}$, is mapped to

$$\alpha_a = [a, -]. \tag{3.10}$$

N.B.: Henceforth and throughout, I will use the term "morphism" and "homomorphism" interchangeably.

We can consider the morphism $\mathscr{G} \to L(\mathscr{G})$ where $L(\mathscr{G})$ is the linear operators on $\mathscr{G}$. We have a representation of our Lie algebra $\mathscr{G}$, called the "**Adjoint Representation**" where $a \mapsto \alpha_a = [a, -]$. We write $\mathrm{ad}_a = \alpha_a$. This is one of the simplest and most important examples of the representation of Lie algebras.

Consider Ker(ad). Then $\alpha_a = 0$. What does it mean that $[a, x] = 0$ for all $x \in \mathscr{G}$? This is precisely the "**Center of** $\mathscr{G}$" denoted by $\mathrm{Ker}(\alpha) = Z$.

Theorem 3.4. *If a finite dimensional Lie Algebra has no center (or a trivial one), then it is isomorphic to a matrix Algebra.*

Proof. We see that Im(ad) consists of a subalgebra of a Lie algebra of matrices since the Lie algebra *IS* a vector space and it is finite dimensional. Thus ad is a matrix algebra. □

Lecture 4

We started discussing the adjoint representation last time. Given a Lie algebra $\mathscr{G}$ and $a \in \mathscr{G}$, we can construct $\alpha_a = [a, -]$, $\alpha_a \colon \mathscr{G} \to \mathscr{G}$ and obeys

$$\alpha_{[a,b]} = [\alpha_a, \alpha_b]. \tag{4.1}$$

The adjoint representation for $\mathbb{R}^3$ equipped with the cross-product form Lie(SO(3)) — the Lie algebra for SO(3). The derivations are defined completely if we know the derivations of the generators; moreover, for e.g. $\widehat{\imath}$, $\widehat{\jmath}$, $\widehat{k}$ we only need to think about $\widehat{\imath}$, $\widehat{\jmath}$ since $\widehat{\imath} \times \widehat{\jmath} = \widehat{k}$.

Definition 4.1. The "**Structure Constants**" for a Lie algebra $\mathscr{G}$ (with generators e_1, ..., e_n which form a basis of the vector space) are specified by $[e_\alpha, e_\beta] = f_{\alpha\beta}{}^\gamma e_\gamma$, where $f_{\alpha\beta}{}^\gamma$ are the structure constants.

Remark 4.2. If we have a real Lie algebra, the structure constants are real; on the other hand, for complex Lie algebra, the structure constants are complex.

Definition 4.3. Let $\mathscr{G}$ be a real Lie algebra. Its "**Complexification**" consists of a Lie algebra denoted by $\mathbb{C}\mathscr{G}$ constructed by making the structure constants complex.

So complexification is some "mapping"

$$(\text{Real Lie Algebras}) \to (\text{Complex Lie Algebras})$$
$$x^\alpha e_\alpha \mapsto z^\alpha e_\alpha$$

where $z^\alpha = x^\alpha + iy^\alpha \in \mathbb{C}$, $x^\alpha \in \mathbb{R}$. So we can write $\mathbb{C}\mathscr{G} = \mathscr{G} \oplus i\mathscr{G}$ where we have this direct sum be the direct sum of vector spaces. This induces a notion of multiplication.

Example 4.4. Consider the Lie algebra $\mathfrak{so}(n)$ which is a real Lie algebra consisting of antisymmetric n-by-n matrices:

$$A^T + A = 0, \tag{4.2}$$

where $A \in \mathrm{Mat}_n(\mathbb{R})$. We want to complexify it, which is very easy We consider the same condition but take matrices with complex entries $A \in \mathrm{Mat}_n(\mathbb{C})$ which obey Eq (4.2). So we write $\mathfrak{so}(n, \mathbb{C}) = \mathbb{C}\mathfrak{so}(n, \mathbb{R})$.

Example 4.5. The algebra $\mathfrak{u}(n)$ of anti-Hermitian matrices $A + A^\dagger = 0$ where $A^\dagger$ is the Hermitian conjugate (i.e. conjugate transpose of A). We have $A \in \mathrm{Mat}_n(\mathbb{C})$ but what is its complexification? Observe

$$\mathbb{C}\mathfrak{u}(n) = \mathfrak{gl}(n, \mathbb{C}), \tag{4.3}$$

why? Well, for $X \in \mathrm{Mat}_n(\mathbb{C})$ we have

$$X = \frac{1}{2}\underbrace{(X - X^\dagger)}_{\text{anti-Hermitian}} + \frac{1}{2}\underbrace{(X + X^\dagger)}_{\text{Hermitian}} \tag{4.4}$$

but observe if A is anti-Hermitian, we have $A + A^\dagger = 0$, then iA is Hermtian since

$$(\mathrm{i}A)^\dagger + \mathrm{i}A = \mathrm{i}(-A^\dagger + A) = 0. \tag{4.5}$$

So $\mathscr{G}$ is anti-Hermitian, $\mathrm{i}\mathscr{G}$ is Hermitian, and $\mathbb{C}\mathscr{G}$ is everything.

We like to work in $\mathbb{C}$ since it is simpler than working in $\mathbb{R}$.

Theorem 4.6. *Complex representations of real Lie algebras are in one-to-one correspondence with the complex representations of its complexification.*

A representation maps basis vectors to linear operators, requiring us to solve

$$[\widehat{e}_\alpha, \widehat{e}_\beta] = f_{\alpha\beta}{}^{\gamma}\widehat{e}_\gamma, \tag{4.6}$$

where $\varphi(e_\alpha) = \widehat{e}_\alpha$. For the complexified Lie algebra, we do *precisely the same thing!*

We will define matrix groups, then matrix Lie algebras. It will not be a general definition.

Definition 4.7. A "**Matrix Group**" is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$.

Theorem 4.8. *The tangent space to the matrix group at the point $I = e = 1$ the identity is a Lie algebra called the Lie algebra of the matrix group.*

Suppose we have a curve $x(t) \in \mathbb{R}^m$ or in any topological space. Well, since x is a curve, it's a mapping

$$x\colon [\alpha, \beta] \to \mathbb{R}^m \tag{4.7}$$

from an interval of the real line $[\alpha, \beta]$ to the space, the tangent vector is

$$\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=\alpha} = x'(\alpha). \tag{4.8}$$

The tangent space at $x_0 \in M$ for a manifold M is the vector space of all tangent vectors at x_0. If the surface is given by

$$f(x^1, \dots, x^n) = 0, \tag{4.9}$$

we promote $x \mapsto x^i(t)$ to be components of a curve, implying

$$f(x^1(t), \dots, x^n(t)) = 0. \tag{4.10}$$

We thus have by the chain rule

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} f(x^1(t), \dots, x^n(t))\right|_{t=\alpha} = \left.\frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=\alpha} = 0. \tag{4.11}$$

If we use the implicit function theorem, we can consider A^i vectors such that

$$\left.A^i\frac{\partial f}{\partial x^i}\right|_{t=\alpha} = 0. \tag{4.12}$$

Example 4.9. Consider $O(n) = \{A \mid A^T A = I\}$ the group of n-by-n orthogonal matrices. We take $A(t)$ to be a curve in $O(n)$ such that $A(0) = I$ is the identity. So $A(t) = I + a(t)$. Consider then

$$(I + a(t))(I + a(t)^T) = I + a(t) + a(t)^T + \mathcal{O}(t^2), \tag{4.13}$$

then we can deduce the structure of the Lie algebra for "infinitesimal $a(t)$" to be precisely the matrices X such that

$$X + X^T = 0. \tag{4.14}$$

That is, all antisymmetric matrices.

If one has forgotten the implicit function theorem, here it is reproduced:

Implicit Function Theorem. *Let $f\colon \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function. Fix $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n+m}$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^m$ such that $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}$ where $\boldsymbol{c} \in \mathbb{R}^m$. If the matrix*

$$J_i{}^j = \frac{\partial f_i(\boldsymbol{x}, \boldsymbol{y})}{\partial y^j} \tag{4.15}$$

is invertible, then there exists an open set U containing $\boldsymbol{x}$, an open set V containing $\boldsymbol{b}$, and a unique continuously differentiable function $g\colon U \to V$ such that

$$\{(\boldsymbol{x}, g(\boldsymbol{x}))\} = \{(\boldsymbol{x}, \boldsymbol{y}) | f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}\} \cap (U \times V). \tag{4.16}$$

Lecture 5: Classical Lie Groups and Algebras

The first example is $\mathrm{GL}(n, \mathbb{R})$ and the corresponding Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ which consists of all matrices. It is important to consider $\mathfrak{gl}_n(\mathbb{R}) \subseteq \mathfrak{gl}_n(\mathbb{C})$ the complexification of the algebra. We will denote $\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C}\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{gl}_n$.

Another example $\mathrm{SL}_n(\mathbb{R})$ the group of n-by-n matrices satisfying the property of having unit determinant. To compute the Lie algebra, consider elements close to I or more precisely a curve

$$A(\tau) = I + a(\tau). \tag{5.1}$$

Now we should like to consider the tangent vector by Taylor expanding the curve and using the coefficient of the first order term as the tangenet vector

$$A(\tau) = I + \tau X + \mathcal{O}(\tau^2), \tag{5.2}$$

so we want

$$\det\left(A(\varepsilon)\right) = \det(I + \varepsilon X) = I + \varepsilon\,\mathrm{Tr}(X) + \mathcal{O}(\varepsilon^2). \tag{5.3}$$

We see immediately that the condition $\mathrm{Tr}(X) = 0$ is the condition for elements of the Lie algebra. So we see that

$$\mathrm{Lie}(\mathrm{SL}(n, \mathbb{R})) = \mathfrak{sl}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_n \mid \mathrm{Tr}(X) = 0\}. \tag{5.4}$$

We may consider the complexification

$$\mathbb{C}\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n. \tag{5.5}$$

N.B. $\mathfrak{sl}_n$ is an ideal in $\mathfrak{gl}_n$ and $\mathfrak{gl}_n = \mathbb{C} \oplus \mathfrak{sl}_n$ is the direct sum of the trivial Lie algebra $\mathbb{C}$ and $\mathfrak{sl}_n$ since $A = \alpha \cdot I + A'$ where $A' \in \mathfrak{sl}_n$ so $\mathrm{Tr}\,A = \alpha \cdot \dim$.

We also see $\mathrm{O}(n) = \{A \in \mathrm{GL}(n) \mid A^T A = I\}$. The Lie algebra is obtained by considering $A = I + \varepsilon X$ where $\varepsilon^2 \approx 0$ is an "infinitesimal"[3]. The Lie algebra is obtained by

$$A^T A = (I + \varepsilon X)^T (I + \varepsilon X) \tag{5.6a}$$

[3]Although this is a mathematically unrigorous notion, we can still use it for computational and heuristic purposes.

$$= I + \varepsilon(X^T + X) + \mathcal{O}(\varepsilon^2) \tag{5.6b}$$
$$= I \quad \Longleftrightarrow \quad X^T + X = 0. \tag{5.6c}$$

This is the condition for the Lie algebra of $\mathrm{O}(n)$ which is denoted

$$\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n) \mid X + X^T = 0\}. \tag{5.7}$$

We are interested in $\mathfrak{so}(n) = \mathfrak{so}(n, \mathbb{C}) = \mathbb{C}\mathfrak{so}(n, \mathbb{R})$. We see the Lie group

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n) \tag{5.8}$$

has unit determinants. We can also quickly compute and find that $\mathrm{SO}(n)$ is the connected part of $\mathrm{O}(n)$ which contains the identity.

The group $\mathrm{O}(n)$ has elements $A \in \mathrm{O}(n)$ such that $\det(A)^2 = 1$, so it has two separate components. This is seen in figure 1. This means that the group $\mathrm{O}(n)$ is disconnected, there is no continuous path connecting e.g. an element $X \in \mathrm{O}(n)$ with $\det(X) = -1$ to an element $Y \in \mathrm{O}(n)$ with $\det(Y) = +1$, because the path would have to go through a point with zero determinant. That is a singular matrix, which is not contained in the group $\mathrm{GL}(n)$, and that would imply $\mathrm{O}(n) \not\subseteq \mathrm{GL}(n)$ which is a contradiction.

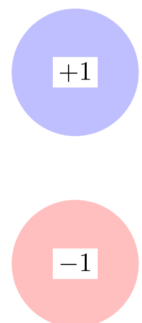


Figure 1: The Two Seperated Components of $\mathrm{O}(n)$.

Remark 5.1. Both $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ are both compact groups, i.e. closed and bounded.

Remark 5.2. Note that $\mathrm{U}(n)$ and $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n)$ are also compact. The condition for $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^\dagger A = I\}$, and the corresponding Lie algebra is $\mathfrak{u}(n)$. The condition for it is

$$(I + \varepsilon A^\dagger)(I + \varepsilon A) = I + \varepsilon(A^\dagger + A) + \mathcal{O}(\varepsilon^2) \tag{5.9a}$$
$$= I \quad \Longleftrightarrow \quad A^\dagger + A = 0. \tag{5.9b}$$

So $\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid A^\dagger + A = 0\}$. We have for $\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) \mid \mathrm{Tr}(A) = 0\}$. We see that $\mathbb{C} = \mathfrak{gl}_n(\mathbb{C})$, $\mathbb{C}\mathfrak{su}(n) = \mathfrak{sl}_n$ are the complexifications.

The last classical group we would like to consider preserves some skew-symmetric inner product. That is to say, $\langle x, y\rangle = -\langle y, x\rangle$ more generally however we will use a Bilinear form B which is antisymmetric

$$B(x, y) = -B(y, x). \tag{5.10}$$

We write

$$B(x, y) = x^T B y \tag{5.11}$$

if B is an antisymmetric matrix. We want to find matrices A such that $B(Ax, Ay) = B(x, y)$, i.e.

$$(Ax)^T BAy = x(A^T BA)y = xBy \tag{5.12}$$

or equivalently $A^T BA = B$. **N.B.** if $B = I$ we recover the orthogonal group. We get a group $\mathrm{Sp}(n) = \{A \in \mathrm{GL}(2n) \mid A^T BA = B\}$. the Lie algebra is of the form

$$(I + \varepsilon X)^T B(I + \varepsilon X) = B \quad \Rightarrow \quad X^T B + BX = 0. \tag{5.13}$$

If B is nondegenerate, then

$$\mathfrak{sp}(n) = \{X | X^T B + BX = 0\} \tag{5.14}$$

is the Lie Algebra for $\mathrm{Sp}(n)$. This is a noncompact group. We can get a compact group by examining the intersection $\mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{U}(n) = \mathrm{Sp}_n$, and $\mathbb{C}\,\mathrm{Lie}(\mathrm{Sp}_n) = \mathfrak{sp}(n)$ is the original Lie Algebra.

Theorem 5.3. *All simple Lie algebras that are Lie algebras of compact groups are classical Lie algebras or one of 5 exceptional Lie algebras.*

There is a related notion of semisimple Lie algebra. A semisimple Lie algebra is a direct sum of noncommutative simple Lie algebras. There is an important class of "**Solvable**" Lie algebras.

Remember (e.g. from Lang's *Algebra* chapter I §3) that a group G is "**Solvable**" if we have a tower of groups

$$G \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_m = \{e\} \tag{5.15}$$

such that for $G_{n-1} \supseteq G_n$ we have G_{n-1}/G_n be Abelian.

The notion of a solvable Lie algebra is the same except we have $\mathfrak{g}_{n-1} \supseteq \mathfrak{g}_n$ be an ideal. We also can doodle this cute diagram

$$\begin{tikzcd} \text{Group} & G \arrow[r, phantom, "\triangleright"] & H \\ \text{Algebra} & \mathrm{Lie}(G) \arrow[u, "\exp"] \arrow[r, phantom, "\supseteq"] & \mathrm{Lie}(H) \arrow[u, "\exp"'] \text{ is an ideal} \end{tikzcd} \tag{5.16}$$

In a sense, these two notions of solvability and semisimplicity are "complementary" — an arbitrary Lie group has a semisimple part and a reductive part. A compact Lie group has a semisimple Lie algebra.

5.1 Exercises

▸ **EXERCISE 3**

Find Lie algebras of the following matrix groups

1. The group of real upper triangular matrices
2. The group of real upper triangular matrices with diagonal entries equal to 1.
3. The group T_k of real $n \times n$ matrices obeying $a_{ii} = 1$, $a_{ij} = 0$ if $j - i < k$ and $j = i$.

▸ **EXERCISE 4**

Check that the groups of Problem 3 and corresponding Lie algebras are solvable.

Lecture 6

Theorem 6.1. *Consider a matrix group $G \subseteq \mathrm{GL}(n)$. Consider $\mathscr{G}$ the tangent space to G at 1. Then $\mathscr{G}$ is a Lie algebra with respect to commutator of elements.*

Proof. Consider a curve

$$x\colon [0,1] \to G, \qquad x(\tau) = 1 + a\tau + \mathcal{O}(\tau^2), \tag{6.1}$$

we obtain the tangent vector

$$x'(0) = \lim_{\tau\to 0} \frac{x(\tau) - x(0)}{\tau} = a. \tag{6.2}$$

We are proving this is a vector space and a ring.

Vector Space. If we have two curves

$$x(\tau) = 1 + a\tau + \cdots, \qquad y(\tau) = 1 + b\tau + \cdots, \tag{6.3}$$

we multiply together to find

$$x(\tau)y(\tau) = 1 + (a+b)\tau + \mathcal{O}(\tau^2), \tag{6.4}$$

and its derivative at 0 is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(x(\tau)y(\tau))\bigg|_{\tau=0} = a + b. \tag{6.5}$$

This implies that $a + b \in \mathscr{G}$. We can similarly show

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau} x(\lambda\tau)\right|_{\tau=0} = \lambda a \in \mathscr{G} \tag{6.6}$$

for all $\lambda \in \mathbb{R}$. Thus it's a vector space.
Ring. The commutator $[a, b]$ is obtained by the group commutator. We first let

$$x(\tau) = 1 + a\tau + \alpha\tau^2 + \cdots, \qquad y(\tau) = 1 + b\tau + \beta\tau^2 + \cdots, \tag{6.7}$$

and for the inverses use primed coefficients

$$x^{-1}(\tau) = 1 + a'\tau + \alpha'\tau^2 + \cdots, \qquad y^{-1}(\tau) = 1 + b'\tau + \beta'\tau^2 + \cdots. \tag{6.8}$$

Then we plug it into the definition of the group commutator:

$$x(\tau)y(\tau)x^{-1}(\tau)y^{-1}(\tau) = (1 + a\tau + \cdots)(1 + b\tau + \cdots)(1 + \alpha\tau + \cdots)(1 + \beta\tau + \cdots) \tag{6.9}$$

and we demand that

$$x(\tau)x^{-1}(\tau) = 1 \tag{6.10}$$

produces the conditions that $a + a' = 0$, and $aa' + \alpha' + \alpha = 0$ from the first and second order terms respectively. We see similar reasoning for $y(\tau)y^{-1}(\tau) = 1$ produces $b + b' = 0$ and $bb' + \beta' + \beta = 0$ as conditions on the coefficients. By carrying out multiplication, we find

$$\begin{aligned}(1 &+ a\tau + \alpha\tau^2 + \cdots)(1 + b\tau + \beta\tau^2 + \cdots)(1 + a'\tau + \alpha'\tau^2 + \cdots)(1 + b'\tau + \beta'\tau^2 + \cdots)\\ &= 1 + (a + b + a' + b')\tau\\ &\quad + (\alpha + \beta + \alpha' + \beta' + ab + aa' + ab' + ba' + bb' + a'b')\tau^2 + \cdots\end{aligned} \tag{6.11}$$

The first order term vanishes identically. The second order terms can be factored as

$$[(\alpha + \alpha' + aa') + (\beta + \beta' + bb') + ab + ab' + ba' + a'b']\tau^2 \tag{6.12}$$

which can be rewritten as

$$[ab + (-a)(-b) + a(-b) + (-a)b]\tau^2 = [ab - ba]\tau^2. \tag{6.13}$$

Thus

$$x(\tau)y(\tau)x(\tau)^{-1}y(\tau)^{-1} = 1 + [a, b]\tau^2 + \mathcal{O}(\tau^3). \tag{6.14}$$

We can introduce a new parameter $\sigma = \sqrt{\tau}$ and rewrite our equations to first order in σ which implies $[a, b] \in \mathscr{G}$. □

6.1 Derivations as Infinitesimal Automorphisms

We have an algebra $\mathscr{A}$, all automorphisms of $\mathscr{A}$ form a group $\mathrm{Aut}(\mathscr{A})$. We may say that derivations are "tangent vectors" to automorphisms, or in other words are infinitesimal automorphisms. Consider a continuous family of automorphisms $x(\tau)$, then

$$x(\tau)(ab) = x(\tau)(a) \cdot x(\tau)(b). \tag{6.15}$$

We take its derivative, if $x'(\tau) = \alpha_\tau$, then

$$\alpha_\tau(ab) = \alpha_\tau(a)x(\tau)(b) + x(\tau)(a)\alpha_\tau(b). \tag{6.16}$$

We assume $x(0) = 1$, then

$$\alpha_0(ab) = \alpha_0(a)b + a\alpha_0(b), \tag{6.17}$$

in other words derivations are infinitesimal automorphisms. We can say that derivations form the tangent space to $e \in \mathrm{Aut}(\mathscr{A})$.

N.B. *We cheated!* We assumed $\mathscr{A}$ was equipped with a topology, so a curve being "continuous" (much less differentiable!) made sense.

Consider a smooth manifold M, consider $C^\infty(M)$ the algebra of smooth functions on M. Any change of variables, i.e. smooth map, $\varphi\colon M \to M$ will generate an automorphism. Diffeomorphisms of M may be considered as automorphisms of $C^\infty(M)$. Then $\mathrm{Diff}(M) \subseteq \mathrm{Aut}(C^\infty(M))$.

To be completely precise we should write $f(\varphi^{-1}(x))$. We may say the diffeomorphisms form an infinite dimensional topological group. If M is compact, there is a natural notion of topology on $\mathrm{Diff}(M)$. If we have a family of diffeomorphisms $\varphi_t(x)$, differentiation with respect to t is obvious. Vector fields are derivations of this algebra; that is $\mathrm{Der} = \mathrm{Lie}\,\mathrm{Aut}\,\mathscr{A}$ and

$$\mathrm{Vect}(M) \subseteq \mathrm{Lie}\,\mathrm{Diff}(M). \tag{6.18}$$

We are of course a bit sloppy here....we won't go into details on infinite dimensional groups.

Suppose that G acts on M, its action can be a smooth action. That is we have a morphism of $G \to \mathrm{Diff}(M)$ and at the level of Lie algebras consider $\mathrm{Lie}(G) \to \mathrm{Vect}(M)$.

Lecture 7

We took a matrix group G and considered the tangent space at $e \in G$, $T_eG = \mathrm{Lie}(G)$ and showed it was the Lie algebra for G. If we took a curve $x\colon [0,1] \to G$ with $x(0) = e$, then we took $x'(0) \in T_eG$. This tangent space is closed under the commutator, and addition, so it's a Lie algebra.

Question. Given a Lie Algebra, could we "restore" all elements of the group?

Yes, lets take $x\colon [0,1] \to G$ without the condition $x(0) = e$. If $x(\tau_0) = e$, then

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\bigg|_{\tau=\tau_0} \in \mathrm{Lie}(G). \tag{7.1}$$

The value of the parameter is completely irrelevant. What if we take $x(\tau_0) \neq e$? Then nothing dangerous! Because look, we can take another curve

$$y(\tau) = x(\tau_0)^{-1}x(\tau) \tag{7.2}$$

which is still a curve in the group. Now $y(\tau_0) = e$. Therefore we can say that $y'(\tau_0) \in \mathrm{Lie}(G)$. Or by plugging in the definition, we see $y'(\tau_0) = x(\tau_0)^{-1}x(\tau_0) \in \mathrm{Lie}(G)$. This is due to the product rule and $x(\tau_0)$ being a constant. So *for any curve* in the group $x(\tau) \in G$ we have $x(\tau)^{-1}x'(\tau) \in \mathrm{Lie}(G)$. Let

$$\xi(\tau) = x(\tau)^{-1}x'(\tau). \tag{7.3}$$

So a curve in G will generate a curve in the Lie Algebra of G by means of this simple way. But now we would like to have a curve in $\mathrm{Lie}(G)$ give rise to a curve in G. We know

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = x(\tau)\xi(\tau) \tag{7.4}$$

gives a system of differential equations. To solve it, we need initial conditions. If we take $x(0) = e$, then a solution exists.

How to restore the Lie group? Take all the curves in the Lie algebra, then we get all the curves in the Lie group starting at 1, then...we get the whole matrix group? Not really, only the "**Connected Part**"!!! We use the notion of path connected, that any two points of the group have a continuous path connecting them.

Example 7.1. $\mathrm{O}(3)$ is not connected, since $\det(X) = \pm 1$ for all $X \in \mathrm{O}(3)$. We cannot connect two matrices $X, Y \in \mathrm{O}(3)$ if $\det(XY) = -1$. But $\mathrm{SO}(3)$ is the connected component of $\mathrm{O}(3)$ since $\det(X) = 1$ for all $X \in \mathrm{SO}(3)$.

Moral: We still have a way to get back the connected part of the group.

We get $x(\tau)^{-1}x'(\tau) = \xi(\tau) \in \mathrm{Lie}(G)$. We can take very simple curves in this algebra, namely constants! So $\xi(\tau)$ are constant matrices. Then we get $\xi(\tau) = A$ implies $x(\tau)^{-1}x'(\tau) = A$ if and only if

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = x(\tau)A, \tag{7.5}$$

with $x(0) = e$. Then $x(\tau) = \exp(A\tau)$. One way to define $\exp(A\tau)$ is as a solution to this differential equation. The other way is as a series

$$x(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}(A\tau)^n \tag{7.6}$$

Corollary 7.2. *If* $A \in \mathrm{Lie}(G)$*, then* $\exp(A) \in G$*.*

In reality this is almost sufficient. We have the "**Exponential Map**"

$$\exp\colon \mathrm{Lie}(G) \to G, \qquad A \mapsto \exp(A). \tag{7.7}$$

It maps a neighborhood of $O \in \mathrm{Lie}(G)$ onto a neighborhood of $1 \in G$. Why is this true? Because look, this is definitely true if the matrix group is a Lie group. Why? Because look, what we have is a map of the tangent spaces to the manifold. It is obvious this matrix is nondegenerate. It is really easy to check for classical groups.

If $B = \exp(A)$, then $A = \log(B)$. So we have

$$\exp\colon \mathrm{Lie}(G) \to G \tag{7.8}$$

and

$$\log\colon G \to \mathrm{Lie}(G) \tag{7.9}$$

which exists in the neighborhood of 1. In reality matrix groups are the only thing that are interesting. However we should also consider other groups. This requires a general definition of Lie algebras.

Take any Lie group G (so take $1 = e \in G$, we can introduce a coordinate system in the neighborhood of 1, $(x^1, ..., x^n)$, so it is topologically equivalent to the unit ball, and moreover permits functions to be differentiable). Take the tangent space at the identity $T_{\mathbf{1}}G$ and introduce an operation in such a way that makes it a Lie algebra. We consider curves in G, $x(\tau)$, such that $x(0) = 1$ and

$$\left.\frac{\mathrm{d}x}{\mathrm{d}\tau}\right|_{\tau=0} \in \mathrm{Lie}(G). \tag{7.10}$$

Although it is coordinate dependent, we know how to change coordinates $x \to y(x)$, $y^i = y^i(x)$, and

$$\frac{\mathrm{d}y^i}{\mathrm{d}\tau} = \frac{\partial y^i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}, \tag{7.11}$$

so we could define curves in any coordinate system and obtain the curve in any coordinate system.

We still need to define the Lie Bracket $[\xi, \eta]$ for $\xi, \eta \in T_{\mathbf{1}}G$. We would like it to be compatible with the commutator of matrices. So what to do?

We will draw

$$\xi = \left.\frac{\mathrm{d}x}{\mathrm{d}\tau}\right|_{\tau=0}, \qquad \text{and} \qquad \eta = \left.\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau}\right|_{\tau=0} \tag{7.12}$$

where x, $\widetilde{x}$ are curves in the group. We introduce the commutator

$$x(\sqrt{\tau})\widetilde{x}(\sqrt{\tau})x(\sqrt{\tau})^{-1}\widetilde{x}(\sqrt{\tau})^{-1}, \tag{7.13}$$

we would like

$$[\xi, \eta] = \frac{\mathrm{d}}{\mathrm{d}\tau} x(\sqrt{\tau})\widetilde{x}(\sqrt{\tau})x(\sqrt{\tau})^{-1}\widetilde{x}(\sqrt{\tau})^{-1}\bigg|_{\tau=0}. \tag{7.14}$$

We have already proven when we work with matrix groups, this is the exact same thing as the commutator for matrices.

This is not quite the end of the story. But from this definition, we can derive a lot of important stuff. Namely, if G, G' are two Lie groups and $\Phi\colon G \to G'$ is a Lie group morphism, then $\mathrm{Lie}(\Phi)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(G')$ is a Lie Algebra morphism.

Proposition 7.3. Lie *is a functor.*

Note that the Lie group morphism maps $\Phi(e) = e'$, $\Phi_*(T_{\mathbf{1}}G) = T_{\mathbf{1}'}G'$ which is then a Lie algebra morphism.

N.B. If the group is simply connected, Lie algebra morphisms induce Lie group morphisms.

7.1 Exercises

Lie algebra $\mathfrak{sl}(n)$ (denoted also by the symbol A_{n-1}) consists of traceless $n \times n$ complex matrices. The symbol $E_{i,j}$ denotes a matrix with only one non-zero entry that is equal to 1 and located in i-th row and j-th column.

▸ **EXERCISE 5**

Check that the matrices $E_{i,j}$ for $i = j$ and the matrices $h_i = E_{i,i} - E_{i+1,i+1}$ form a basis of $\mathfrak{sl}(n)$. Find the structure constants in this basis.

▸ **EXERCISE 6**

Check that subalgebra $\mathfrak{h}$ of all diagonal matrices is a maximal commutative subalgebra. Prove that there exists a basis of $\mathfrak{sl}(n)$ consisting of eigenvectors for elements of $\mathfrak{h}$. (This means that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n)$.)

▸ **EXERCISE 7**

Check that $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ form a system of multiplicative generators of $\mathfrak{sl}(n)$. Prove relations

$$[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0, \tag{7.15a}$$

$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \tag{7.15b}$$

$$(\operatorname{ad} e_i)^{-a_{ij}+1} e_j = 0, \qquad (\operatorname{ad} f_i)^{-a_{ij}+1} f_j = 0 \tag{7.15c}$$

for some choice of matrix a_{ij}.

We use here the notation $\mathrm{ad}(x)$ for the operator transforming y into $[x, y]$.

Lecture 8

8.1 Main Theorems

We have a group G and we can construct the corresponding Lie Algebra $\mathrm{Lie}(G)$ constructed by examining the tangent space at the identity $e = I \in G$:

$$\mathrm{Lie}(G) = T_e G. \tag{8.1}$$

We discussed obtaining the Lie algebra *from* the Lie group, and if we have a group morphism

$$\varphi\colon G \to G', \tag{8.2}$$

we can construct a morphism of the corresponding Lie Algebras

$$\varphi_*\colon \mathrm{Lie}(G) \to \mathrm{Lie}(G'). \tag{8.3}$$

That is to say the following diagram commutes

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Lie}(G) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_*} \\ G' & \longrightarrow & \mathrm{Lie}(G') \end{array} \tag{8.4}$$

we would like to consider going the other way. That is given a Lie algebra $\mathscr{G}$, we would like to construct a corresponding group, and show that Lie algebra morphisms $\mathscr{G} \to \mathscr{G}'$ generate group morphisms. But the group needs to be simply connected.

Consider $SU(2)/\mathbb{Z}_2 = SO(3)$. However $SU(2)$ has the same Lie algebra as $SO(3)$; when we identify the algebra topologically as the sphere, this quotient identifies opposite points as the same. This doesn't affect the Lie algebra.

If we drop the condition of being simply connected, a Lie algebra may give rise to two different Lie groups. Simply connected permits us to continuously deform one path to another. Every closed curved is contractible iff the space is simply connected.

Theorem 8.1. *For each Lie algebra there exists a unique simply connected Lie group.*

If $g\colon [0,1] \to G$ is a differentiable curve on the group, we may construct a curve

$$\gamma(t) = g(t)^{-1}\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\log\big(g(t)\big) \tag{8.5}$$

on the algebra. We have

$$\left.\frac{\mathrm{d}g(t)}{\mathrm{d}t}\right|_t \in T_{g(t)}G, \tag{8.6}$$

which is not necessarily in the Lie algebra T_eG. However for some $g \in G$ we have $g \cdot 1 = g$, so this is a translation which sends $1 \mapsto g$. We have a map

$$g_*\colon T_eG \to T_gG, \tag{8.7}$$

and so the right formula would be

$$\gamma(t) = (g_*(t))^{-1}\,\frac{\mathrm{d}g(t)}{\mathrm{d}t}. \tag{8.8}$$

But we will abuse notation and write

$$\gamma(t) = g(t)^{-1}\frac{\mathrm{d}g(t)}{\mathrm{d}t}. \tag{8.9}$$

We have a correspondence between curves in the Lie Algebra and curves in the Lie group. We obtain a system of differential equations

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = g(t)\gamma(t) \tag{8.10}$$

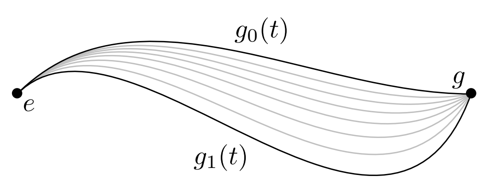
which has a unique solution for $g(0) = e$.

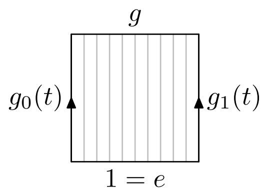
Consider a curve $g_0(t)$, $g_1(t)$ in the group, where

$$g_0(0) = g_1(0) = e, \tag{8.11}$$

and we have

$$g_0(1) = g_1(1) = g. \tag{8.12}$$





We assume that G is simply connected and we will deform the path, thus obtaining a family of paths $g_\tau(t) = g(\tau, t)$ such that $g(0,t) = g_0(t)$ and $g(1,t) = g_1(t)$. We can draw a diagram (seen on the left), we have two variables to consider τ and t, which to differentiate? Both of them! Consider

$$\xi(\tau, t) = g(\tau, t)^{-1} \frac{\partial g(\tau, t)}{\partial t} \tag{8.13}$$

and

$$\eta(\tau, t) = g(\tau, t)^{-1} \frac{\partial g(\tau, t)}{\partial \tau}. \tag{8.14}$$

So what do we know? Well, the curves in the Lie algebra are

$$\gamma_0(t) = g_0(t)^{-1} \frac{\mathrm{d}g_0(t)}{\mathrm{d}t} = \xi(0, t) \tag{8.15}$$

and

$$\gamma(1)(t) = g_1(t)^{-1} \frac{\mathrm{d}g_1(t)}{\mathrm{d}t} = \xi(1, t). \tag{8.16}$$

We also know $g_\tau(0) = e = 1$, and $g_\tau(1) = g$, for every τ. We see then that

$$\eta(\tau, 0) = \eta(\tau, 1) = 0. \tag{8.17}$$

We have

$$\partial_t g(\tau, t) = g(\tau, t)\xi(\tau, t) \tag{8.18}$$

and

$$\partial_\tau g(\tau, t) = g(\tau, t)\eta(\tau, t). \tag{8.19}$$

We deduce from

$$\partial_\tau \partial_t g(\tau, t) = \partial_t \partial_\tau g(\tau, t) \tag{8.20}$$

that

$$(\partial_\tau g)\xi + g\partial_\tau \xi = (\partial_t g)\eta + g(\partial_t \eta), \tag{8.21}$$

and by multiplying on the right by $g(\tau, t)^{-1}$ we have

$$\underbrace{\left(g^{-1}\partial_\tau g\right)}_{=\eta} \xi + \partial_\tau \xi = \underbrace{\left(g^{-1}\partial_t g\right)}_{=\xi} \eta + \partial_t \eta \qquad \Longrightarrow \qquad \partial_t \eta - \partial_\tau \xi = \eta\xi - \xi\eta. \tag{8.22}$$

We have an equation of r the form

$$\frac{\partial \eta(\tau, t)}{\partial t} - \frac{\partial \xi(\tau, t)}{\partial \tau} = [\xi(\tau, t), \eta(\tau, t)]. \tag{8.23}$$

Now, what should we do with this? In reality, we've done everything already. What is our goal? We'd like to restore the group knowing the Lie Algebra. We get points in the group by considering equivalence classes of paths in the Lie Algebra.

8.2 Exercises

8.2.1 Algebra D_n

The Lie algebra D_n consists of $2n \times 2n$ complex matrices L obeying

$$(FL)^T + FL = 0 \tag{8.24}$$

where, in block form,

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{8.25}$$

▸ **EXERCISE 8**

Check that D_n is isomorphic to the complexification of the Lie algebra of the orthogonal group $\mathrm{O}(2n)$.

▸ **EXERCISE 9**
Check that the matrices

$$e_{ij} := \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix} \tag{8.26}$$

together with the matrices

$$f_{pq} := \begin{bmatrix} 0 & E_{pq} - E_{qp} \\ 0 & 0 \end{bmatrix}, \qquad g_{pq} := \begin{bmatrix} 0 & 0 \\ E_{pq} - E_{qp} & 0 \end{bmatrix} \tag{8.27}$$

form a basis of D_n.

Here $i, j = 1, \ldots, n$, $1 \le p < q \le n$, and $E_{i,j}$ has only one nonzero entry that is equal to unity located in the i^{th} row and j^{th} column.

▸ **EXERCISE 10**
Check that the subalgebra $\mathfrak{h}$ of all matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \tag{8.28}$$

(where A is a diagonal matrix) is a maximal commutative subalgebra, and prove that there exists a basis of D_n consisting of eigenvectors for elements of $\mathfrak{h}$ acting on D_n by means of adjoint representation. (This means that $\mathfrak{h}$ is a Cartan subalgebra of D_n.)

▸ **EXERCISE 11**
Check that $e_i = e_{i,i+1}$ for $i = 1, ..., n-1$ and $e_n = f_{n-1,n}$; $f_i = e_{i+1,i}$ for $i = 1, ..., n-1$ and $f_n = g_{n-1,n}$ form a system of multiplicative generators of D_n. Prove the relations

$$[e_i, f_j] = \delta_{ij} h_i \tag{8.29a}$$

$$[h_i, h_j] = 0 \tag{8.29b}$$

$$[h_i, e_j] = a_{ij} e_j \tag{8.29c}$$

$$[h_i, f_j] = -a_{ij} f_j \tag{8.29d}$$

$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0 \qquad \text{when } i \neq j \tag{8.29e}$$

$$(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0 \qquad \text{when } i \neq j \tag{8.29f}$$

We use here the notation $(\mathrm{ad}\, x)$ for the operator transforming y into $[x, y]$.

8.2.2 Algebra C_n

Consider the Lie algebra C_n consisting of $2n \times 2n$ complex matrices obeying

$$(FL)^T + FL = 0 \tag{8.30}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{8.31}$$

▸ **EXERCISE 12**
Check that C_n is isomorphic to the complexification of the Lie algebra of the compact group $\mathrm{Sp}(2n) \cap \mathrm{U}(2n)$ where $\mathrm{Sp}(2n)$ stands for the group of linear transformations of $\mathbb{C}^{2n}$ preserving non-degenerate anti-symmetric bilinear form and $\mathrm{U}(2n)$ denotes unitary group.

▸ **EXERCISE 13**
Check that the matrices

$$e_{ij} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix} \tag{8.32a}$$

$$f_{pq} = \begin{bmatrix} 0 & E_{pq} + E_{qp} \\ 0 & 0 \end{bmatrix} \tag{8.32b}$$

$$g_{pq} = \begin{bmatrix} 0 & 0 \\ E_{pq} + E_{qp} & 0 \end{bmatrix} \tag{8.32c}$$

form a basis of C_n, where $i, j = 1, ..., n$ and $1 \le p \le q \le n$.

▸ **EXERCISE 14**
Check that the subalgebra $\mathfrak{h}$ of all matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \tag{8.33}$$

where A is a diagonal matrix, is a maximal commutative subalgebra. Prove there exists a basis of C_n consisting of eigenvectors for elements of $\mathfrak{h}$ acting on C_n by means of adjoint representation.

▸ **EXERCISE 15**
Check that $e_i = e_{i,i+1}$ $(i = 1, ..., n-1)$ and $e_n = f_{n,n}$; $f_i = e_{i+1,i}$ $(i = 1, ..., n-1)$ and $f_n = g_{n,n}$ form a system of generators of C_n. Prove

$$[e_i, f_j] = \delta_{ij} h_i, \tag{8.34a}$$
$$[h_i, h_j] = 0 \tag{8.34b}$$
$$[h_i, e_j] = a_{ij} e_j, \tag{8.34c}$$
$$[h_i, f_j] = -a_{ij} f_j, \tag{8.34d}$$
$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad i \neq j \tag{8.34e}$$
$$(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0, \qquad i \neq j \tag{8.34f}$$

8.2.3 Algebra B_n

The algebra B_n consists of $(2n+1) \times (2n+1)$ complex matrices obeying

$$L^T F + F L = 0 \tag{8.35}$$

where

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \tag{8.36}$$

I_n is the $n \times n$ identity matrix, and we have written F in block form.

▸ **EXERCISE 16**
Show that B_n is isomorphic to the complexified Lie algebra of $\mathrm{O}(2n+1)$.

▸ **EXERCISE 17**
Check that the subalgebra $\mathfrak{h}$ of all matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -A \end{bmatrix} \tag{8.37}$$

(where A is a diagonal matrix) is a maximal Abelian subalgebra, and prove there is a basis of B_n consisting of eigenvectors for elements of $\mathfrak{h}$ acting on B_n by the adjoint representation.

▸ **EXERCISE 18**
Find a system e_i, f_j of multiplicative generators of B_n obeying

$$[e_i, f_j] = \delta_{ij} h_i \tag{8.38a}$$
$$[h_i, h_j] = 0 \tag{8.38b}$$
$$[h_i, e_j] = a_{ij} e_j \tag{8.38c}$$
$$[h_i, f_j] = -a_{ij} f_j \tag{8.38d}$$
$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad i \neq j \tag{8.38e}$$
$$(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0, \qquad i \neq j \tag{8.38f}$$

for “some” matrix a_{ij}.

▸ **EXERCISE 19**
Describe the roots and root vectors of A_n, B_n, C_n, D_n.

Lecture 9

Let G be a Lie group, consider $\mathrm{Lie}(G)$ its Lie algebra. Then there is a correspondence between curves in the group and curves in the Lie algebra. So if we have two curves in the Lie algebra, we have two curves in the Lie group, then for simply connected groups we may deform two curves $g_1(t)$, $g_2(t)$ with

$$g_1(0) = g_2(0) = g_0, \quad \text{and} \quad g_1(1) = g_2(1) = g_1 \tag{9.1}$$

by introducing a family of curves $g_\tau(t)$ which has a corresponding family of curves in the Lie algebra. We have

$$\xi(\tau,t) = g_\tau(t)^{-1}\frac{\mathrm{d}g_\tau(t)}{\mathrm{d}t} \tag{9.2a}$$

$$\eta(\tau,t) = g_\tau(t)^{-1}\frac{\mathrm{d}g_\tau(t)}{\mathrm{d}\tau} \tag{9.2b}$$

and we have the relation

$$\frac{\partial\eta}{\partial t} - \frac{\partial\xi}{\partial t} = [\xi,\eta] \tag{9.3}$$

which occurs in the Lie algebra. Observe that

$$\xi(t,0) = \gamma_0(t), \quad \xi(t,1) = \gamma_1(t), \quad \eta(0,\tau) = \eta(1,\tau) = 0. \tag{9.4}$$

So given these conditions that, for

$$\frac{\partial\eta(\tau,t)}{\partial t} - \frac{\partial\xi(\tau,t)}{\partial\tau} = [\xi(\tau,t),\eta(\tau,t)] \tag{9.5}$$

with boundary conditions

$$\xi(t,0) = \gamma_0(t) \quad \text{and} \quad \xi(t,1) = \gamma_1(t) \tag{9.6a}$$

$$\eta(1,\tau) = \eta(0,\tau) = 0 \tag{9.6b}$$

can we get information induced in the group? We have

$$\xi(t,\tau) = g(t,\tau)^{-1}\frac{\partial g(t,\tau)}{\partial t}, \tag{9.7}$$

where $g(0,\tau) = 1$. We can restore $g(t,\tau)$ since there is a unique solution to eq (9.7).

Suppose we have $\mathrm{Lie}(G) \to \mathrm{Lie}(G')$ be a Lie algebra morphism; how can we induce a Lie group morphism? Well, how we do it makes heavy use of this curve voodoo. The basic correspondence we have is that "points in the group" corresponds to "curves in the Lie Algebra", and "multiplication in the group" corresponds to "concatenation of paths in the Lie algebra." Group curve concatenation can be performed, for

$$g_1 : [0,b] \to G \tag{9.8}$$

and

$$g_2 : [b,a] \to G, \tag{9.9}$$

as

$$g(t) = \begin{cases} g_1(t) & t \in [0,b] \\ g_2(b)^{-1}g_2(t) & t \in [b,a]. \end{cases} \tag{9.10}$$

If the paths $g_1(t)$, $g_2(t)$ are not loops, i.e. $g_1(0) \neq g_1(b)$ and $g_2(b) \neq g_2(a)$, then

$$g(t) = \begin{cases} g_1(t) & t \in [0,b] \\ g_1(b)g_2(b)^{-1}g_2(t) & t \in [b,a]. \end{cases} \tag{9.11}$$

We see that $g(b)$ is in the first case equal to $g_1(b)$, and in the second case

$$g(b) = g_1(b)g_2(b)^{-1}g_2(b) = g_1(b). \tag{9.12}$$

Thus the two cases agree on the overlap.

The corresponding curve in the Lie algebra is

$$\gamma(t) = \begin{cases} g_1(t)^{-1}\dfrac{\mathrm{d}g_1(t)}{\mathrm{d}t} & t \in [0,b] \\ (g_2(b)^{-1}g_2(t))^{-1}\left(g_2(b)^{-1}\dfrac{\mathrm{d}g_2(t)}{\mathrm{d}t}\right) & t \in [b,a] \end{cases} \tag{9.13}$$

up to a constant (i.e. $g_1(b)$) in the second case. It doesn't play a significant role, as it is factored out. We end up with

$$\gamma(t) = \begin{cases} g_1(t)^{-1}\dfrac{\mathrm{d}g_1(t)}{\mathrm{d}t} & t \in [0,b] \\ g_2(t)^{-1}\dfrac{\mathrm{d}g_2(t)}{\mathrm{d}t} & t \in [b,a] \end{cases} \tag{9.14}$$

We will consider the construction of Lie groups from Lie algebra next time...

We proved there exists a one-to-one correspondence between simply connected Lie groups and finite dimensional Lie algebras. If we have a discrete normal subgroup $N \subseteq G$, then the Lie algebra of $G/N \cong \mathrm{Lie}(G)$. This is because there is a neighborhood $\mathcal{U}$ of $1 \in G$ such that $\mathcal{U} \cap N = \{1\}$.

Theorem 9.1. *If G is simply connected, and $\mathrm{Lie}(G) \cong \mathrm{Lie}(G')$, then $G' \cong G/N$ where N is a discrete normal subgroup of G.*

Example 9.2. $\mathbb{R}$ equipped with addition has trivial commutators in Lie algebra, but $\mathrm{Lie}\big(U(1)\big) \cong \mathrm{Lie}(\mathbb{R})$ so $U(1) \cong \mathbb{R}/\mathbb{Z}$.

Lecture 10

We will finish the construction of the Lie group from the Lie algebra. There is an important formula called the "**Baker-Campbell-Hausdorff formula.**" Recall we considered a correspondence between a curve in the Lie group and a curve in the Lie algebra. The question is can we only consider curves $\exp(tA)$ one-parameter families of the group; in the neighborhood of the identity, the correspondence is one-to-one. It would follow that multiplication in the group

$$\mathrm{e}^A \cdot \mathrm{e}^B = \mathrm{e}^{C(A,B)} \tag{10.1}$$

goes to some operation in the algebra. We have

$$C(A,B) = \log(\mathrm{e}^A \cdot \mathrm{e}^B) \tag{10.2}$$

we know

$$\mathrm{e}^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}, \tag{10.3}$$

but there is no such series for the logarithm. There is some expansion for the logarithm *in some neighborhood.* We need to be careful about the order of multiplication, we can express the series in terms of the commutators

$$C(A,B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] - \frac{1}{24}[B,[A,[A,B]]] + \cdots. \tag{10.4}$$

Really, the most important part of this formula is

$$C(A,B) = A + B + \frac{1}{2}[A,B] + \cdots \tag{10.5}$$

We expect *ab initio* that $C(A,B) \in \mathrm{Lie}(G)$ is in the Lie algebra. This formula permits us to construct a "**local Lie group**", i.e. a group with induced group operation in the neighborhood of the unit element.

There is a special situation with this being global. Consider a nilpotent Lie algebra. The Baker-Campbell-Hausdorff formula becomes a polynomial with finitely many terms, which gives rise to a Lie group from the Lie algebra.

10.1 Representations of $\mathfrak{sl}(2)$

This is really quite important in math and in physics. We know

$$\mathbb{C} \otimes \mathfrak{su}(2) \cong \mathfrak{sl}(2). \tag{10.6}$$

So representations of $\mathfrak{su}(2)$ may be studies by representations of $\mathfrak{sl}(2)$. We know

$$SO(3) \cong SU(2)/\mathbb{Z}_2, \tag{10.7}$$

so we can study representations of $\mathfrak{so}(3)$ too!

For $\mathfrak{sl}(2)$, we have generators e, f, h with the commutation relations

$$[e,f] = h \tag{10.8a}$$

$$[h,e] = 2e \tag{10.8b}$$

$$[h,f] = -2f. \tag{10.8c}$$

We would like to describe all representations of $\mathfrak{sl}(2)$. For a general Lie Algebra, we take its Cartan subalgebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$. Here h is a generator of the Cartan subalgebra. We will take any rep

$$\varphi\colon \mathscr{G} \to \mathfrak{gl}(n). \tag{10.9}$$

So

$$f \mapsto \varphi(f) = F, \quad e \mapsto \varphi(e) = E, \quad h \mapsto \varphi(h) = H, \tag{10.10}$$

and the commutation relations are

$$[E,F] = H \tag{10.11a}$$

$$[H,E] = 2E \tag{10.11b}$$

$$[H,F] = -2F. \tag{10.11c}$$

We need to find 3 such matrices. We will consider eigenvectors of H called "**Weight Vectors**"

$$H\vec{x} = \lambda \cdot \vec{x}. \tag{10.12}$$

Once we have one weight vector $\vec{x}$, we can construct others via use of E and F. We have from eq (10.11b)

$$HE = E(H+2) \tag{10.13}$$

which, when applied to the weight vector, yields

$$HE\vec{x} = E(H+2)\vec{x} = (\lambda+2)E\vec{x}. \tag{10.14}$$

This implies that $E\vec{x}$ is also an eigenvector of H with eigenvalue $\lambda+2$. Thus we have infinitely many weight vectors, right? Well, this is **wrong** since $E\vec{x}$ could vanish! If $E\vec{x} = 0$, then $\vec{x}$ is called the "**Highest Weight Vector**".

We also have

$$H(F\vec{x}) = (\lambda-2)F\vec{x} \tag{10.15}$$

by the exact same reasoning. This means that $F\vec{x}$ is also a weight vector. We will now describe all finite dimensional representations of $\mathfrak{sl}(2)$.

Remark 10.1. In finite dimensional representations, H always has an eigenvector.

Lets apply E to $\vec{x}$ many times, so we get new eigenvectors. Then at some moment

$$HE^k\vec{x} = 0 \tag{10.16}$$

for some k since we cannot have an infinite number of distinct eigenvectors. Let

$$\vec{v} := E^{k-1}\vec{x} \tag{10.17}$$

be the highest weight vector, so

$$E\vec{v} = 0. \tag{10.18}$$

Let

$$H\vec{v} = m\vec{v}. \tag{10.19}$$

Let

$$\vec{v}_k = F^k\vec{v}, \tag{10.20}$$

we know

$$H\vec{v}_k = (m-2k)\vec{v}_k. \tag{10.21}$$

This is a weight vector. We have

$$F\vec{v}_k = \vec{v}_{k+1} \tag{10.22}$$

by definition. We should apply

$$E\vec{v}_k = EF\vec{v}_{k-1} = (FE+H)\vec{v}_{k-1}. \tag{10.23}$$

We can guess that E *raises* the eigenvalue. That is

$$E\vec{v}_k = \gamma_k\vec{v}_k \tag{10.24}$$

where γ_k is some factor.

We can compute

$$\begin{aligned}
E\vec{v}_k &= FE\vec{v}_{k-1} + H\vec{v}_{k-1} && \text{(10.25a)}\\
&= F(\gamma_{k-1}\vec{v}_{k-2}) + (m+2-2k)\vec{v}_{k-1} && \text{(10.25b)}\\
&= \gamma_{k-1}F\vec{v}_{k-2} + (m+2-2k)\vec{v}_{k-1} && \text{(10.25c)}\\
&= (m+2-2k+\gamma_{k-1})\vec{v}_{k-1} && \text{(10.25d)}\\
&= \gamma_k\vec{v}_{k-1} && \text{(10.25e)}
\end{aligned}$$

This implies that

$$\gamma_k = \gamma_{k-1} + m + 2 - 2k \tag{10.26}$$

a recursion relation which permits us to compute γ_k, an arithmetic progression. We have our representation be irreducible if and only if

$$\operatorname{span}\{\vec{v}_k\} \cong \mathbb{C}^n. \tag{10.27}$$

We have everything, we just need to compute the γ_k constants. It turns out that the weights range from m, $m-2$, $\dots$, $-m$.

Lecture 11

We consider representations of $\mathfrak{sl}(2) = A_1$. We analyzed completely the finite dimensional representations; the only place where finite dimensions were used was in proving the existence of the highest weight vector. We reasoned h has eigenvectors. Then we applied e to the eigenvectors of h, which produced a different eigenvector or zero.

We had the eigenvector

$$h\vec{v} = \lambda\vec{v}, \tag{11.1}$$

then

$$h(e^k\vec{v}) = (\lambda + 2k)e^k\vec{v}, \tag{11.2}$$

but in the finite dimensional case we get at some moment

$$e^n\vec{v} = 0 \tag{11.3}$$

for some n. So in finite dimensions, such a vector always exists. In the infinite dimensional case, we will assume a highest weight vector exists. Then we will describe all irreducible representations, we did this basically. Let $\vec{v}$ be such that

$$e\vec{v} = 0, \tag{11.4}$$

then let

$$\vec{v}_k = f^k\vec{v}, \tag{11.5}$$

so

$$h\vec{v}_k = (m - 2k)\vec{v}_k \tag{11.6}$$

and we have "ladder relations"

$$f\vec{v}_k = \vec{v}_{k+1} \tag{11.7}$$

and

$$e\vec{v}_k = \gamma_k\vec{v}_{k-1}, \tag{11.8}$$

where γ_k is some coefficient which requires solving a recursive formula. We have

$$\gamma_k = k(m - k + 1). \tag{11.9}$$

Lets prove this is an irreducible representation. What does this mean? It doesn't have any nontrivial subrepresentations. Suppose we do have some nontrivial subrepresentation, it should contain at least one vector. Suppose this one vector is of the form

$$\sum c_k\vec{v}_k = \vec{w}. \tag{11.10}$$

Lets apply to this vector $e^s\vec{w}$, what happens? It is pretty clear it should be $\vec{w} = \vec{v}_{s'}$ where s' is some index, this is due to h having an eigenvector in any representation.

We can now apply e and f to $\vec{w}$, we end up recovering

$$e^{s'}\vec{w} \propto \vec{v}_0. \tag{11.11}$$

We made the mistake that

$$e\vec{v}_k = k(m - k + 1)\vec{v}_{k-1} \tag{11.12}$$

exists, i.e. $m - k + 1 \neq 0$. If $m \notin \mathbb{Z}$, then this is an irreducible representation. But if $m \in \mathbb{Z}$, more specifically

$$k = m + 1, \tag{11.13}$$

then

$$e\vec{v}_{m+1} = 0. \tag{11.14}$$

So we get an irreducible subrepresentation spanned by $\vec{v}_0$, $\vec{v}_1$, ..., $\vec{v}_m$. So for each $m \in \mathbb{N}$, we have on irreducible representation of dimension $m + 1$, so we have $m + 1 = 1, 2, 3, \ldots$.

Now lets discuss the group $SU(2)$, recall $\mathfrak{su}(2)$ consists of traceless anti-Hermitian matrices; recall unitary matrices satisfy

$$A^{\dagger}A = I. \tag{11.15}$$

The rows and columns form an orthonormal basis:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{11.16}$$

so

$$|a|^2 + |b|^2 = 1 \quad \text{and} \quad |c|^2 + |d|^2 = 1 \quad \text{and} \quad ad - bc = 1. \tag{11.17}$$

If we know a and b, we can deduce that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}. \tag{11.18}$$

We can consider the topology here, if

$$a = a_0 + ia_1 \quad \text{and} \quad b = b_0 + ib_1 \tag{11.19}$$

then we are working with a 4-dimensional sphere. We can say that topologically SU(2) is compact and simply connected. The representations of SU(2) may be identified by representations of its Lie algebra. We recall

$$\mathbb{C}\,\mathrm{Lie}\big(SU(2)\big) = \mathfrak{sl}(2). \tag{11.20}$$

We may describe the representations directly.

Remark 11.1. $\mathfrak{su}(2)$ has a scalar representation, i.e. the most boring representation imaginable (everything is represented by the unit matrix). It's trivial, and has dimension 1.

Now we have the vector or "fundamental" representation. It is 2-dimensional. Every matrix is represented by itself. Let $V = \mathbb{C}^2$ be the representation of SU(2). For this special situation, we can work with polynomials of $x, y \in \mathbb{C}$.

That is to say, if $\{(x, y)\} = V$, we may take the space of functions on V. We introduce

$$\psi_g \colon \varphi(z) \mapsto \varphi(g^{-1}z) \tag{11.21}$$

which "deforms" $\varphi(z)$ into $\varphi(g^{-1}z)$. If we have

$$\psi_h \colon \varphi(z) \mapsto \varphi(h^{-1}z), \tag{11.22}$$

then we demand

$$\varphi_h \circ \varphi_g \colon \varphi(z) \mapsto \varphi\big((hg)^{-1}z\big). \tag{11.23}$$

Functions of V are contravariant functors.

This is a reducible representation, since we may restrict our focus to polynomials over V. Is the space of polynomials an irreducible representation? No! Why? We can have the subspace of homogeneous polynomials of degree m. So it would be irreducible and spanned by x^m, $x^{m-1}y$, ..., xy^{m-1}, y^m which is of dimensions $m+1$. We can deduce the representation of the Lie algebra. Observe for us in $\mathfrak{su}(2)$,

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{11.24}$$

so it corresponds to

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \in \mathrm{SU}(2). \tag{11.25}$$

How h acts on the basis is that

$$h\colon x \mapsto ux, \quad y \mapsto u^{-1}y. \tag{11.26}$$

So in effect,

$$x^{m-k}y^k \mapsto u^{m-2k}x^{m-k}y^k, \tag{11.27}$$

with the Lie algebra $u = 1 + \varepsilon$ where $|\varepsilon|^2 \ll 1$. So

$$u^m = 1 + m\varepsilon \tag{11.28}$$

and so on.

Lecture 12: Representations

Today we will talk about representations, and this is the main part of the class. First some things about representations. We have a representation be a morphism $G \to \mathrm{GL}(n,\mathbb{C})$ for Lie groups, a Lie algebra representation is a morphism $\mathscr{G} \to \mathfrak{gl}(n,\mathbb{C})$. How do we classify representations? Well, the most important group representations are orthogonal or unitary in the sense of using an inner product $\langle \cdot, \cdot \rangle$, i.e. morphisms

$$G \to \mathrm{O}(n) \quad \text{and} \quad G \to \mathrm{U}(n) \tag{12.1}$$

respectively. However, in the language of Lie algebras

$$A = 1 + a \tag{12.2}$$

and the condition is as follows

$$\langle ax, y \rangle + \langle x, ay \rangle = 0, \tag{12.3}$$

which for algebras results in morphisms

$$\mathscr{G} \to \mathfrak{so}(n) \quad \text{and} \quad \mathscr{G} \to \mathfrak{u}(n) \tag{12.4}$$

respectively for orthogonal and unitary representations.

Theorem 12.1. *Unitary and orthogonal representations are completely reducible (i.e. a direct sum of irreps).*

Lemma 12.2. *If W is an invariant subspace, then $W^\perp$ is also an invariant subspace.*

Lemma 12.2. We have $AW \subseteq W$, where $A = \varphi(X)$ for some $X \in G$. If $x \in W^\perp$,

$$\langle x, w \rangle = 0 \tag{12.5}$$

for all $w \in W$. But

$$\langle x, Aw \rangle = 0 \tag{12.6}$$

since $Aw \in W$. But

$$\langle x, Aw \rangle = \langle A^\dagger x, w \rangle = 0 \tag{12.7}$$

if and only if

$$A^\dagger x \in W^\perp \tag{12.8}$$

for every A in our representations. But $A^\dagger = A^{-1} = \varphi(X)^{-1} = \varphi(X^{-1})$, so we are done. If $\langle x, x \rangle \geq 0$ for all x, then $V = W \oplus W^\perp$. ☐

Theorem 12.1. We have by our lemma, if our representation is irreducible we're done; otherwise, we write

$$V = W \oplus W^{\perp}. \tag{12.9}$$

We iterate on our summands, until we end up writing

$$V = \bigoplus_{i \in I} V_i \tag{12.10}$$

for our representation as a direct sum of mutually orthogonal irreps. For finite dimensional V, there is only finitely many recursions; **note** this is the first time finite dimensionality is used. There is an analogous notion for infinite dimensionality, but we need to use the direct integral. □

Now are all representations unitary or orthogonal? Or...equivalent to unitary or orthogonal representations? The answer is "No" for the simple reason they are not completely reducible!

Example 12.3. Consider the one-dimensional Lie algebra $\mathbb{R}$. It has one generator e, and its commutator

$$[e, e] = 0. \tag{12.11}$$

Now to construct a representation of this, how to do this? Send this generator to somewhere satisfying the commutation relations...but they are *always* satisfied. So representations of this Lie algebra are merely all linear operators.

We should classify these representations. Two representations are "involutive" if they are, as matrices, similar

$$E' = A^{-1}EA. \tag{12.12}$$

Every matrix may be written in Jordan normal form, and that's it! We've classified everything. We get a series of eigenspaces embedded in each other, but no direct sum of invariant subspaces for the simple reason every invariant subspace should have an eigenvector. So this representation *is not* equivalent to a unitary or orthogonal representation, and it is not completely irreducible. Only when all the Jordan cells are one dimensional, then the representation is completely reducible.

The corresponding Lie group to $\mathbb{R}$ is $\mathbb{R}^{\times}_{>0}$, but there is one more group which corresponds to the same Lie algebra viz. $\mathrm{U}(1)$. Geometrically $\mathrm{U}(1)$ is a circle, and its elements are of the form $\exp(\mathrm{i}\varphi)\ldots$ there is a whole in this circle, it's not simply connected. So the representations of the Lie algebra $\mathbb{R}$ does not coincide with representations of the Lie group $\mathrm{U}(1)$. They are related but do not coincide. This group has representations that are equivalent to unitary representations, moreover are completely reducible; if $z = \exp(\mathrm{i}\varphi)$, then the representations are given by

$$z \mapsto z^n. \tag{12.13}$$

How to prove all this stuff?

First how to prove all representations of this group are equivalent to unitary ones. Lets consider any representation

$$\varphi\colon \mathrm{U}(1) \to V \tag{12.14}$$

and introduce an inner product on V, i.e. $\langle\cdot,\cdot\rangle$. There are two possibilities: inner product is invariant (we're happy, the representation is unitary), or it's not invariant (it's not equivalent to a unitary representation). Then we can make the inner product invariant, namely by introducing a new inner product

$$\langle\langle v_2, v_1\rangle\rangle = \int \langle\varphi(g)v_1, \varphi(g)v_2\rangle dg \tag{12.15}$$

which is already invariant with respect to the group G. Namely

$$\langle\!\langle \varphi(h)v_2, \varphi(h)v_1 \rangle\!\rangle = \int \langle \varphi(g)\varphi(h)v_1, \varphi(g)\varphi(h)v_2 \rangle dg \tag{12.16a}$$

$$= \int \langle \varphi(gh)v_2, \varphi(gh)v_1 \rangle dg \quad \text{g'=gh, dg'=dg} \tag{12.16b}$$

$$= \int \langle \varphi(g')v_2, \varphi(g')v_1 \rangle dg' \tag{12.16c}$$

$$= \langle\!\langle v_2, v_1 \rangle\!\rangle \tag{12.16d}$$

For us $g = \exp(i\varphi)$, $gh = \exp(\mathrm{i}\varphi)\exp(\mathrm{i}\theta) = \exp\big(\mathrm{i}(\varphi+\theta)\big)$ is merely a shift. So we need a measure on the circle which is invariant with respect to translations; we have it. We proved more, really, we almost proved the theorem:

Theorem 12.4. *Every complex representation of a compact group is equivalent to a unitary representation.*

The proof is exactly the same, the only thing missing is a lemma.

Lemma 12.5. *Every compact group can be equipped with an invariant measure (measure invariant with respect to shifts).*

Really a measure invariant under right (or left) shifts always exists, but may not be finite for noncompact groups. To fix the measure, fix it in the tangent spaces, but we know the tangent spaces for Lie groups! For compact groups, left invariant measure coincides with right measure, and are finite.

Lecture 13

Today we will talk about compact Lie groups. First a few remarks about characters. Suppose we have

$$\varphi\colon G \to \mathrm{GL}(n) \tag{13.1}$$

be a group morphism. We have

$$\mathrm{Tr}\,\big(\varphi(g)\big) = \chi_\varphi(g) \tag{13.2}$$

be the character at the point g. The character is an invariant of a representation. If φ, φ' are two isomorphic representations, so

$$\varphi'(g) = A\varphi(g)A' \tag{13.3}$$

then

$$\chi_\varphi(g) = \chi_{\varphi'}(g) \tag{13.4}$$

for every $g \in G$. The character is a class function, it doesn't depend on conjugation

$$\chi_\varphi(aga^{-1}) = \chi_\varphi(g). \tag{13.5}$$

This is basically everything we need. In general, for compact groups G, the characters determine everything.

Theorem 13.1. *If* $\rho\colon G \to \mathrm{GL}(n)$, $\rho'\colon G \to \mathrm{GL}(n)$ *are two representations such that* $\chi_\rho(g) = \chi_{\rho'}(g)$ *for each* $g \in G$, *then* $\rho \cong \rho'$.

For a compact group, we have the invariant volume be

$$\int 1 \cdot dv = 1 \tag{13.6}$$

which can be normalized. For a finite group, this integral averaging a function is merely a sum

$$\overline{f} = \int f(v)dv \quad \text{for compact groups} \tag{13.7a}$$

$$= \frac{1}{N}\sum_{g\in G} f(g) \quad \text{for finite groups} \tag{13.7b}$$

We have

$$\langle f, f_1\rangle = \int f^*(v)f_1(v)dv = \overline{f^* f_1} \tag{13.8}$$

we can compute the norm of characters

$$\|\chi\| = \sqrt{\langle\chi,\chi\rangle} = \begin{cases} 1 & \text{if the rep is irreducible} \\ 0 & \text{otherwise} \end{cases} \tag{13.9}$$

So we have

$$\langle\chi_i, \chi_j\rangle = \delta_{ij} \tag{13.10}$$

so we have orthonormal characters from an orthonormal basis. Where? For class functions!

Note class functions only depend on conjugacy classes. For example, consider $G = \mathrm{U}(n)$, we can diagonalize any unitary matrix by means of unitary transformations. We have then diagonal matrices consists of

$$D = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \tag{13.11}$$

where $z_k = \exp(i\varphi_k)$, so $T \cong \mathrm{U}(1)^n$ the n-torus.

What's relevant is that T forms a commutative subgroup of $\mathrm{U}(n)$. This is the maximal commutative subgroup for $\mathrm{U}(n)$, or the maximal torus. Since every element is conjugate to this stuff, if we want to know the character, we only need to be concerned about the character on the torus. But can the character be an arbitrary function here? No, it can't. So the characters $\chi(z_1, \ldots, z_n)$ is a symmetric function on T, so it is invariant under any permutation.

Let G be a connected, compact Lie group. Let T be the maximal torus, i.e. the maximal Abelian subgroup. It will always be a Torus, always a product of $\mathrm{U}(1)$. Then every element of $g \in G$ is conjugate (conjugated in G) to an element of T. Not every function on the torus is a character. We should consider elements of G that form the normalizer of T, i.e.

$$N(T) = \{x \in G \mid xT = Tx\}. \tag{13.12}$$

We know $T \subseteq N(T)$ trivially, so to examine the nontrivial part of the normalizer we should consider the quotient

$$N(T)/T = W \tag{13.13}$$

called the "**Weyl Group**". The Weyl group acts on the Torus. It is obvious that characters should be invariant with respect to this group.

Now to Lie algebras. We will work with complex Lie algebras. We will take G to be a compact Lie group, $\mathbb{C}\,\mathrm{Lie}(G)$ the complexification of its Lie algebra. The Lie algebras obtained in this way are "**Reductive Lie Algebras**". They have an invariant inner product. Every representation of a compact group is equivalent to a unitary representation (if complex), or an orthogonal representation (if real). Lets take a Lie group G, lets take its (real) Lie algebra

$$\mathscr{G} = \mathrm{Lie}(G) = T_e G \tag{13.14}$$

which is the tangent space to the group at $e \in G$. The group G acts on T_eG in the obvious way. For some $x \in G$, then

$$gxg^{-1} \in G \tag{13.15}$$

is an inner automorphism. It maps $e \mapsto e$. So every curve starting at the identity goes to a curve also starting at the identity, so we can define an action of G on tangent vectors. This is called the "**Adjoint Representation of** G", this is probably the most important representation. We had a notion of the adjoint representation for Lie algebras, and really it's related to adjoint representations of Lie groups. We should consider

$$g = 1 + \gamma \tag{13.16}$$

where γ is "small." Then the adjoint group representation is

$$(1+\gamma)x(1+\gamma)^{-1} = x + \gamma x - x\gamma + \cdots \tag{13.17a}$$

$$= x + [\gamma, x] + \cdots \tag{13.17b}$$

We are concluding that the adjoint representation of the group corresponds to the adjoint representation of the algebra. Adjoint representation for compact group is equivalent to an orthogonal representation. There exists an inner product $\langle x, y\rangle$ in $\mathscr{G}$ which is invariant under

$$\langle \mathrm{Ad}_g x, \mathrm{Ad}_g y\rangle = \langle x, y\rangle, \tag{13.18}$$

we see

$$\mathrm{Ad}_{1+\gamma+\cdots} x = x + [\gamma, x] + \cdots = x + \mathrm{ad}_\gamma x + \cdots . \tag{13.19}$$

(The convention is adjoint representation for the Lie group is written as "Ad" but for the Lie algebra is "ad".)

We get

$$\langle [\gamma, x], y\rangle + \langle x, [\gamma, y]\rangle = 0. \tag{13.20}$$

If G is compact, then $\mathrm{Lie}(G)$ is equipped with a nondegenerate positive invariant product (which is precisely the Eq (13.20) condition). We can extend this to the complexified Lie algebra $\mathbb{C}\,\mathrm{Lie}(G)$ but it is a nondegenerate invariant inner product. This is a general result.

Examples. Consider $\mathfrak{gl}(n)$, we have an invariant inner product $\langle x, y\rangle = \mathrm{Tr}(xy)$.

Lecture 14

We have defined a reductive Lie algebra as the complexification of the Lie algebra for a compact group. The Cartan subalgebra (usually denoted $\mathfrak{h}$) corresponds to the maximal torus in the group $T \subseteq G$ is the maximal, Abelian, connected subgroup. Diagrammatically

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\exp"] & G \\ \mathfrak{h} \arrow[u] \arrow[r, "\exp"] & T \arrow[u] \end{tikzcd} \tag{14.1}$$

It is obvious that $T = \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$.

We can see $\mathbb{R}$ is the Lie algebra for $\mathbb{R}^\times_{>0}$ positive reals equipped with multiplication. Note that

$$\mathbb{R}^\times_{>0} \cong \mathbb{R}^+ \tag{14.2}$$

is an isomorphism for $\mathbb{R}^+$ the reals equipped with addition. This is not a unique Lie algebra, we should factorize with respect to a discrete subgroup. To get a compact group, $\mathrm{U}(1)^n$ is the only choice.

Let G be a compact group. Let $\mathscr{G} = \mathrm{Lie}(G)$, since $T \subseteq G$, we see $\mathbb{G}\,\mathrm{Lie}(T) = \mathscr{H}$ is the Cartan subalgebra. This is its definition.

It is tempting to define the Cartan subalgebra as the maximal Abelian subalgebra of $\mathscr{G}$, but this is wrong. It is clear the Cartan subalgebra is really important, since the character restricted to the maximal Torus permits us to reconstruct all information of the representation. We will consider a representation of the Lie algebra

$$\varphi\colon \mathscr{G} \to \mathfrak{gl}(V), \tag{14.3}$$

and we will assume the representation comes from a compact Lie group

$$\Phi\colon G \to \mathrm{GL}(V). \tag{14.4}$$

This is true for compact groups, and the corresponding representation of the corresponding Lie algebra. Consider a representation for an Abelian Lie algebra, it is not necessarily semisimple; not every representation of Abelian Lie algebras comes from U(1), it comes from $\mathbb{R}^+$. We have for any z, a representation of U(1) is $z \mapsto z^n$, $n \in \mathbb{Z}$. We can restrict the representation Φ to the maximal torus, and it is completely reducible. The same may be said about

$$\varphi\colon \mathscr{H} \to \mathfrak{gl}(V) \tag{14.5}$$

the representation of the Cartan subalgebra, it is completely reducible. What are the irreps of Ablian Lie algebras? It is very easy to see irreps of Abelian Lie Algebras are one-dimensional; it may be proven in precisely one million different ways.

Now lets introduce the notion of a "**Weight Vector**" of a representation φ, it is a vector $x \in V$ such that

$$\varphi(h)x = \lambda(h)x \tag{14.6}$$

it is an eigenvector for the Cartan subalgebra, we call $\lambda(h)$ the weight (it is a linear functional for the vector space underlying $\mathscr{H}$). That is to say $\lambda \in \mathscr{H}^*$.

Proposition 14.1. *There exists a basis of V consisting of weight vectors.*

If we know all the weights, we may compute the character of the representation. Suppose we write

$$\mathscr{H} = \left\{ \sum_k \xi^k h_k \right\}, \tag{14.7}$$

then any linear function $h_k \mapsto \alpha_k$ acts as

$$\sum \xi^k h_k \mapsto \sum \xi^k \alpha_k. \tag{14.8}$$

There is an exponential map

$$A \mapsto \exp(A) \tag{14.9}$$

that maps $A \in \mathscr{G}$ to $\exp(A) \in G$. If we have a one-parameter family in the Lie algebra tA, then we have a one-parameter family in the subgroup $\exp(tA)$.

Suppose the coordinates of the Lie group is $\{z^1, \dots, z^k\}$, then the basis for the Lie algebra would be

$$\xi^j = \log(z^j), \tag{14.10}$$

to recover the group you should exponentiate. We get the maximal torus, and then we can compute the trace.

We have the adjoint representation

$$\alpha_i(h)E_i = [h, E_i] \tag{14.11}$$

where E_i is the weight vector for $\alpha_i(h)$. Nonzero weights for the adjoint representation are called "**Roots**" and weight vectors for the adjoint representation are called "**Root Vectors**".We can give a different definition for the Cartan subalgebra:

Definition 14.2. The "**Cartan Subalgebra**" is a maximal Abelian subalgebra that is completely reducible in (all representations, but in particular) the Adjoint representation.

Consider $\mathrm{U}(n)$, and $\mathfrak{gl}(n) = \mathbb{C}\,\mathrm{Lie}\big(\mathrm{U}(n)\big)$, the maximal torus here consists of all diagonal matrix

$$H = \begin{bmatrix} \exp(i\varphi_1) & & \\ & \ddots & \\ & & \exp(i\varphi_n) \end{bmatrix} \tag{14.12}$$

is the maximal Torus. Here the basis of the weight vectors may be taken to be the standard basis, the representation is

$$\rho\colon \mathrm{U}(n) \to \mathrm{Aut}(\mathbb{C}^n). \tag{14.13}$$

So we have n weights which are functionals on $\mathscr{H}$.

Lecture 15

Today we will start with some general examples. First some simple constructions of representations which are quite general. We will consider a representation

$$\varphi\colon G \to \mathrm{GL}(V). \tag{15.1}$$

If we have one representation, we may consider many others related to it. We may use any natural construction, any functor, will give you something. For example we may consider the dual space

$$V^* = \{v\colon V \to \mathbb{F}\} \tag{15.2}$$

This is a contravariant functor. Remember $\varphi(g) \in \mathrm{GL}_n$ if V is finite dimensional; duality is related by the transpose $\varphi(g)^T$, we may ask ourselves if it is a representation?

We see immediately *no it isn't!* Because we may say the transpose

$$\big(\varphi(g)\varphi(h)\big)^T \neq \varphi(g)^T\varphi(h)^T \tag{15.3}$$

therefore we do not have a representation. It is simple to cure this, we take

$$\big(\varphi(g)^T\big)^{-1} = \big(\varphi(g)^{-1}\big)^T \tag{15.4}$$

which is the "**Dual Representation**", i.e. the representation on the dual space. We demand then that

$$\big(\varphi(g)^{-1}\big)^T = \varphi(g^{-1})^T \tag{15.5}$$

and then the character of the dual representation is

$$\chi_{\mathrm{dual}}(g) = \mathrm{Tr}\left(\varphi(g^{-1})^T\right) = \mathrm{Tr}\left(\varphi(g^{-1})\right) \tag{15.6}$$

so

$$\chi_{\mathrm{dual}}(g) = \chi(g^{-1}). \tag{15.7}$$

There is another operation that is important, namely taking the tensor product. Suppose V has basis $(v_1, \ldots, v_m)$ and W has basis $(w_1, \ldots, w_n)$, then $V \otimes W$ has basis

$$(v_1 \otimes w_1, \ldots, v_1 \otimes w_n, \ldots, v_m \otimes w_1, \ldots, v_m \otimes w_n) \tag{15.8}$$

and a vector in $V \otimes W$ is of the form

$$z = z^{ij} v_i \otimes w_j = \sum_{i=1}^{m}\sum_{j=1}^{n} z^{ij} v_i \otimes w_j \tag{15.9}$$

where we use Einstein summation conventions where indices upstairs are summed over the indices downstairs. The dependence on the choice of basis is fictitious. If we have a change of coordinates in V have the components transform by

$$x^i \mapsto \widetilde{x}^i = a^i{}_j x^j \tag{15.10}$$

and we consider some arbitrary vector

$$v = x^j v_j \quad \text{in } V \tag{15.11}$$

and if we do likewise consider a change of coordinates in W by

$$y^l \mapsto \widetilde{y}^l = b^l{}_k y^k \tag{15.12}$$

where we implicitly sum over k, then we have

$$w = y^k w_k \tag{15.13}$$

describe an arbitrary element. What is the transformation in the coordinates of the tensor product? It is very simple. We obtain them by

$$\widetilde{z}^{il} = a^i{}_j b^l{}_k z^{jk} \tag{15.14}$$

so if

$$a = a(g) \quad \text{and} \quad b = b(g) \tag{15.15}$$

for some group element $g \in G$, then

$$a \otimes b = (a \otimes b)(g) \tag{15.16}$$

depends on g too. This gives rise to a tensor product of representations, which is a representation by functoriality.

If we have

$$\varphi\colon G \to \mathrm{GL}(V) \tag{15.17}$$

and

$$\psi\colon G \to \mathrm{GL}(W) \tag{15.18}$$

be representations, then we have the tensor product of representations as

$$(\varphi \otimes \psi)_g (v \otimes w) = \big(\varphi(g)v\big) \otimes \big(\psi(g)w\big). \tag{15.19}$$

What about vectors that are not basis vectors? We can use distributivity, if

$$v = x^i v_i \quad \text{and} \quad w = y^j w_j \tag{15.20}$$

then by definition

$$v \otimes w = x^i y^j (v_i \otimes w_j). \tag{15.21}$$

In other words, if V and W are G-modules, then $V \otimes W$ is a G-module. We may iterate for as many G-modules tensored together as possible. We may recall

$$W \otimes V \cong V \otimes W \tag{15.22}$$

naturally.

We may construct more representations via some gadget called an "**Intertwiner**" which is a morphism of G-modules (i.e. preserves commutator, group operation). Sometimes we use shorthand

$$\varphi_g v = gv \tag{15.23}$$

for the group action. Then a morphism is

$$\alpha(gv) = g(\alpha v). \tag{15.24}$$

If an intertwiner is invertible, we have an equivalence of representations.

If we consider $V \otimes V$, then we have a natural intertwiner namely

$$v \otimes w \mapsto w \otimes v. \tag{15.25}$$

This is a natural isomorphism of representations, so nothing changes. If we have $V^{\otimes n}$, then we have the symmetric group S_n consisting of intertwiners. So a permutation is an intertwiner. We may consider vectors x such that

$$\alpha(x) = x \tag{15.26}$$

is invariant under such permutations; they form a subspace. More precisely

$$x = z^{ij} v_i \otimes v_j \tag{15.27}$$

and the coefficients are tensors, what we do is consider symmetric tensors which are fixed points of the intertwiner which implies the coefficients obey

$$z^{ij} = z^{ji} \tag{15.28}$$

for all i, j.

We may consider the subspace obeying

$$\alpha(x) = -x \tag{15.29}$$

then the coefficients are

$$z^{ij} = -z^{ji} \tag{15.30}$$

antisymmetric tensors. The symmetric one is denoted by $\mathrm{Sym}^2 V$ and the antisymmetric by $\bigwedge^2 V$. We generalize to the tensor product of n spaces

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \tag{15.31}$$

then we get guys with n indices $z^{i_1 \cdots i_n}$. We can apply various demands of indices. We use the notation

$$z^{[ij]} = \frac{1}{2!}(z^{ij} - z^{ji}) \tag{15.32}$$

and

$$z^{(ij)} = \frac{1}{2!}(z^{ij} + z^{ji}). \tag{15.33}$$

We may also take tensor products including the dual space and the vector space, for example

$$V^{\otimes m} \otimes (V^*)^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{n \text{ times}} \tag{15.34}$$

which results in guys

$$a^{i_1 \cdots i_m}_{j_1 \cdots j_n} \tag{15.35}$$

with mixed indices.

We have been talking about groups, but we may consider analogous gadgetry for the Lie algebra. If we have

$$(\varphi_g \otimes \psi_g)(u \otimes v) = \big(\varphi(g)(u)\big) \otimes \big(\psi(g)(v)\big) \tag{15.36}$$

for the Lie group, and we take

$$g = 1 + \gamma \tag{15.37}$$

where γ is "small." We obtain from the Lie group representation $\varphi_{1+\gamma}$ a Lie algebra representation $\widetilde{\varphi}_\gamma$, but how does the representation behave under the tensor product of Lie algebra representations? We have

$$(\widetilde{\varphi}_\gamma \otimes \widetilde{\psi}_\gamma)(u \otimes v) = \big(\widetilde{\varphi}_\gamma(u)\big) \otimes v + u \otimes \big(\widetilde{\psi}_\gamma(v)\big). \tag{15.38}$$

Why? Well observe that

$$(\varphi_{1+\gamma} \otimes \psi_{1+\gamma}) = (\mathbf{1} + \widetilde{\varphi}_\gamma) \otimes (\mathbf{1} + \widetilde{\psi}_\gamma) \tag{15.39a}$$

$$= \mathbf{1} + \underbrace{\widetilde{\varphi}_\gamma \otimes \mathbf{1} + \mathbf{1} \otimes \widetilde{\psi}_\gamma}_{\text{Lie Algebra rep.}} + \mathcal{O}(\varepsilon^2) \tag{15.39b}$$

where ε is the "magnitude" of γ, which is negligibly small in comparison to 1, and $\mathbf{1}$ is the identity operator.

Consider the simplest example $\mathrm{U}(n)$ and its fundamental representation $\mathrm{GL}(\mathbb{C}^n)$. The maximal torus is

$$T = \left\{ \begin{bmatrix} e^{i\varphi_1} & & \\ & \ddots & \\ & & e^{i\varphi_n} \end{bmatrix} \right\}, \tag{15.40}$$

this corresponds to the Cartan subalgebra consisting of diagonal matrices. The weight vectors are the standard basis

$$v_i = e_i \tag{15.41}$$

which is 1 for the i^{th} component, 0 for all others. Now it is clear what are the weights, merely the corresponding components. We may consider the tensor product of two such representations. The basis is by our definition $v_i \otimes v_j$, and it is very easy to understand

$$v_i \otimes v_j \mapsto (\varphi_i + \varphi_j)(v_i \otimes v_j) \tag{15.42}$$

is a weight vector. There is a rule for the characters

$$\chi_{\varphi \otimes \psi} = \chi_\varphi \chi_\psi \tag{15.43}$$

using the characters of the "component" representations. We also have a representation of symmetric tensors with the basis

$$v_i \otimes v_j + v_j \otimes v_i \tag{15.44}$$

and for a representation of antisymmetric tensors

$$v_i \otimes v_j - v_j \otimes v_i \tag{15.45}$$

up to some overall factor of 1/2. Both bases have almost the same weights $\varphi_i + \varphi_j$, but for the antisymmetric tensors we require $i \neq j$.

Lecture 16

If we have

$$\varphi\colon \mathscr{G} \to \mathfrak{gl}(V) \tag{16.1}$$

a representation, and $\mathscr{H} \subset \mathscr{G}$ is the Cartan subalgebra, then we recall a weight vector v is an eigenvector

$$\varphi(h)v = \lambda(h)v \tag{16.2}$$

for every $h \in \mathscr{H}$. We have an adjoint representation

$$\mathrm{ad}\colon \mathscr{G} \to \mathfrak{gl}(\mathscr{G}) \tag{16.3}$$

where

$$(\mathrm{ad}\, x)v = [x, v] \tag{16.4}$$

and the weight vectors for this representation are called root vectors, and the weights are called roots. To define roots and root vectors for $\mathscr{G}$ we are solving

$$[h, v] = \alpha(h)v. \tag{16.5}$$

We can construct new root vectors from a given root vector by applying e_i, f_j to it. We have

$$[h, e] = \alpha(h)e \iff he = eh + \alpha(h)e \tag{16.6a}$$

$$\iff h(ev) = \big(eh + \alpha(h)e\big)v \tag{16.6b}$$

but

$$hv = \lambda(h)v \qquad \Longrightarrow \qquad h(ev) = \big(\lambda(h) + \alpha(h)\big)ev. \tag{16.7}$$

This is either zero or another distinct eigenvector.

Proposition 16.1. *If v is a weight vector with weight λ and e is a root vector with root α, then ev is a weight vector with weight $\lambda + \alpha$ unless $ev = 0$.*

Now we will introduce a definition. Well several definitions. First we introduce a notion of a Cartan Matrix which is presented differently in different papers.

Definition 16.2. A "**Cartan Matrix**" is a matrix $A = [a_{ij}]$ such that

1. $a_{ii} = 2$ are the diagonal components;
2. $a_{ij} \in \mathbb{Z}$ for any i, j;
3. $a_{ij} \leq 0$ for off-diagonal components;
4. although not necessarily symmetric, if $a_{ij} = 0$ then $a_{ji} = 0$;
5. it should be symmetrizable, i.e. we have a diagonal matrix B such that $AB = D$ is also diagonal.

Remark 16.3. Most of the time we will work with A nondegenerate, i.e.

$$\det(A) \neq 0 \tag{16.8}$$

But this is not a necessary condition, so we do not make it part of the definition.

For every classical Lie Algebra, the matrix a_{ij} is nondegenerate

$$\det(a_{ij}) \neq 0. \tag{16.9}$$

We have explicitly computed this, so we should look at our answers and nothing more.

The only thing that needs discussion is "Why is A symmetrizable?" We know there exists a nondegenerate invariant inner product on classical Lie algebras. The adjoint representation is orthogonal with respect to this inner product, i.e.

$$\big\langle [h, x], y \big\rangle + \big\langle x, [h, y] \big\rangle = 0 \tag{16.10}$$

where $h, x, y \in \mathscr{G}$. This could be viewed as a consequence of compact Lie groups having unitary representations giving invariant inner product.

We can introduce the Killing form as

$$\langle x, y\rangle = \mathrm{Tr}\left(\mathrm{ad}_x\, \mathrm{ad}_y\right) \tag{16.11}$$

which is an invariant inner product. We may introduce an invariant inner product on the group

$$\langle Ux, Uy\rangle = \langle x, y\rangle \tag{16.12}$$

where $U \in G$, but when $U = \mathbf{1} + u\varepsilon$ where ε is "small", then we get

$$\langle \varphi(u)x, y\rangle + \langle x, \varphi(u)y\rangle = 0 \tag{16.13}$$

where φ is a morphism. But as a representation we have

$$\Big\langle [u,x], y\Big\rangle + \Big\langle x, [u,y]\Big\rangle = 0. \tag{16.14}$$

If we let $u = e_i$, $x = f_i$, $y = h_j$ we get

$$\Big\langle [u,x], y\Big\rangle + \Big\langle x, [u,y]\Big\rangle = \Big\langle [e_i, f_i], h_j\Big\rangle + \Big\langle f_i, [e_i, h_j]\Big\rangle \tag{16.15a}$$

$$= \langle h_i, h_j\rangle + \langle f_i, -a_{ji}e_i\rangle \tag{16.15b}$$

This holds if and only if

$$\begin{array}{ccc} \langle h_i, h_j\rangle & = & a_{ji}\langle f_i, e_i\rangle \\ \| & & \| \\ \langle h_j, h_i\rangle & = & a_{ij}\langle f_j, e_i\rangle \end{array} \tag{16.16}$$

Using the inner product on the group, we may construct the matrix $B = \mathrm{diag}\langle e_i, f_i\rangle$ which implies AB is symmetric.

For every Cartan matrix, we may construct a Lie algebra called a Kac–Moody algebra. Really simple Lie Algebras are Kac–Moody algebras with additional condition that the Cartan matrix is positive definite. We will now describe all irreducible representations of classical Lie Algebras; this is true for all Lie Algebras related to compact groups, and reductive Lie Algebras.

In reality we may say for every compact group, the corresponding Lie algebras have precisely the right generators. Moreover, we may classify algebras of compact groups. This gives us a general theorem for representations of compact Lie algebras. We would like to explain the notion of a highest weight vector in this situation. Namely the highest weight vector v is such that

$$\varphi(e_i)v = 0 \tag{16.17}$$

for all $e_i \in \mathscr{G}$. Of course this means that

$$\varphi(h)v = \lambda(h)v \tag{16.18}$$

for all $h \in \mathscr{H}$, then this λ is called the highest weight.

First of all, what is $\lambda(-)$? It is a linear functional $\lambda \in \mathscr{H}^*$, i.e.

$$\lambda\colon \mathscr{H} \to \mathbb{F} \tag{16.19}$$

it is a linear functional acting on the Cartan subalgebra. Then:

1. Irreducible representations contain not more than one highest weight vector, up to a constant factor;

2. Finite dimensional irreducible representation $\iff$ finite dimensional representation with one highest weight vector;

3. For every $\lambda \in \mathscr{H}^*$ one can construct a unique irreducible representation with highest weight λ but this representation can be infinite dimensional;

4. This representation is finite dimensional if and only if $\lambda(h_i)$ is a non-negative integer.

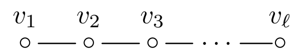
(4) is the most important point!

This gives us a complete description of finite dimensional irreducible representations.

16.1 Dynkin Diagrams

It is a very convenient way to depict Cartan matrices. Namely first of all the dimension of the Cartan algebra is called the "**Rank of the Lie Algebra**". We have A_ℓ, B_ℓ, C_ℓ, D_ℓ all of rank ℓ.

The Dynkin diagram for A_ℓ we draw ℓ vertices and we draw edges. We have the number of edges connecting vertices v_i to v_j be given by the formula using the Cartan matrix



$$n_{ij} = a_{ij}a_{ji}. \tag{16.20}$$

Suppose we know B, its diagonal so we only need to keep track of one index really. Since we suppose we know B, then

$$a_{ij}b_j = a_{ji}b_i \tag{16.21}$$

implies

$$b_i = \frac{a_{ij}b_j}{a_{ji}} \tag{16.22}$$

we get

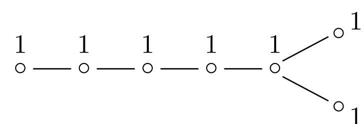
$$n_i b_i = {a_{ij}}^2 b_j, \tag{16.23}$$

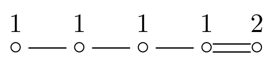
or equivalently

$${a_{ij}}^2 = \frac{n_i b_i}{b_j}. \tag{16.24}$$

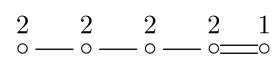
What is the conclusion? If we know $[n_i]$ and $[b_i]$, we can compute $[a_{ij}]$.

Let us write down the Dynkin diagrams for the classical Lie groups we have considered. For D_n we have the diagram drawn on the right for the case when $n = 7$ (observe there are 7 vertices). The Cartan matrix for D_n is symmetric. One can observe this by considering the adjacency matrix for the graph.

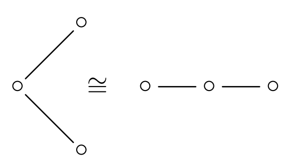




For C_n we see the Cartan matrix is not symmetric, but we can symmetrize it. We find that $a_{n-1,n}a_{n,n-1} = 2$. For B_n we see the Dynkin diagram is "the same" as for C_n but with different labels for the vertices.



Almost all of these groups are simple and almost all of them are not isomorphic. But almost all. For example, in D_2 we have two disconnected vertices for the Dynkin diagram. So D_2 is not simple, it is the direct product of SU(2) at the level of Lie Algebras, and *almost* the direct product at the level of Lie groups. So we may examine the Dynkin diagram for D_3 to find:



So we see this is the same Dynkin diagram as for A_3 which implies at the level of Lie Algebras

$$\mathrm{SU}(4) \cong \mathrm{SO}(6) \tag{16.25}$$

but only at the level of Lie Algebras. We similarly have $\mathsf{B}_2 \cong \mathsf{C}_2$ by inspection of the Dynkin diagrams, but again it is an isomorphism at the level of Lie Algebras.

Box 16.1 Dynkin Diagrams

The problem we are facing is really two-fold: (a) given a Dynkin diagram obtain the Cartan matrix, and (b) given the Cartan matrix obtain the Dynkin diagram. This box is really based off of §4.7 of Kac's book *Infinite Dimensional Lie Algebras*. No secrets among friends: Kac provides the method of, given a Cartan matrix, producing the Dynkin diagram. We review this, and provide the algorithm going in the opposite direction. We also consider examples. Throughout $A = [a_{ij}]$ is the Cartan matrix.

Given Cartan Matrix Obtain Dynkin Diagram. The basic idea is that we will have an $n \times n$ matrix A. The Dynkin diagram is a graph that will have n vertices, which are labeled by integers $i = 1, \dots, n$. If

$$a_{ij}a_{ji} \leq 4 \quad \text{and} \quad |a_{ij}| \geq |a_{ji}| \tag{16.26}$$

then vertices i and j are connected by $|a_{ij}|$ lines; moreover if $|a_{ij}| > 1$, then these lines are equipped with an arrow pointing towards vertex j.

Why do we need an arrow? The idea is that the Cartan matrix is not symmetric, but has a weaker condition that $a_{ij} \neq 0$ implies $a_{ji} \neq 0$. Since we know the product by the number of lines, we know the values by considering which direction the arrow points.

Given a Dynkin Diagram Obtain Cartan Matrix. This occurs more often in practice (at least, for physicists). What can we know immediately from the properties of a Cartan matrix? Well, we know

$$a_{ii} = 2 \tag{16.27}$$

for all i. We know that the number of vertices n gives information about the number of rows, and the number of columns, of the Cartan matrix — i.e. A is an $n \times n$ matrix. We also know if $i \neq j$ that

$$a_{ij} \leq 0 \quad \text{and} \quad a_{ij} \in \mathbb{Z}. \tag{16.28}$$

The rest we need to find from the diagram.

If vertex i and j are connected by k lines, then $a_{ij} < 0$. What values can this component be? Well, if $k = 1$, then

$$a_{ij} = a_{ji} = -1 \tag{16.29}$$

since there is no arrow, it must be -1. If there are multiple lines, we have an arrow to indicate which entry

Remark 16.4. Note that in these examples, the vertices are labeled by *indices* to keep track of which we are discussing. Usually, the labels of a vertex are the relative (squared) lengths of the fundamental roots as Gilmore describes it [see Robert Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* Dover Publications (2002) Ch 8 §III.2 pp 306 *et seq.*].

N.B. the method we have described are used to deduce a *generalized* Cartan matrix from a Dynkin diagram. So if we restrict focus to Dynkin diagrams corresponding to "strict" Cartan matrices, we recover precisely the same information. But we can do more! We can consider "closed loops" in our approach! The only requirement we have for our considerations is that there are less than 4 edges connecting any pair of vertices.

Example 16.5. Consider the Dynkin diagram given by

$$\overset{1}{\circ} - \overset{2}{\circ} \Rightarrow \overset{3}{\circ} - \overset{4}{\circ}$$

We see that there are 4 vertices, so immediately we know that the Cartan matrix is 4×4 and we can write:

$$A = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}. \tag{16.30}$$

We also see that there is one line connecting vertex 1 to vertex 2, so that means we can write

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}. \tag{16.31}$$

We then observe that there are no other edges connected to 1, so

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & & \\ 0 & & 2 & \\ 0 & & & 2 \end{bmatrix}. \tag{16.32}$$

Similar reasoning holds for vertex 4, it's connected by a single edge to vertex 3

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & & \\ 0 & & 2 & -1 \\ 0 & & -1 & 2 \end{bmatrix}. \tag{16.33}$$

There are no other edges that connect vertex 4 to any other vertex, so

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & & 0 \\ 0 & & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}. \tag{16.34}$$

We see that there are *two lines* connecting vertex 2 to vertex 3 and there is an arrow. The arrow means that

$$a_{23} \neq a_{32}. \tag{16.35}$$

The arrow points towards 3, so

$$|a_{32}| < |a_{23}|. \tag{16.36}$$

Then we use the fact that there are two edges means that

$$|a_{23}| = 2 \tag{16.37}$$

This is sufficient information to conclude

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}. \tag{16.38}$$

Thus we conclude our example.

Example 16.6. Consider the Dynkin diagram given by

$$\overset{1}{\circ} \Rrightarrow \overset{2}{\circ}$$

We see that there are 2 vertices, so immediately we know that the Cartan matrix is 2×2 and we can write:

$$A = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix}. \tag{16.39}$$

The two vertices are connected by 3 edges. There is an arrow pointing from vertex 1 to vertex 2. This implies that

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}. \tag{16.40}$$

Observe that if the arrow pointed the other way, we would merely have the transpose of our matrix.

16.2 Returning to Representations

The representations are described by means of highest weight. We had previously

$$\varphi(e_i)x = 0 \tag{16.41}$$

where x is our highest weight vector, and the highest weight is described by

$$\varphi(h)x = \lambda(h)x \tag{16.42}$$

where $\lambda \in \mathscr{H}^*$ is the highest weight. We should demand $\lambda(h_i) \geq 0$, and $\lambda(h_i) \in \mathbb{Z}$. We will now turn our attention to examples.

We will consider the fundamental representations of $\mathsf{A}_{\ell+1} = \mathfrak{sl}(\ell+1)$. The fundamental representation is the representation by $(1+\ell) \times (1+\ell)$ matrices. We found

$$e_i = E_{i,i+1} \tag{16.43}$$

where $E_{i,j}$ has zero components everywhere except at i, j it is 1. The Cartan subalgebra is

$$\mathscr{H} = \left\{ \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \text{ such that } \lambda_1 + \cdots + \lambda_{\ell+1} = 0 \right\} \tag{16.44}$$

What are the weight vectors here? It is quite clear that the weight vectors $u_1, \ldots, u_{\ell+1}$ are the standard basis vectors. Observe

$$hu_i = \lambda_i u_i. \tag{16.45}$$

What is the highest weight vector of this representation? We see that

$$e_i u_j = 0 \tag{16.46}$$

unless $j = i+1$ we have

$$e_i u_{i+1} = u_i \tag{16.47}$$

The highest weight vector is clearly u_1 because the shift goes down and there is no way down. This implies the representation is irreducible as the highest weight vector is unique up to some coefficient. We see that

$$\lambda(h_i) = \delta_{i1} \tag{16.48}$$

also holds.

What about the tensor product of representations. We find the basis to be $u_j \otimes u_k$ and

$$h_i(u_j \otimes u_k) = (h_i u_j) \otimes u_k + u_j \otimes (h_i u_k) \tag{16.49a}$$

$$= (\lambda_j + \lambda_k)(u_j \otimes u_k) \tag{16.49b}$$

We found all the weight vectors... well not really since $u_1 \otimes u_2$ is a weight vector with the same weight as $u_2 \otimes u_1$, so $u_1 \otimes u_2 \pm u_2 \otimes u_1$ is again a weight vector. We find the highest weight vector to be $u_1 \otimes u_1$, but we see that

$$e_1(u_1 \otimes u_2) = u_1 \otimes u_1 \tag{16.50}$$

and

$$e_1(u_2 \otimes u_1) = u_1 \otimes u_1 \tag{16.51}$$

so it follows that $u_1 \otimes u_2 - u_2 \otimes u_1$ is again a highest weight vector... so we have 2 distinct highest weight vectors! This cannot be an irreducible representation. This we know, we may consider the symmetric and antisymmetric parts of the representation.

Lecture 17

Recall we start with a compact Lie group G, and we are interested in complex representations of this group. If the group is simply connected, this is the same as representations of its Lie Algebra

$$\mathbb{C}\,\mathrm{Lie}(G) = \mathscr{G} \tag{17.1}$$

which is called "**Reductive**" if it is obtained in this way. If we have a compact group, every representation is reducible (i.e. it can be written as the direct sum of irreducible representations). Moreover if we consider the adjoint representation of this reductive Lie Algebra, then we may consider it as a matrix Lie Algebra... well not entirely. The adjoint action is

$$\mathrm{ad}_a\, x = [a, x] \tag{17.2}$$

but the center is mapped to zero. Well, the center's not interesting, so we factorize the reductive algebra by its center, and we obtain a semisimple Lie Algebra. The maximal Abelian subgroup is known as the maximal Torus in the group. The corresponding notion for Lie Algebras is the Cartan subalgebra $\mathscr{H}$.

$$\text{A reductive Lie Algebra} = (\text{center}) \oplus (\text{semisimple part}) \tag{17.3}$$

We wish to work with completely reductive representations, so we will talk about semisimple Lie Algebras. We are looking at representations of $\mathscr{H} \subseteq \mathscr{G}$ the Cartan subalgebra, which will be the direct sum of irreducible representations. But they'd be 1-dimensional, since $\mathscr{H}$ is Abelian. So we have a set of eigenvectors

$$\varphi(h)X_\lambda = \lambda(h)X_\lambda \tag{17.4}$$

called "**Weight Vectors**" where we label X with an index λ its "**Weight**". These X_λ form a basis; now we can consider as a representation the adjoint representation. And then simply we repeat the formulas

$$[h, E_i] = \alpha_i(h)E_i \tag{17.5}$$

which are weights and weight vectors for the adjoint representation. Well, if

$$\alpha_i(h) \neq 0 \tag{17.6}$$

then $\alpha_i(h)$ are called the "**Roots**" and the E_i are called the "**Root Vectors**". Loosely we have

$$\text{basis of } \mathscr{G} = \text{root vectors} + \text{basis of } \mathscr{H}. \tag{17.7}$$

Now we will deviate. We would like to give a definition for a simple root, or more precisely a simple system of roots. Recall if we take a root vector and act by means of root vector on the weight vector $\varphi(E_i)x$ we get *either* a weight vector *or* we get zero. By considering the adjoint representation, we have $[E_i, E_j]$ be either a root vector iff

$$\alpha_i + \alpha_j \neq 0 \tag{17.8}$$

or an element of $\mathscr{H}$ iff

$$\alpha_i + \alpha_j = 0, \tag{17.9}$$

or it could be zero. We can try to minimize our work, and find the roots with the property that all other roots are obtained by means of a linear combination of these guys. These roots are called "**Simple Roots**".

How many simple roots would there be? Well, if

$$\ell = \dim(\mathscr{H}) = \mathrm{rank}(\mathscr{G}) \tag{17.10}$$

is the number of simple roots α_1,...,α_ℓ which form a basis of $\mathscr{H}^*$. We also require that every other root has the form

$$\alpha = m_1\alpha_1 + \cdots + m_\ell\alpha_\ell \tag{17.11}$$

where the coefficients are all positive, or all negative, but **not mixed** coefficients. There is a challenge that such things exist, but it does; moreover we have *done* this for all classical groups. If we have a simple system of roots α_1, ..., α_ℓ we can form a good basis.

First of for each root, we have a corresponding root vector denoted by e_1, ..., e_ℓ. Then we have $-\alpha_1$, ..., $-\alpha_\ell$ with corresponding root vectors f_1, ..., f_ℓ. This is not the end of the story. We've listed root vectors of this kind. We may consider commutators of

$$[e_i, f_i] = h_i. \tag{17.12}$$

They are sitting in $\mathscr{H}$ since their weight is

$$\alpha_i - \alpha_i = 0. \tag{17.13}$$

We can take

$$[e_i, f_j] = 0 \tag{17.14}$$

since it has a weight of

$$\alpha_i - \alpha_j \neq 0 \tag{17.15}$$

which cannot appear *by definition* we have **no mixed** coefficients. We have

$$[h, e_j] = \alpha_j(h)e_j \tag{17.16}$$

by construction, since e_j are root vectors. Lets denote

$$[h_i, e_j] = a_{ij}e_j \tag{17.17}$$

where $a_{ij} = \alpha_j(h_i)$. In the same way,

$$[h_i, f_j] = -a_{ij}f_j. \tag{17.18}$$

So we've obtained this stuff, and this matrix a_{ij} is called the "**Cartan Matrix**" of the Lie Algebra. We came to the same conclusion from a different starting point. We have introduced the notion of a simple system of roots, and proven a theorem that such manipulations work.

Recall we are working with classical groups, which are matrix groups. We are finding the weight vectors of the fundamental representation. We will start with

$$\mathfrak{sl}(n+1) = \mathsf{A}_n. \tag{17.19}$$

This is generated by matrices, the Cartan subalgebra consists of diagonal matrices, with φ_i on the diagonal such that

$$\sum_i \varphi_i = 0 \tag{17.20}$$

since the matrices are traceless. We can see that the standard basis are the weight vectors, we will let u_i denote the canonical basis. We see that

$$\begin{bmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_{n+1} \end{bmatrix} u_i = \varphi_i u_i \tag{17.21}$$

so we see that φ_i is a functional on $\mathscr{H}$.

What are the roots? The weights of the adjoint representation, we will skip this part. We need to look at other Lie Algebras. They are defined in a similar way, namely as matrices preserving some Bilinear form. That is

$$(x, y) = x^T \omega y, \tag{17.22}$$

the demand of invariance amounts to

$$(Ax, Ay) = x^T A^T \omega A y = (x, y) \quad \iff \quad A^T \omega A = \omega \tag{17.23}$$

but if we consider

$$A = \mathbf{1} + a \tag{17.24}$$

for “infinitesimal” a we get

$$\begin{aligned} A^T \omega A &= (\mathbf{1} + a)^T \omega (\mathbf{1} + a) && (17.25\text{a}) \\ &= (\omega + a^T \omega)(\mathbf{1} + a) && (17.25\text{b}) \\ &= \omega + (\omega a + a^T \omega) + \underbrace{a^T \omega a}_{\approx 0} && (17.25\text{c}) \end{aligned}$$

So with summation convention we explicitly have

$$a^i_j \omega_{ik} + \omega_{ij} a^i_k = 0, \tag{17.26}$$

or if

$$b_{jk} = \omega_{ji} a^i_k \tag{17.27}$$

then our invariance condition amounts to

$$b_{jk} = \mp b_{kj} \tag{17.28}$$

for orthogonal groups this is antisymmetric, for the symplectic group it is symmetric. We have a one-to-one correspondence between b_{jk} and a^i_j, so the conclusion is:

1. for $\mathfrak{so}(n)$ we have its adjoint representation antisymmetric square of its fundamental representation
2. for $\mathfrak{sp}(n)$ we have its adjoint representation be the symmetric square of its fundamental representation.

That is, the symmetric or antisymmetric parts of $V \otimes V$.

Lecture 18

I was absent due to family reasons.

Lecture 19

Consider $\mathfrak{gl}(n)$ where the Cartan subalgebra is simply the diagonal matrices, which we'll denote by $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ or simply $(\lambda_1, \ldots, \lambda_n)$. By definition the weight vectors are merely the canonical basis, denoted by $\vec{u}_1, \ldots, \vec{u}_n$ with the corresponding weights $\lambda_1, \ldots, \lambda_n$. This is the fundamental representation. Also $-\lambda_1, \ldots, -\lambda_n$ are the weights of the dual fundamental representation, or covector representation.

The adjoint representation consists of matrices (i.e. a tensor with 1 upper and 1 lower index):

$$a^i_j \in V \otimes V^*. \tag{19.1}$$

Consider the action of

$$(V \otimes V^*) \times V \to V \tag{19.2}$$

defined in the obvious way using the map

$$V^* \times V \to \mathbb{F}. \tag{19.3}$$

At any rate the adjoint representation is precisely this tensor product. So we see that $(\lambda_i - \lambda_j)$ are the weights of the adjoint representation. If it is nonzero we call them "**Roots**" with the condition that $i \neq j$. Now $\mathfrak{gl}(n)$ is not simple, so we should talk about $\mathfrak{sl}(n)$. We have in $\mathfrak{sl}(n)$ the Cartan subalgebra satisfy

$$\lambda_1 + \cdots + \lambda_n = 0. \tag{19.4}$$

The roots are (again) the same.

We would like to find simple roots (recall we can decompose roots into positive and negative, the minimal set of positive roots are called "**Simple Roots**"). We consider the situation when $i > j$ and $i < j$, one should be called "positive", the other "negative". We choose $i > j$ to be negative roots, $i < j$ to be positive roots. Thus $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_3$, $\ldots$, $\lambda_{n-1} - \lambda_n$ are the simple roots of $\mathfrak{sl}(n) = \mathsf{A}_{n-1}$, there are $n-1$ simple roots.

Now, for $\mathrm{SO}(2n)$ we have the fundamental representation be $2n$-dimensional. For the Lie algebra business $\mathfrak{so}(2n)$ the weights are $\lambda_1, -\lambda_1, \ldots, \lambda_n, -\lambda_n$ (weights of the fundamental vector representation, to be precise). What are the weights of the adjoint representation? We know the adjoint representation for $\mathfrak{so}(2n)$ is

$$V \otimes V^* = V \otimes V \tag{19.5}$$

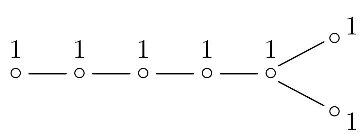
since

$$V^* = V \tag{19.6}$$

as far as the representations are concerned. The antisymmetric part preserves the inner product. The weights of the adjoint representation, we consider $\vec{u}_i$ $(i = 1, \ldots, 2n)$. When we take the tensor product we should get $\vec{u}_i \otimes \vec{u}_j$, and for $\bigwedge^2 V$ consider

$$\vec{u}_i \otimes \vec{u}_j - \vec{u}_j \otimes \vec{u}_i \quad \text{for } i \neq j. \tag{19.7}$$

The weights should be $\pm\lambda_\alpha \pm \lambda_\beta$. The nonzero guys are the "**Roots**". We say that $\lambda_\alpha + \lambda_\beta$ are the positive roots, $-\lambda_\alpha - \lambda_\beta$ are the negative roots, and $\lambda_\alpha - \lambda_\beta$ be positive iff $\alpha < \beta$. We have $\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n$ be the simple roots... but that is not sufficient, we have $n-1$ roots and $\dim(\mathscr{H}) = n$. So we should add $\lambda_1 + \lambda_n$ to our list of simple roots. We can obtain everything by considering these guys. The Dynkin diagram looks like:



As previously noted in Lecture 16, the Cartan matrix here is symmetric.

The next Lie algebra we will consider is $\mathsf{B}_n = \mathfrak{so}(2n+1)$. The Cartan subalgebra is

$$\mathscr{H} = \left\{ \begin{bmatrix} 1 & & & \\ & [\mathfrak{so}(2)] & & \\ & & \ddots & \\ & & & [\mathfrak{so}(2)] \end{bmatrix} \right\} \tag{19.8}$$

What are the weights of the fundamental vector representation? They are 0, λ_1, $-\lambda_1$, ..., λ_n, $-\lambda_n$. The adjoint representation is again $\bigwedge^2(V)$ and again we are doing the same thing. We are adding the weights for $i \neq j$: $\lambda_i + \lambda_j$, $\lambda_i - \lambda_j$, $-\lambda_i - \lambda_j$ but we have more, we can take $0 + \lambda_i$ and $0 - \lambda_i$. What are the positive weights? They are: λ_i, $\lambda_i + \lambda_j$, and $\lambda_i - \lambda_j$ for $i < j$. What are the simple roots? We take $\lambda_1 - \lambda_2$, ..., $\lambda_{n-1} - \lambda_n$, and λ_n. These are the n simple roots. But in this situation, the Dynkin diagram is

$$\overset{2}{\circ} - \overset{2}{\circ} - \overset{2}{\circ} - \overset{2}{\circ} = \overset{1}{\circ}$$

Remember the number of lines connecting the nodes ij are

$$n_{ij} = a_{ij} a_{ji}. \tag{19.9}$$

We will get

$$a_{ij} \neq a_{ji} \tag{19.10}$$

the diagonal matrix $\operatorname{diag}(2, 2, \ldots, 2, 1)$ symmetrizes the Cartan matrix. The root vectors e_α have positive roots, and f_α has negative roots. We obtain

$$[e_\alpha, f_\alpha] = h_\alpha \in \mathscr{H} \quad \text{and} \quad [h_\alpha, e_\beta] = a_{\alpha\beta} e_\beta \tag{19.11}$$

and so on.

For $\mathsf{C}_n = \mathfrak{sp}(n)$, the weights are λ_α, $-\lambda_\alpha$ and the adjoint representation has $\lambda_\alpha + \lambda_\beta$, $\lambda_\alpha - \lambda_\beta$, $-(\lambda_\alpha + \lambda_\beta)$ for all α, β. The descriptions of the simple roots begins with the same stuff. The simple roots are $\lambda_1 - \lambda_2$, ..., $\lambda_{n-1} - \lambda_n$, $2\lambda_n$. We are allowed to add 2 guys, which is how we get $2\lambda_n$. The Dynkin diagram is:

$$\overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} - \overset{1}{\circ} = \overset{2}{\circ}$$

Observe that this resembles B_n's Dynkin diagram, but it is different. The diagonalization here requires the matrix $\operatorname{diag}(1, \ldots, 1, 2)$.

Lecture 20

We formulated a theorem of the structure of semisimple algebras, then considered examples. Lets go back. Recall we considered the situation when we had e_α, f_α, h_α (members of the Lie Algebra) with the relations that

$$[h_\alpha, h_\beta] = 0 \tag{20.1a}$$
$$[e_\alpha, f_\beta] = h_\alpha \delta_{\alpha\beta} \tag{20.1b}$$
$$[h_\alpha, e_\beta] = a_{\alpha\beta} e_\beta \tag{20.1c}$$
$$[h_\alpha, f_\beta] = -a_{\alpha\beta} f_\beta. \tag{20.1d}$$

We see that e_α, f_α are root vectors, so

$$[h, e_\alpha] = \lambda_\alpha(h) e_\alpha \tag{20.2}$$

and similarly

$$[h, f_\beta] = -\lambda_\beta(h) f_\beta. \tag{20.3}$$

We recall that a mapping

$$\lambda\colon \mathscr{H} \to \mathbb{F} \tag{20.4}$$

is called a "**Root**", it's a linear functional on $\mathscr{H}$. We see that

$$\lambda_\beta(h_\alpha) = a_{\alpha\beta}. \tag{20.5}$$

Moreover $a_{\alpha\beta}$ should be the Cartan matrix, so

$$a_{\alpha\alpha} = 2 \tag{20.6a}$$

$$a_{\alpha\beta} \leq 0 \quad \text{for } \alpha \neq \beta \tag{20.6b}$$

$$a_{\alpha\beta} \text{ is symmetrizable} \tag{20.6c}$$

We also assume that

$$\det(a_{\alpha\beta}) \neq 0 \tag{20.7}$$

i.e. the Cartan matrix is nondegenerate. We can find the Cartan matrix for semisimple Lie Algebras. An ideal corresponds to an invariant subspace of the Lie algebra under the adjoint representation. We know a simple Lie Algebra is simple iff it has only trivial ideals. For semisimple Lie Algebras, we can partition roots into positive and negative roots. Positive roots contain a subset that generates all roots, we call this subset "**Simple Roots**".

What may be said of representations with this data? We take $\mathscr{G}$ a Lie algebra, we take a representation

$$\varphi\colon \mathscr{G} \to \mathfrak{gl}(V) \tag{20.8}$$

for some vector space V, and we may consider the weights of this Lie Algebra

$$\varphi(h)\vec{v} = \alpha(h)\vec{v} \tag{20.9}$$

and weight vectors $\vec{v}$ (where we take $h \in \mathscr{H}$). The root vectors act on weight vectors, namely $\varphi(e_k)\vec{v}$ is a weight vector (supposing that $\vec{v}$ was initially a weight vector) with weight $\alpha + \lambda_k$ provided that it is nonzero. Similarly $\varphi(f_j)\vec{v}$ is a weight vector with weight $\alpha - \lambda_j$, so $\varphi(f_j)$ lowers the weights. The highest weight vector is annihilated by applying $\varphi(e_i)\vec{v} = 0$ for all i. The highest weight vector always exists in in finite dimensional representations, although this is not necessarily true for infinite dimensional representations.

Theorem 20.1. (The highest weight vector exists in finite dimensional representations.) *If a finite dimensional representation is reducible, then the highest weight vector is not unique.*

Why? Well, at least one exists in the finite dimensional case. Why? Trivially, because in linear algebra the eigenvalue problem has no solution if the matrix is all zeros. We cannot have that for a nontrivial representation.

Now suppose there exists a representation that is a subrepresentation which will be irreducible and contains a different highest weight vector. Let us suppose we have highest weight vector, then we may construct a subrepresentation consisting of $f_{\alpha_1}(\cdots)f_{\alpha_n}\vec{v}$ which is a subrepresentation — it is highest weight since

$$e_\beta\vec{v} = 0 \tag{20.10}$$

for all β. Suppose our representation is reducible. If this is so, there is a highest weight vector in the subrepresentation.

Remark 20.2. To prove a representation is irreducible, it is sufficient to prove the uniqueness of the highest weight vector.

How to classify, to describe representations (especially irreducible representations). This is a simple thing, namely take this highest weight vector

$$\varphi(h)\vec{v} = \alpha(h)\vec{v} \tag{20.11}$$

where $\alpha \in \mathscr{H}^*$, and we should calculate $\alpha(h_1)$, …, $\alpha(h_n)$ for all basis elements of the Cartan subalgebra. We will prove that $\alpha(h_i) \geq 0$ and $\alpha(h) \in \mathbb{Z}$. To prove this is extremely simple, because – look – we have these commutation relations

$$\operatorname{span}\{h_i, e_i, f_i\} \cong \mathfrak{sl}(2) \tag{20.12}$$

for fixed i. For this algebra $\mathfrak{sl}(2)$ we know *everything*, in particular all finite dimensional irreducible representations, which is *precisely* the guys we are interested in. So the representation is characterized by n non-negative numbers. So can we take thse numbers in any way we want? Yes we can, we'll prove it in the next lecture. We will merely check this for A_n. This is interesting by itself. We will later check this for B_n, C_n; the proof will be constructive.

For A_n, we have $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$. We see

$$h_i = E_{i,i} - E_{i+1,i+1} \tag{20.13}$$

What to do? Well, we have first of all n numbers $\alpha(h_1)$, …, $\alpha(h_n)$. We will prove these numbers may be taken by considering the standard basis in $\mathbb{R}^n$.

We will call the name for these representations "**Elementary Representations**". First it is sufficient to find elementary representations, they represent $\mathscr{G}$ in spaces V_1, …, V_n. We will take $V_1^{\otimes m_1} \otimes \cdots \otimes V_n^{\otimes m_n}$, and the highest weights α_1, …, α_n with the corresponding highest weight vectors $\vec{v}_1$, …, $\vec{v}_n$, then the corresponding weight vectors in $V_1^{\otimes m_1} \otimes \cdots \otimes V_n^{\otimes m_n}$ have weights $m_1\alpha_1 + \cdots + m_n\alpha_n$. If we analyze these weights, we may consider any representation constructed from the elementary representations.

What to do? Construct the elementary representation, which is very easy…we take the fundamental representation. If $(\varphi_1, \ldots, \varphi_n)$ are the coordinates of the Cartan subalgebra (bear in mind because we work with A_n we have $\varphi_1 + \cdots + \varphi_n = 0$ and we work with diagonal matrices), then the weights are simply φ_1, …, φ_n. The highest weight correspond to φ_1. We see

$$\alpha_1(h_k) = \begin{cases} 1 & k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{20.14}$$

We would like to now note this corresponds to $(1, 0, \ldots, 0)$.

We want to consider $(0, 1, 0, \ldots, 0)$. This is constructed by considering $\bigwedge^2(V)$, the antisymmetric part of $V \otimes V$. The highest weight vector is $v_1 \otimes v_2 - v_2 \otimes v_1$, and the corresponding weight is $\alpha_1 + \alpha_2$. We see

$$(\alpha_1 + \alpha_2)(h_k) = \begin{cases} 0 & k \neq 2 \\ 1 & k = 2 \end{cases} \tag{20.15}$$

This corresponds to the desired $(0, 1, 0, \ldots, 0)$.

The general case we have the highest weight be $\alpha_1 + \cdots + \alpha_k$, which corresponds to the representation $\bigwedge^k(V)$ — the antisymmetric part of $V^{\otimes k}$. The highest weight vector is then $\vec{v}_{[1} \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_{k]}$.

Lecture 21

Let us discuss some notions important in themselves, namely, the notion of a "**Universal Enveloping Algebra**". Lets start with a Lie Algebra $\mathscr{G}$, and the commutation relations

$$[e_\alpha, e_\beta] = c^\gamma_{\alpha\beta} e_\gamma. \tag{21.1}$$

We want to embed this into an associative unital algebra

$$\mathscr{G} \subseteq \mathcal{U}(\mathscr{G}) \tag{21.2}$$

in such a way that

$$[a, b] \mapsto ab - ba. \tag{21.3}$$

We take $\mathcal{U}(\mathscr{G})$ to be unital and associative with generators $e_1, \ldots, e_n$, and with relations

$$e_\alpha e_\beta - e_\beta e_\alpha = c^\gamma_{\alpha\beta} e_\gamma. \tag{21.4}$$

It is clear in $\mathcal{U}(\mathscr{G})$ we have part which consists of linear guys $c^\alpha e_\alpha$ which is the Lie algebra with respect to the commutator, but this is only a small part of the Universal Enveloping Algebra. It has, first of all, an arbitrary guy written as

$$c \cdot \mathbf{1} + c^\alpha e_\alpha + c^{\alpha\beta} e_\alpha e_\beta + \cdots \tag{21.5}$$

with quadratic, cubic, and higher order terms. We can consider commutators of these guys, but note that

$$e_\alpha e_\beta = e_\beta e_\alpha + c^\gamma_{\alpha\beta} e_\gamma \tag{21.6}$$

so not all of these guys are unique. We may assume that always $\alpha \leq \beta \leq \gamma \leq \cdots$. This sum is finite. The alternative to doing them ordered, we may do them symmetrically.

Remark 21.1. The universal enveloping algebra is an infinite dimensional algebra, even for the simplest case! Consider $c^\gamma_{\alpha\beta} = 0$ for all α, β, γ. So $[e_\alpha, e_\beta] = 0$, then $\mathcal{U}(\mathscr{G})$ is a commutative polynomial algebra of n-variables.

We would like to say

$$\operatorname{Hom}\big(\mathscr{G}, \mathfrak{gl}(n)\big) = \operatorname{Hom}\big(\mathcal{U}(\mathscr{G}), \mathrm{Mat}_n\big). \tag{21.7}$$

We represent the generator $e_\alpha \in \mathscr{G}$ by E_α such that

$$E_\alpha E_\beta - E_\beta E_\alpha = c^\gamma_{\alpha\beta} E_\gamma \tag{21.8}$$

which specifies (by $c^\gamma_{\alpha\beta}$) the representation of the Lie Algebra. That's pretty important in physics, we considered such a thing even if not in this vocabulary. Consider M_x, M_y, M_z which are all the angular momentum operators, which are precisely a representation $\mathfrak{so}(3)$. The commutator of these guys are precisely the commutation relations from the Lie Algebra. We have the total angular momentum

$$M^2 = M_x^2 + M_y^2 + M_z^2 \tag{21.9}$$

but this isn't in $\mathscr{G}$, it's in $\mathcal{U}(\mathscr{G})$.

The universal enveloping algebra is useful in many relations. What is important is to consider the center of $\mathcal{U}(\mathscr{G})$. When considering stuff that commutes with everything; it is sufficient to consider the stuff that commutes with the generators.

Schur's Lemma. *If we have an operator that commutes with all operators in the irreducible representations of a Lie Algebra, then it is a scalar times the identity.*

As a corollary the central element is a scalar, if not it's an irreducible representation. We may consider the eigensubspaces, which form a decomposition into irreducible representations.

Now we will talk about the highest weight representation. We have generators e_α, f_α, h_α (in principle these are either multiplicative or linear basis elements); we may consider the subalgebras

1. $\mathscr{G}_+$ generated by e_α (it's basis as a vector space is all positive roots, although in principle we can use simple roots since we can obtain all other roots this way);

2. $\mathscr{G}_-$ generated by f_α;

3. $\mathscr{H}$ generated by h_α.

When generating the universal enveloping algebra, we use all roots. We use wither the simple roots or all the roots for the Verma module. We may consider as vector spaces

$$\mathscr{G} = \mathscr{G}_{+} \oplus \mathscr{G}_{-} \oplus \mathscr{H} \tag{21.10}$$

How to construct the highest weight representation? We take the highest weight vector $\vec{v}$, and we may apply the operators e_α and we get zero

$$e_\alpha \vec{v} = 0 \tag{21.11}$$

for all α. We can apply $h \in \mathscr{H}$, we get

$$h\vec{v} = \lambda(h)\vec{v} \tag{21.12}$$

Now we apply $f_{\alpha_1}, \ldots, f_{\alpha_n} \in \mathscr{G}_{-}$ and

$$\mathrm{span}\{f_{\alpha_1} \cdots f_{\alpha_n} \vec{v}\} \tag{21.13}$$

forms an invariant subspace. Further it forms a representation. Why is this an invariant subspace? For the simple reason that

$$[e_\alpha, f_\beta] = \delta_{\alpha\beta} h_\beta, \tag{21.14}$$

in the representation it is

$$[E_\alpha, F_\beta] = \delta_{\alpha\beta} H_\beta \tag{21.15}$$

and so we may interchange e and f at some price. We can push all the e's to the right, etc.

So we have some subspace. What if we don't have it? Then we will *construct* it. We will take $\mathcal{U}(\mathscr{G}_{-})$, or more precisely $\mathcal{U}(\mathscr{G}_{-})\vec{v}$ which are the combinations

$$\sum_n c^{\alpha_1 \cdots \alpha_n} f_{\alpha_1} \cdots f_{\alpha_n} \vec{v} \tag{21.16}$$

where $\alpha_1 \leq \cdots \leq \alpha_n$. We can define the action of $\mathscr{G}$ on this set, and this is called a "**Verma Module**". There is another construction which is more or less immediate.

So for every λ we may construct an infinite dimensional module, an infinite dimensional representation with highest weight λ... and λ is *completely arbitrary.* It is infinite dimensional. Why? Well, consider $\mathfrak{sl}(2)$ where we have e, f, h. We consider all elements of the form $f^n\vec{v}$, then

$$h(f^n\vec{v}) = (\lambda - n) f^n \vec{v} \tag{21.17}$$

and in principle it's an infinite dimensional representation since we are completely arbitrary here. But is this representation irreducible? We know for $\mathfrak{sl}(2)$, the representation is reducible when $\lambda \geq 0$ and $\lambda \in \mathbb{Z}$. At some moment we would get

$$h(f^n\vec{v}) = 0 \tag{21.18}$$

and we would get more importantly

$$e(f^n\vec{v}) = 0 \tag{21.19}$$

we have 2 highest weight vector in the same representation. Which means we can do the following: we can factorize our representation with respect to this subrepresentation and get a finite dimensional and irreducible representation.

So for every λ we have an infinite dimensional representation with highest weight λ called the Verma Module. What can we do? This is not necessarily reducible. If $\lambda(h_i) \in \mathbb{Z}$ and $\lambda(h_i) \geq 0$, then the Verma module is reducible. Let us take the largest subrepresentations (well largest nontrivial representations), then the quotient

$$\text{(Verma Module)/(Largest Nontrivial Subrepresentation)} \tag{21.20}$$

is a finite dimensional irreducible representation.

The fact it is an irreducible representation is trivial, since we factorized by the largest nontrivial subrepresentation; if we have a subrepresentation in the quotient, then there's a larger nontrivial subrepresentation in the Verma module, which is impossible. The nontrivial part is the finite dimensionality of the irreducible representation.

Lecture 22

Now we will give another construction of the Verma module. Really, it will be a general case of our particular construction called "**Induced Representations**". We will talk about representations of associative algebras, it is more general than representations of Lie Algebras (since we may construct for any Lie Algebra $\mathscr{G}$ with its universal enveloping algebra $\mathcal{U}(\mathscr{G})$). So representations of $\mathscr{G}$ and $\mathcal{U}(\mathscr{G})$ are precisely the same. Therefore we will consider representations of an associative algebra $\mathscr{A}$. To ever $x\in\mathscr{A}$ we assign a linear operator

$$\widehat{x}\colon V\to V. \tag{22.1}$$

We have

$$\widehat{xy}=\widehat{x}\widehat{y}. \tag{22.2}$$

We would like to say that representations of $\mathscr{A}$ is the same as a left $\mathscr{A}$-module. An $\mathscr{A}$-module is a vector space V and we have multiplication by elements of $\mathscr{A}$ (so we may consider $x\vec{v}$ for $x\in\mathscr{A}$ and $\vec{v}\in V$). We may think of it as a vector space over $\mathscr{A}$. We should have the standard relations for a sort of associativity

$$(xy)\vec{v}=x(y\vec{v}). \tag{22.3}$$

This is precisely what we have if we write

$$x\vec{v}=\widehat{x}\vec{v}. \tag{22.4}$$

We see

$$(xy)\vec{v}=x(y\vec{v})\qquad\Longleftrightarrow\qquad\widehat{xy}=\widehat{x}\widehat{y} \tag{22.5}$$

If we can talk about left modules, we may talk about *right* modules. What does it mean? Well $\mathscr{A}$-scalar multiplication occurs on the right, i.e. we have

$$(\vec{v}x)y=\vec{v}(xy) \tag{22.6}$$

We write

$$\vec{v}x=\widetilde{x}\vec{v} \tag{22.7}$$

is it a representation? Not really, observe

$$\widetilde{xy}=\widetilde{y}\widetilde{x}. \tag{22.8}$$

We may say a right module is a representation of $\mathscr{A}^{op}$, where $\mathscr{A}^{op}$ has multiplication defined by $x\cdot y=yx$.

Example 22.1. A left module could be $\mathscr{A}$, but this is also a right module. Or we may say that $\mathscr{A}$ is a bimodule, i.e. a left and a right module such that

$$(xa)y=x(ay) \tag{22.9}$$

for all $x,y,a\in\mathscr{A}$.

Now we would like to define the tensor product of algebras. If we have a right module $V_{\mathscr{A}}$ and a left module ${}_{\mathscr{A}}W$, both over the associative algebra $\mathscr{A}$, then we may take the tensor product as modules

$$V_{\mathscr{A}} \underset{\mathscr{A}}{\otimes} {}_{\mathscr{A}}W = \frac{V \otimes W}{Va \otimes W \sim V \otimes aW} \tag{22.10}$$

as vector spaces, and we identify

$$\vec{v}a \otimes \vec{w} \sim \vec{v} \otimes a\vec{w} \tag{22.11}$$

which is the definition of taking the tensor product over $\mathscr{A}$. The tensor product describes bilinear functions

$$f\colon V \otimes W \to Z \tag{22.12a}$$
$$v \otimes w \mapsto f(v, w) \tag{22.12b}$$

be bilinear.

This means that bilinear functions of $\mathscr{A}$-modules would be such that

$$f(va, w) = f(v, aw). \tag{22.13}$$

A really nice picture occurs when we work with bimodules, or $(\mathscr{A}, \mathscr{B})$-modules. We have $\mathscr{A}$, $\mathscr{B}$ be associative algebras. An $(\mathscr{A}, \mathscr{B})$-module is simultaneously a left $\mathscr{A}$-module *and* a right $\mathscr{B}$-module. The corresponding operation should commute. We have

$$(av)b = a(vb) \tag{22.14}$$

be such a condition. In particular, what is an $(\mathscr{A}, \mathscr{A})$-module? It is something we called a "bimodule". Suppose we have an $(\mathscr{A}, \mathscr{B})$-module ${}_{\mathscr{A}}V_{\mathscr{B}}$ and a $(\mathscr{B}, \mathscr{C})$-module ${}_{\mathscr{B}}W_{\mathscr{C}}$ then we may consider

$${}_{\mathscr{A}}V_{\mathscr{B}} \underset{\mathscr{B}}{\otimes} {}_{\mathscr{B}}W_{\mathscr{C}} \tag{22.15}$$

by identifying

$$vb \otimes w \sim v \otimes bw. \tag{22.16}$$

We can say that $V \underset{\mathscr{B}}{\otimes} W$ is an $(\mathscr{A}, \mathscr{C})$-module. We see that

$$a(v \otimes w) = (av) \otimes w. \tag{22.17}$$

Is it possible to do this with the relation

$$vb \otimes w \sim v \otimes bw? \tag{22.18}$$

We assert the multiplication by a is compatible with this equivalence because, well, it is:

$$a(vb \otimes w) = a(vb) \otimes w \tag{22.19a}$$
$$= (av)b \otimes w \tag{22.19b}$$
$$\sim av \otimes bw \tag{22.19c}$$
$$= a(v \otimes bw) \tag{22.19d}$$

Example 22.2. Lets take a simple example $\mathscr{B} \subseteq \mathscr{A}$ a subalgebra. Suppose we have a $\mathscr{B}$-module ${}_{\mathscr{B}}V$. We would like to get an $\mathscr{A}$-module. How to do this? Look, we consider $\mathscr{A}$ our algebra as an $(\mathscr{A}, \mathscr{B})$-module, now we take the tensor product of these guys over $\mathscr{B}$

$${}_{\mathscr{A}}\mathscr{A}_{\mathscr{B}} \underset{\mathscr{B}}{\otimes} {}_{\mathscr{B}}V \tag{22.20}$$

which is an $\mathscr{A}$-module.

We have $\mathscr{G} \supseteq \mathscr{G}'$. Let $\mathscr{A} = \mathcal{U}(\mathscr{G})$, $\mathscr{B} = \mathcal{U}(\mathscr{G}')$. It is obvious that

$$\mathscr{A} \supseteq \mathscr{B} \tag{22.21}$$

Suppose we have a representation of $\mathscr{G}'$ we recall this is a $\mathscr{G}'$-module, which is precisely the same as a $\mathcal{U}(\mathscr{G}')$-module; lets call this thing V. We get the

$$\mathcal{U}(\mathscr{G}) \underset{\mathscr{B}}{\otimes} V \tag{22.22}$$

a $\mathscr{G}$-module. This is precisely the notion of an "**Induced Representation of Lie Algebras**".

For the Verma module, we have

$$\mathscr{G} = \mathscr{G}_+ \oplus \mathscr{H} \oplus \mathscr{G}_- \tag{22.23}$$

there is a highest weight vector $\vec{v}$. We have for $h \in \mathscr{H}$,

$$h\vec{v} = \lambda(h)\vec{v}. \tag{22.24}$$

For

$$\mathscr{G}_+\vec{v} = 0 \tag{22.25}$$

So we write

$$\mathscr{G}' = \mathscr{G} \oplus \mathscr{H} \tag{22.26}$$

which is a subalgebra. The one-dimensional $\mathscr{G}'$-module is span$\{\vec{v}\}$. This gives us a representation of $\mathscr{G}_+$ too. We can write that the Verma module is an induced module

$$\mathcal{U}(\mathscr{G}) \bigotimes_{\mathcal{U}(\mathscr{G}')} V_\lambda \tag{22.27}$$

where V_λ is precisely the span$\{\vec{v}\}$, λ is precisely the highest weight. The only that remains is to prove that (22.27) is precisely the Verma module. We see that

$$\mathcal{U}(\mathscr{G}_-)\vec{v} = \mathcal{U}(\mathscr{G}) \bigotimes_{\mathcal{U}(\mathscr{G}')} V_\lambda \tag{22.28}$$

we did prove something before (although perhaps not in this name): the Poincaré-Birkhoff-Witt theorem. We proved something about the structure of $\mathcal{U}(\mathscr{G})$, namely

$$\mathscr{G} = \{E_1, \dots, E_n\} \tag{22.29}$$

so

Poincaré-Birkhoff-Witt theorem

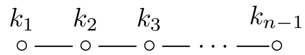
$$\mathcal{U}(\mathscr{G}) = \{c_0 + c^i E_i + c^{ij} E_i E_j + \cdots\} \tag{22.30}$$

but such a representation is not unique: it may be made unique by (1) demanding $i \leq j$; or (2) that c^{ij}, c^{ijk}, and all the other coefficients, be symmetric in its indices. This is the Poincaré-Birkhoff-Witt theorem. Then it is clear that $\mathcal{U}(\mathscr{G})$ is on the left hand side of the tensor product; what is not clear is this identity. We speak of all possible roots $E = \{e_i, f_i, h_i\}$.

Lecture 23

Today we will talk about spinor representations.

Now we will repeat what we know about $\mathfrak{sl}(n)$. For this Lie algebra, we can construct all irreducible representations as tensor representations. What does this mean? We have the fundamental representation V, we can take tensor powers $V \otimes \cdots \otimes V$ and there are a lot of invariant subspaces there. All irreducible representations are equivalent to representations in one of these invariant subspaces, we can give a more precise result. The algebra $\mathfrak{sl}(n)$ has Cartan subalgebra of rank $n-1$, we can draw the Dynkin diagram for A_{n-1} as



We assign to each node a number. When we take $n-2$ nodes to be zero, and one node to be 1, we can get an elementary representation. This corresponds to $\bigwedge^k V$. Here we have the highest weight vector

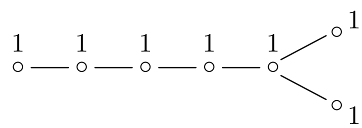
$$\vec{v}_k \in \bigwedge^k V \tag{23.1}$$

then we take

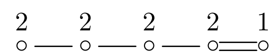
$$\bigotimes (\bigwedge^k V)^{\otimes n_k} \quad \text{and} \quad \bigotimes \vec{v}_k^{\otimes n_k} \tag{23.2}$$

will be the highest weight vector with weight $(n_1, \ldots, n_k, \ldots)$. This representation is reducible, but we can get an irreducible part.

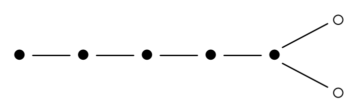
We can restrict our attention to any subgroup of $\mathrm{SL}(n)$, in particular $\mathrm{SO}(n)$, or more precisely $\mathfrak{so}(n)$. Not all representations of $\mathfrak{so}(n)$ are tensor representations, not all of them may be embedded in this tensor representation. Remember for $\mathsf{D}_n = \mathrm{SO}(2n)$ the Dynkin diagram is of the form



and for $\mathsf{B}_n = \mathrm{SO}(2n+1)$ the Dynkin diagram is



There are some special nodes. We have really from the left $(n-2)$ nodes in D_n:



the solid nodes are precisely when we get the same situation as A_{n-2}. For the special nodes, putting 1 as the value on the black nodes gives special representations called "**Spinor Representations**". We want to construct spinor representations. The main tool will be Clifford Algebras.

What is the Clifford Algebra? It is a unital associative algebra with generators $e_1, \ldots,$ e_n and relations

$$e_\alpha e_\beta + e_\beta e_\alpha = 2\eta_{\alpha\beta} \cdot \mathbf{1} \tag{23.3}$$

where $\eta_{\alpha\beta}$ are numbers and form a symmetric matrix

$$\eta_{\alpha\beta} = \eta_{\beta\alpha}. \tag{23.4}$$

We will require the matrix be degenerate

$$\det(\eta) \neq 0 \tag{23.5}$$

but really we can impose the opposite condition

$$\eta = 0 \tag{23.6}$$

and we get what is called a "**Grassmann Algebra**". We can reduce everything to these two opposite conditions. We can consider any field, but we will work with $\mathbb{C}$.

What is important is that $\eta_{\alpha\beta}$ may be diagonalized. For complex numbers this shows all Clifford Algebras of a given dimension are isomorphic. So we may choose $\eta_{\alpha\beta}$ as we'd like. And we choose

$$\eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{23.7}$$

when $n = 2m$ (for some $m \in \mathbb{N}$). We have two sorts of generators, lets denote generators by symbols $a_i^\dagger$, a_i. Then the defining relations will be of the following form

$$a_i a_j^\dagger + a_j^\dagger a_i = 2\delta_{ij} \tag{23.8a}$$

$$a_i^2 = 0 \tag{23.8b}$$

which gives us

$$a_i a_j + a_j a_i = a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0. \tag{23.9}$$

Physicists give a different name for this stuff, referring to it as "**Canonical Anticommutation Relations**", and a_i, $a_i^\dagger$ play the role of creation and annihilation operators. What may be said of the representations for this Clifford Algebra. It is extremely simple: when $n = 2m$, $C\ell(n)$ has only one irreducible representation. By the way, Physicists also have another name for representations of $C\ell(n)$: representations of Clifford Algebras are called "**Dirac Matrices**". We are sending

$$e_\alpha \mapsto \Gamma_\alpha \tag{23.10}$$

generators to matrices which obey

$$\Gamma_\alpha \Gamma_\beta + \Gamma_\beta \Gamma_\alpha = 2\eta_{\alpha\beta} \cdot \mathbf{1}. \tag{23.11}$$

We would now like to prove: if we require irreducibility, then we will have only one irreducible representation of Clifford Algebras.

So how to prove this? It is very simple and more or less based on the idea of the highest weight vector, although in this situation it is named the vacuum vector ϕ. The Fock Vacuum vector ϕ is annihilated by all a_i

$$a_i \phi = 0 \tag{23.12}$$

for all i. Here we can consider $a_{i_1}^\dagger \cdots a_{i_n}^\dagger \phi$, we may take arbitrary linear combinations of these guys. We can define the action of a on these guys. We just use the relations

$$a_i a_j^\dagger = 2\delta_{ij}\mathbf{1} - a_j^\dagger a_i \tag{23.13}$$

and we are golden.

We have some questions. Maybe this vacuum vector does not exist at all. We construct this sum

$$\widehat{N} = \sum_i a_i^\dagger a_i \tag{23.14}$$

observe that we may consider eigenvectors of $\widehat{N}$:

$$\widehat{N}\psi = N\psi \tag{23.15}$$

which always exist for non-trivial finite-dimensional representations. We see

$$a_i^\dagger \colon N \mapsto N+1 \quad \text{and} \quad a_i \colon N \mapsto N-1. \tag{23.16}$$

We have

$$\widehat{N}(a_\alpha \psi) = \sum_i a_i^\dagger a_i a_\alpha \psi \tag{23.17a}$$

$$= a_\alpha (N-1)\psi \tag{23.17b}$$

We cannot go down indefinitely, and the number operator does indeed exist. The representation is irreducible iff the vacuum vector is unique (it is more or less obvious, if we had reducibility we'd have another vacuum vector). It is very easy to show that there is only one irreducible representation, and all other representations are direct sums of irreducible representations.

Remark 23.1. How to realize this representation? We can consider the Grassmann algebra with n variables, we have "functions of anticommuting variables" in the jargon of physicists. We may multiply by α_i and differentiate with respect to them $\partial/\partial\alpha_i$. Differentiation amounts to "canceling after moving to the left." These two operators give a Clifford Algebra.

The question is why do we need this? Elements of the orthogonal group generate automorphisms of the Clifford algebra. What does this mean? It means if we take generators e_i of the Clifford algebra, and we use the transformation

$$\widetilde{e}_j = a_j{}^i e_i \tag{23.18}$$

and we assume $a_j{}^i$ is an orthogonal matrix, then it is clear that the commutation relations for $\widetilde{e}$ are precisely the same as for e. We see

$$\widetilde{e}_i\widetilde{e}_j + \widetilde{e}_j\widetilde{e}_i = 2\eta_{ij} \cdot \mathbf{1} \tag{23.19}$$

In other words, if we have Dirac Matrices, then we can obtain "new" Dirac matrices by using precisely the same orthogonal transformation:

$$\widetilde{\Gamma}_i = a_i{}^j \Gamma_j. \tag{23.20}$$

We only have one irreducible representation, so

$$\widetilde{\Gamma}_i \sim \Gamma_i \tag{23.21}$$

should be similar. Is this matrix unique? That is we have matrices U_a indexed by elements of the orthogonal group such that

$$\widetilde{\Gamma}_i = U_a \Gamma_i U_{a^{-1}}, \tag{23.22}$$

we can replace

$$U_a \mapsto \lambda U_a. \tag{23.23}$$

We demand

$$U_a U_b = \lambda U_{ab} \tag{23.24}$$

for the simple reason that we may perform 2 change of coordinates. This is a small problem, the U_a specify a representation of the orthogonal group called the "**Spinor Representation**" of the orthogonal group. It is a projective representation due to this pesky λ.

Lecture 24

We have a Clifford algebra over any field $\mathbb{F}$, we take some nondegenerate matrix η, and we look for generators obeying

$$\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha = 2\eta_{\alpha\beta} \cdot \mathbf{1} \tag{24.1}$$

This is a representation of a Clifford algebra, they may be classified quite easily:

1. all representations are irreducible;
2. the symmetric matrix $\eta_{\alpha\beta}$ is a symmetric (m,n)-form, which corresponds to $C\ell(m,n,\mathbb{R})$.

We have orthogonal groups $\mathrm{O}(m,n,\mathbb{R})$ for the metric which is similar to

$$g = \mathrm{diag}(\underbrace{+1,\dots,+1}_{m\ +1\text{'s}}, \underbrace{-1,\dots,-1}_{n\ -1\text{'s}}) \tag{24.2}$$

If we take

$$\widetilde{\Gamma}_\alpha = a_\alpha{}^\beta \Gamma_\beta \tag{24.3}$$

(using Einstein summation convention) and $a_\alpha{}^\beta \in \mathrm{O}(\eta,\mathbb{F})$, then the commutation relations are preserved.

Classification Theorem for Representations of Clifford Algebras. *There is only one irreducible representation for $C\ell(2n)$ for each dimension, and other irreducible representations in the same dimension are equivalent:*

$$\widetilde{\Gamma}_\alpha = U_a \Gamma_\alpha {U_a}^{-1} \quad \textit{and} \quad U_{ab} = (\textit{constant}) U_a U_b \tag{24.4}$$

and we cannot set this constant to be unity.

Consider SO(3), odd-dimensional guys. The situation is a little different. How to construct representations of the group in odd dimension? Recall

$$\mathrm{SO}(2n, \mathbb{C}) \subseteq \mathrm{SO}(2n+1, \mathbb{C}) \tag{24.5}$$

is an embedding. We may construct $\Gamma_1, \ldots, \Gamma_{2n}$ which obey the desired relation. After that, the theorem is: we may add the last matrix in such a way that the commutation relations are satisfied. The construction is very simple:

$$\Gamma_{2n+1} = \Gamma_1(\cdots)\Gamma_{2n} \tag{24.6}$$

up to some constant. We should quickly verify that Γ_{2n+1} satisfy the desired properties, e.g.

$$\Gamma_{2n+1}^2 = 1. \tag{24.7}$$

It is very clear that

$$\Gamma_{2n+1}^2 = \pm 1 \tag{24.8}$$

so we merely choose the coefficient to make it unity. Therefore we may extend any irreducible representation of the Clifford Algebra over $2n$-dimensions to be an irreducible representation of $C\ell(2n+1)$, with an appropriate choice of coefficient. This may be done in two ways really (up to sign).

Lets compute the dimensions. $C\ell(2n)$ has one irreducible representation. What is its dimensions? It is

$$2^n = \dim\big(C\ell(2n)\big) \tag{24.9}$$

Why? We have our guys divided into 2 parts, creation and annihilation operators, and we may apply them to a vacuum state. The rest follows trivially.

Consider $C\ell(2n+1)$ we have 2 irreducible representations both of dimension 2^n, since we considered irreducible representations and added 1 Dirac matrix generator. For, e.g., $n = 1$, we get $C\ell(3)$ and moreover we have an irreducible representation of SO(3) of dimension $2^1 = 2$. But a 2-dimensional representation of SO(3) does not exist! What does exist is

$$\mathrm{SU}(2) \to \mathrm{SO}(3) \tag{24.10}$$

which is a 2-sheeted covering. We may take the inverse mapping:

$$\mathrm{SO}(3) \to \mathrm{SU}(2) \tag{24.11}$$

which is a 2-valued representation. This is *precisely* the spinor representation.

24.1 Review of Previous Stuff

Here we will stress several points which should be stressed. There is a notion of a "**Reducible Representation**", i.e. a representation with a nontrivial subrepresentation (an invariant subspace). There is a notion of "**Completely Reducible Representation**" which is the direct sum of irreducible representation. Is an irreducible representation completely reducible? Yes! It is! Really, it is $\rho = \rho$.

Remember unitary and orthogonal representations are completely reducible, representations of a Lie algebra/group are not necessarily reducible (standard example: Abelian Lie algebra, $\mathfrak{u}(1)^k$ which has more representations since U(1) is not simply connected).

Irreducible representations of an Abelian Lie algebra are *always* 1-dimensional, but this doesn't mean that every representation of the Abelian Lie Algebras are direct sums of irreducible representations. For a representation onto a matrix with not normal Jordan form, the representation is not completely reducible; for the representation of the single generator by a diagonalizable matrix, we get the direct sum of 1-dimensional irreducible representations.

Remember

$$\text{Reductive Lie Algebra} = (\text{Center}) \oplus (\text{semisimple part}) \tag{24.12}$$

It has a non-trivial center which is an Abelian Lie Algebra.

The Cartan subalgebra is a maximal Abelian subalgebra, but this is not the end of the story because we should have some sort of requirement of semisimplicity. It was defined as the Lie Algebra of the maximal Torus for a compact group. What does "maximal" mean? If we add anything else, it become non-Abelian. But it is **WRONG** to state it contains every Abelian subalgebra. Consider $\mathfrak{gl}(n)$, the Cartan subalgebra consists of diagonal matrices. . . but we should have a basis. If we change basis, we get a completely different Cartan subalgebra. . . well, a *conjugate* Cartan subalgebra. Really, 2 maximal Tori are conjugate; this is a nontrivial statement. Care needs to be taken when working with a Cartan subalgebra.

We have a similar situation with simple roots, and the highest weight representation. We need additional data to introduce these notions. There is *no* intrinsic information in the Lie Algebra. The first choice is to fix the Cartan subalgebra $\mathscr{H}$. Then we decompose the Lie Algebra into 3 parts:

$$\mathscr{G} = \mathscr{G}_+ \oplus \mathscr{H} \oplus \mathscr{G}_- \tag{24.13}$$

where this direct sum is as vector spaces. Each summand is a subalgebra. We have a basis consisting of 3 types of elements

$$E_i \in \mathscr{G}_+ \tag{24.14a}$$
$$H_k \in \mathscr{H} \tag{24.14b}$$
$$F_j \in \mathscr{G}_- \tag{24.14c}$$

The commutation relations are nontrivial. The computations may not be made component-wise. We can define the highest weight vector $\vec{v}$ by

$$E_i\vec{v} = 0 \tag{24.15}$$

for all $E_i \in \mathscr{G}_+$ **and**

$$H_k\vec{v} = \lambda(H_k)\vec{v} \tag{24.16}$$

where

$$\lambda\colon \mathscr{H} \to \mathbb{F} \tag{24.17}$$

is a linear functional called the "**Highest Weight**". This picture is not terribly convenient, we'd like one with more information. Namely a system of multiplicative generators denoted by lowercase letters: e_α, h_α, and f_α. We take E_i as the roots of Lie Algebra, so

$$[H, \mathscr{G}_+] \subseteq \mathscr{G}_+ \tag{24.18}$$

thus

$$[H_k, E_i] = \alpha(H_k)E_i \tag{24.19}$$

are root vectors E_i and roots $\alpha(H_k)$. We see the commutator of root vectors is a root vector again $[E_i, E_j]$ with root $\alpha_i + \alpha_j$. We consider the e_α sufficient to generate all E_i, and f_j sufficient to generate all F_j.

Lecture 25

The goal here on out is twofold: (1) the Virasoro algebra is important, (2) we will be forced to reconsider the relevant theorems for semisimple Lie algebras. We will touch upon the notion of central extensions explicitly.

Definition 25.1. Lets suppose we have a group (G/N) and $N \subseteq G$ is a normal subgroup, or more precisely $N \subseteq Z(G)$. Let $H := G/N$. Then G is a "**Central Extension**" of H.

Example 25.2. Consider $\mathrm{GL}(V)$, consider its center which consists of scalar matrices

$$N = \{\, cI : c \in \mathbb{C} \,\}. \tag{25.1}$$

We have $\mathrm{PGL}(V)$ be the "projectivization" of $\mathrm{GL}(V)$, and $\mathrm{GL}(V)$ is the central extension of $\mathrm{PGL}(V) = \mathrm{GL}(V)/N$.

The notion is very close to the notion of a "**Projective Representation**" where we have

$$U_a U_b = c(a,b) U_{ab} \tag{25.2}$$

the product be up to a factor $c(a,b)$ which may depend on a and/or b. We have $U_a \in \mathrm{GL}(V)$, denote

$$[U_a] \in \mathrm{PGL}(V) \tag{25.3}$$

Therefore we may write

$$[U_a][U_b] = [U_{ab}] \tag{25.4}$$

We may alternatively think of a projective representation as a morphism

$$\rho\colon G \to \mathrm{PGL}(V). \tag{25.5}$$

One more thing to say is we may consider the situation when G is a central extension of H. This means

$$H = G/N. \tag{25.6}$$

Then we may say an irreducible (complex) representation ρ of a group G gives us a projective representation of H. The image of $\rho(N)$ commutes with the image $\rho(G)$, so by Schur's lemma

$$\rho(N) = \{cI\}. \tag{25.7}$$

for each scalar $c \in \mathbb{C}$. So then $\rho(H)$ is "up to constants" $\rho(G)$. We have this situation be described by demanding the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, shift right] & \mathrm{GL}(V) \\ H \arrow[u, shift right] \arrow[ur, "\widetilde{\varphi}"'] & \end{tikzcd} \tag{25.8}$$

We lift h to G in many ways, we get a multivalued map

$$\widetilde{\varphi}\colon H \to \mathrm{GL}(V) \tag{25.9}$$

This is precisely a projective representation.

There is a parallel notion for Lie Algebras. We have

$$\mathscr{G}/\mathscr{N} = \mathscr{H}, \tag{25.10}$$

and we have that $\mathscr{N} \subseteq$ center of $\mathscr{G}$. Then we have $\mathscr{G}$ be the central extension of $\mathscr{H}$. Everything may be repeated, with small changes of course. For groups it is difficult to lift a quotient group H to the group G; for Lie algebras, we may lift it as linear spaces. If we have a surjective morphism $V \to W$ of vector spaces, we may lift it in th following way: find $\widetilde{W} \subseteq V$ such that $\widetilde{W} \cong W$. Then we have

$$V = \widetilde{W} \oplus (\text{something}) \tag{25.11}$$

this is something special for vector spaces. So therefore for Lie Algebras we may say that

$$\mathscr{G} = \mathscr{H} \oplus \mathscr{N} \tag{25.12}$$

as vector spaces!! Does it work for the direct sum as Lie Algebras?

For simplicity we will consider $\mathscr{N} = \{\lambda c\}$ where $\lambda c \in \mathbb{C}$ and c is one single generator. We can now say that $g \in \mathscr{G}$ may be represented as

$$g = h + \lambda c \tag{25.13}$$

for some $h \in \mathscr{H}$. The general properties are really clear. Suppose we have the commutator

$$[g, g'] = [h + \lambda c, h' + \lambda' c] \tag{25.14a}$$

$$= [h, h'] + \underbrace{[\lambda c, h'] + [h, \lambda' c] + [\lambda c, \lambda' c]}_{=0 \text{ since } c \text{ is in the center}} \tag{25.14b}$$

So we find

$$[h, h']_{\text{new}} = \underbrace{\varphi(h, h')}_{\text{element in } \mathscr{H}} + \overbrace{\lambda(h, h')}^{\text{Element in } \mathscr{N}} \cdot c \tag{25.15}$$

This is our new commutator, if we factorize with respect to $\mathscr{N}$, we get a morphism

$$\mathscr{G} \to \mathscr{H} \tag{25.16}$$

The formula is as follows

$$[h, h']_{\text{new}} = [h, h']_{\text{old}} + \lambda(h, h') \cdot c \tag{25.17}$$

which is precisely our central extension. Note that $[\cdot,\cdot]_{\text{new}}$ is the commutator in $\mathscr{G}$, and $[\cdot,\cdot]_{\text{old}}$ is the commutator in $\mathscr{H}$. Can we use any $\lambda(h, h')$? Of course not, otherwise it'd be too easy. We want the new bracket to form a Lie Algebra. So it needs to obey the condition of antisymmetry (which is easy), but also the Jacobi identity.

25.1 Spinor Representations

How to construct spinor representations in the language of Lie Algebras? Remember the Clifford algebra is an algebra with generators γ_μ and the anticommutator of these generators is

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \cdot 1 \tag{25.18}$$

where $\eta_{\mu\nu}$ is a symmetric, nondegenerate matrix. So a representation of the Clifford algebra is given by the Dirac matrices

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} \cdot 1 \tag{25.19}$$

We know something about the representations of Clifford algebras. We will denote $C\ell(n)$ as the n-dimensional Clifford algebra. Now what we would like to do, since $C\ell(n)$ is an associative algebra, is to note it is also a Lie algebra:

$$[\Gamma_a, \Gamma_b] = \Gamma_a \Gamma_b - \Gamma_b \Gamma_a \tag{25.20}$$

We will divide our Clifford algebra into parts, a standard trick! The general element $a \in C\ell(n)$ can be written as

$$a = a^0 \cdot \mathbf{1} + a^\mu \Gamma_\mu + a^{\mu\nu}\Gamma_\mu\Gamma_\nu + \cdots \tag{25.21}$$

We will now consider elements only up to quadratic terms. Is it a subalgebra? Not exactly... but if we use the Lie algebra operation, we get a Lie subalgebra. Why is this true?

First of all, constant terms drop out of the commutator, and

$$[\Gamma_\mu, \Gamma_\nu] = \text{something quadratic} \tag{25.22}$$

But what about

$$[\Gamma_\mu\Gamma_\nu, \Gamma_\rho\Gamma_\sigma] = ? \tag{25.23}$$

Well, we see first of all that

$$\begin{aligned}
[\Gamma_\alpha, \Gamma_\beta\Gamma_\gamma] &= \Gamma_\alpha\Gamma_\beta\Gamma_\gamma - \Gamma_\beta\Gamma_\gamma\Gamma_\alpha && (25.24a)\\
&= \Gamma_\alpha\Gamma_\beta\Gamma_\gamma - \Gamma_\beta(-\Gamma_\alpha\Gamma_\gamma + 2\eta_{\alpha\gamma}\cdot\mathbf{1}) && (25.24b)\\
&= \Gamma_\alpha\Gamma_\beta\Gamma_\gamma + \Gamma_\beta\Gamma_\alpha\Gamma_\gamma - 2\eta_{\alpha\gamma}\Gamma_\beta && (25.24c)\\
&= (\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha)\Gamma_\gamma - 2\eta_{\alpha\gamma}\Gamma_\beta && (25.24d)\\
&= 2\eta_{\alpha\beta}\Gamma_\gamma - 2\eta_{\alpha\gamma}\Gamma_\beta && (25.24e)
\end{aligned}$$

Then we take advantage of the identity of the commutator

$$[AB, C] = A[B, C] + [A, C]B \tag{25.25}$$

to find

$$\begin{aligned}
[\Gamma_\mu\Gamma_\nu, \Gamma_\alpha\Gamma_\beta] &= \Gamma_\mu[\Gamma_\nu, \Gamma_\alpha\Gamma_\beta] + [\Gamma_\mu, \Gamma_\alpha\Gamma_\beta]\Gamma_\nu && (25.26a)\\
&= 2\Gamma_\mu(\eta_{\nu\alpha}\Gamma_\beta - \eta_{\nu\beta}\Gamma_\alpha) + 2(\eta_{\mu\alpha}\Gamma_\beta - \eta_{\mu\beta}\Gamma_\alpha)\Gamma_\nu && (25.26b)\\
&= 2\eta_{\nu\alpha}(-\Gamma_\beta\Gamma_\mu + 2\eta_{\mu\beta}\mathbf{1}) - 2\eta_{\nu\beta}(-\Gamma_\alpha\Gamma_\mu + 2\eta_{\alpha\mu}\mathbf{1}) + \cdots && (25.26c)\\
&= 4(\eta_{\nu\alpha}\eta_{\mu\beta} - \eta_{\nu\beta}\eta_{\mu\alpha})\mathbf{1}\\
&\qquad + 2(\eta_{\mu\alpha}\Gamma_\beta\Gamma_\mu - \eta_{\mu\beta}\Gamma_\alpha\Gamma_\nu - \eta_{\nu\alpha}\Gamma_\beta\Gamma_\mu + \eta_{\nu\beta}\Gamma_\alpha\Gamma_\mu) && (25.26d)\\
&\sim a\Gamma_\sigma\Gamma_\tau + b\cdot\mathbf{1} && (25.26e)
\end{aligned}$$

We find that $a\cdot\mathbf{1} + \Gamma_\mu\Gamma_\nu a^{\mu\nu}$ is itself a Lie subalgebra. What is this Lie algebra? We may say that $a^{\mu\nu}$ may be restricted, due to the anticommutation relation any symmetric combination disappears. So

$$a^{\mu\nu} = -a^{\nu\mu} \tag{25.27}$$

is an antisymmetric matrix. It is pretty clear it is the orthogonal Lie algebra, as it is described by means of antisymmetric matrices. But we have this extra stuff, so we have no chance for it to be an orthogonal Lie Algebra, but it is the central extension to the orthogonal Lie algebra. Moreover it is the trivial central extension.

Lecture 26

Last time we considered Clifford algebras and its representation by gamma matrices

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu}\cdot\mathbf{1} \tag{26.1}$$

We can consider part of this business which is spanned by

$$a\mathbf{1} + b_{\mu\nu}\Gamma^\mu\Gamma^\nu \tag{26.2}$$

zero order terms and quadratic terms. This is a subalgebra of the associative algebra. We may consider it with an induced Lie algebra structure. We consider the coefficients of the quadratic term

$$b_{\mu\nu} = -b_{\nu\mu} \tag{26.3}$$

to be antisymmetric. Moreover this is an "almost" orthogonal Lie algebra.

We will introduce new notation

$$\Gamma^{\mu\nu} = \frac{1}{2}(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu). \tag{26.4}$$

This implies that

$$\Gamma^{\mu\nu} = -\Gamma^{\nu\mu}. \tag{26.5}$$

We also observe that

$$\Gamma^{\mu\nu} = \frac{1}{2}(\Gamma^\mu\Gamma^\nu + \Gamma^\mu\Gamma^\nu - 2\eta^{\mu\nu}\mathbf{1}) \tag{26.6a}$$

$$= \Gamma^\mu\Gamma^\nu - \eta^{\mu\nu}\cdot\mathbf{1} \tag{26.6b}$$

This is something created in such a way it gives us an orthogonal Lie algebra. We may take $a_{\mu\nu}\Gamma^{\mu\nu}$ requiring[4]

$$a_{\mu\nu} = -a_{\nu\mu}. \tag{26.7}$$

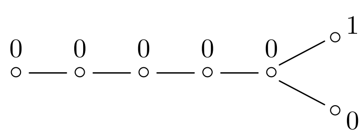
If we take the commutator of

$$[a_{\mu\nu}\Gamma^{\mu\nu}, b_{\rho\sigma}\Gamma^{\rho\sigma}] = C_{\alpha\beta}\Gamma^{\alpha\beta} \tag{26.8}$$

where

$$C = [a, b]. \tag{26.9}$$

The only thing we should check, that's not so easy, is that the scalar part goes away. We see by adding a scalar part, we get a central extension of $\mathfrak{so}(n)$. This is a trivial central extension, we managed to separate it into 2 parts. A representation of Clifford algebra gives us a representation of the orthogonal algebra called the "**Spinor Representation**". For $\mathfrak{so}(2n)$, i.e. D_n, we get a reducible representation — alternatively, for public relations reasons, we say we get 2 irreducible representations. The spinor representation corresponds to the Dynkin diagram:



So lets consider all this stuff in the case when the Lie algebra is $\mathfrak{so}(2n)$. In this case, remember what did we do? We constructed the Clifford algebra in such a way

$$\eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{26.10}$$

and instead of the Γ we took $a^\dagger$ and a with the anti-commutation relations

$$[a_i, a_j]_+ = [a_i^\dagger, a_j^\dagger]_+ = 0 \tag{26.11a}$$

$$[a_i, a_j^\dagger]_+ = 2\delta_{ij} \tag{26.11b}$$

Of course this describes the canonical anticommutation relations; this is of course a Clifford algebra, or more precisely a particularly case of it.

[4]Although this requirement is not really necessary. We can write $a_{\mu\nu}$ as the sum of the symmetric and antisymmetric part. But if $s_{\mu\nu}$ is symmetric, then $s_{\mu\nu}\Gamma^{\mu\nu} = 0$ always.

We have proven in any finite dimensional representation there is (for Clifford algebras) a vacuum vector ϕ such that

$$a_i\phi = 0 \tag{26.12}$$

for all a_i. We have a basis by applying the creation operators to ϕ our vacuum: $a^\dagger_{i_1}(\cdots)a^\dagger_{i_k}\phi$ (where $i_1 < \cdots < i_k$), it follows

$$(a^\dagger_i)^2 = 0 \tag{26.13}$$

for all i, so we need the indices to be strictly ordered. We have a spinor representation here, we should consider these guys $\Gamma^{\mu\nu}$. But really what will we do? We will only write $\Gamma^\mu\Gamma^\nu$, we *should* write the constant part

$$a^\dagger_i a^\dagger_j - \eta_{ij}\mathbf{1} \tag{26.14}$$

but

$$\eta_{ij} = 0 \tag{26.15}$$

in this instance, so we have terms of the form

$$a^\dagger_i a^\dagger_j \quad \text{and} \quad a_i a_j \tag{26.16a}$$

$$\frac{1}{2}(a^\dagger_i a_j - a_j a^\dagger_i) = a^\dagger_i a_j - \frac{1}{2}\delta_{ij} \tag{26.16b}$$

These are the generators of the spinor representation acting on the Fock space; it is reducible. Look at the number of $a^\dagger$'s and a's — they're always even. The parity is preserved. The Fock space is the sum of two parts

$$\mathcal{F} = \mathcal{F}_{\text{odd}} \oplus \mathcal{F}_{\text{even}} \tag{26.17}$$

where even/odd refer to the number of $a^\dagger$'s. So we have 2 representations, one is called the "**Left Spinor Representation**" and the other is surprisingly enough the "**Right Spinor Representation**".

We want to check these representations are irreducible. How to do this? Take the Cartan subalgebra of this stuff. The Cartan subalgebra is generated by

$$h_i = a^\dagger_i a_i - \frac{1}{2}. \tag{26.18}$$

What we see here is something similar to one of the exercises, with D_n we had elements of the form

$$x = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{26.19}$$

written in block form, where

$$B = -B^T \tag{26.20a}$$

$$C = -C^T \tag{26.20b}$$

$$D = -A^T \tag{26.20c}$$

describe the $n \times n$ block components. We know the Cartan subalgebra, we should take the simple roots, although this is unnecessary. We decompose the algebra into three parts:

$$\text{algebra} = (\text{positive}) \oplus (\text{negative}) \oplus (\text{Cartan}). \tag{26.21}$$

Let us take

$$a^\dagger_i a^\dagger_j \sim B \tag{26.22}$$

be the negative part,

$$a_i a_j \sim C \tag{26.23}$$

for the positive part. We have additionally

$$\alpha_{ij}(a_i^\dagger a_j - \frac{1}{2}\delta_{ij}) \tag{26.24}$$

where α_{ij} is the Cartan matrix. There are two places to make a choice: positive guys form a Lie algebra, and the negative guys form a Lie algebra too. These choices should be compatible, how they form Lie algebras should be done compatibly. Now the highest weight vectors must be determined. We have ϕ as the highest weight vector, but we see that

$$a_j(a_i^\dagger\phi) = 0 \tag{26.25}$$

for all $a_i^\dagger$ and a_j. But only one of them is the highest weight vector, really. Since we can decompose $\mathcal{F}$ into two irreducible representations, we have *two* highest weight vectors.

What we would like to do now is consider infinite-dimensional Clifford algebra. This is taught all the time in physics, there are an infinite number of creation/annihilation operators for fermions. What to do? We have creation and annihilation operators $a_i^\dagger$, a_i which obey the canonical anticommutation relations

$$[a_i, a_j]_+ = [a_i^\dagger, a_j^\dagger]_+ = 0 \tag{26.26a}$$

$$[a_i, a_j^\dagger]_+ = \delta_{ij} \tag{26.26b}$$

We may try to consider infinite dimensional representations of this algebra, constructed in exactly the same way. Consider a vector ϕ such that

$$a_i\phi = 0 \tag{26.27}$$

for all a_i. We may consider

$$a_{i_1}^\dagger(\cdots)a_{i_k}^\dagger\phi \tag{26.28}$$

where $i_1 < \cdots < i_k$. We can act by means of annihilation operators a_i on this stuff. Something new happens, namely:

- First of all, we do not know if this vector ϕ exists at all. Maybe it doesn't exist!
- Second we may introduce $a_i^\dagger$, a_i in many ways. These operators are on completely equal footing, and we may interchange them! Or parts of them! Why not? If we do this in finite dimensions, we have a vector annihilated by all $a^\dagger$'s, and we have a vector annihilated by all a's. They the operators are on completely equal footing. Physicists are brave, they apply an infinite number of $a^\dagger$'s. Mathematicians are not so brave.

This Fock space may be definite in the infinite-dimensional case.

We would like to consider analogous generators for an infinite-dimensional spinor representation. We have some problems. We consider

$$A = \sum_{i,j}\alpha_{ij}a_i^\dagger a_j^\dagger + \beta\cdot\mathbf{1} \tag{26.29}$$

we consider operators of this form (we can do it for finite dimensions). Then this is a representation of $\mathrm{GL}(n)$, it is sitting inside of $\mathrm{SO}(n)$. Well $\mathfrak{gl}(n)$ to be precise. We may consider something like this here. We get a nontrivial central extension for infinite-dimensional representations. We will describe an infinite-dimensional general linear Lie Algebra $\mathfrak{gl}(\infty)$. We cannot consider them all, but we will restrict our focus. These guys appear in physics all the time during quantization. We have projective representations as our tool in quantum theory.

Lecture 27

We will discuss some stuff that generalizes what we considered. Last time we considered infinite dimensional Clifford algebras with generators $a_k^\dagger$, a_k such that

$$[a_i, a_j]_+ = [a_i^\dagger, a_j^\dagger]_+ = 0 \tag{27.1a}$$

$$[a_i, a_j^\dagger]_+ = \delta_{ij} \tag{27.1b}$$

We can construct a representation in analogy to the finite dimensional case: take a vector ϕ such that

$$a_i\phi = 0 \tag{27.2}$$

for all a_i. We consider

$$a_{k_1}^\dagger(\cdots)a_{k_n}^\dagger\phi \tag{27.3}$$

where $k_1 < \cdots < k_n$. This gives us a space called the "**Fock Space**" and the vector ϕ is called the "**Vacuum Vector**". We may swap

$$a^\dagger \iff a \tag{27.4}$$

we get another Fock space; we may exchange only parts of the operator, and we still get another Fock space. Maybe these Fock spaces are equivalent, maybe not.

Consider terms of the form $\alpha_{kl}a_k^\dagger a_l$ with Euclidean summation convention[5]. We consider

$$[\alpha_{kl}a_k^\dagger a_l, \beta_{rs}a_r^\dagger a_s] = \alpha_{kl}a_k^\dagger a_l\beta_{rs}a_r^\dagger a_s - \beta_{rs}a_r^\dagger a_s\alpha_{kl}a_k^\dagger a_l \tag{27.5a}$$

$$= \alpha_{kl}\beta_{rs}a_k^\dagger a_l a_r^\dagger a_s - a_r^\dagger a_s a_k^\dagger a_l \tag{27.5b}$$

Well, we deduce

$$a_l a_r^\dagger = -a_r^\dagger a_l + \delta_{rl}. \tag{27.6}$$

We thus get something of the form

$$[\alpha_{kl}a_k^\dagger a_l, \beta_{rs}a_r^\dagger a_s] \sim \gamma_{mn}a_m^\dagger a_n + \operatorname{Tr}[\alpha,\beta]\cdot\mathbf{1}, \tag{27.7}$$

the trace of the commutator is nonzero for infinite dimensional matrices α, β.

What we would like to do is move from formal considerations to reality. When working with infinite sums, it is not terribly clear. But with representations, we may ask questions regarding the sum

$$\left(\sum_{kl}\alpha_{kl}a_k^\dagger a_l\right)(a_{i_1}^\dagger\cdots a_{i_s}^\dagger\phi) \tag{27.8}$$

we may questions. If the sum is infinite, then sometimes it is still well-defined. Suppose for every l, there are finitely many k's which have nonzero components

$$\alpha_{kl} \neq 0 \tag{27.9}$$

(if $k, l \in \mathbb{Z}$, we could say α_{kl} if $|k-l| < N$ for some fixed N). We have finite number of indices; therefore if l is sufficiently large, then we can obtain a vanishing answer. Therefore only finitely many cases are possible, but the expression is well-defined! So everything is fine! But such operators appear in such matrices.

In particular with the Virasoro algebra, we have

$$\ell_k = -z^{k+1}\frac{d}{dz} \tag{27.10}$$

[5] If two indices are repeated, either upstairs or downstairs, sum over it. So e.g. $\alpha_{kl}a_k a_l = \sum_{k,l}\alpha_{kl}a_k a_l$. The term "Euclidean summation convention" is due to Misner, Thorne and Wheeler's *Gravitation* Chapter 12.3: "('Euclidean' index convention: repeated space indices to be summed even if both are down; dot denotes time derivative.)".

If we take the basis z^n, the derivative shifts the index

$$\frac{d}{dz}\colon z^k \mapsto kz^{k-1}, \tag{27.11}$$

multiplication by z shifts the index too:

$$z\colon z^k \mapsto z^{k+1}. \tag{27.12}$$

We have a nontrivial central extension of some algebra $\mathfrak{gl}(\infty)$ the Lie algebra of *good* infinite dimensional matrices (but not *all* infinite dimensional matrices). We can embed the Witt algebra into $\mathfrak{gl}_\infty$.

Remark 27.1. The infinite dimensional Clifford algebra appears to be the limit or colimit of all finite dimensional Clifford algebras; Schwarz says this is not so for the infinite dimensional case; there appears to be canonical embeddings

$$\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2} \hookrightarrow \cdots \tag{27.13}$$

which is the limit of this sort of representation. There are some side-effects with this, especially regarding the central extension — it becomes trivial in this limit!

We would like to describe a very interesting situation when we get an affine algebra, a situation of the Kac–Moody algebra. Take any Lie Algebra $\mathscr{G}$. Now take all Laurent series $\sum a_n z^n$ where $a_n \in \mathscr{G}$ and $n \in \mathbb{Z}$, but we restrict ourselves specifically to Laurent *polynomials* so this sum is *finite*. We can obtain a Lie algebra, defining the Lie bracket as

$$[az^m, bz^n] = [a,b]z^{m+n}, \tag{27.14}$$

and by demanding linearity, distributivity, etc. This is not the most interesting one; the most important thing here is the central extension. One of the ways is by means of matrices and infinite matrices, then perform the central extension.

We will merely write the answer. (This is related to the answer to one of the problems on the final!) We will define a *new* Lie Algebra denoted $\widehat{\mathscr{G}}$ in the following way: take the central extension of the algebra in the following manner. Generators are of the form az^n, the central extension c. We assume

$$[az^n, c] = 0 \tag{27.15}$$

We don't really have a choice! Define the new commutator by

$$[az^m, bz^n]_{\text{new}} = [a,b]_{\text{old}} z^{m+n} + \underbrace{m\langle a,b\rangle\delta_{m+n,0}c}_{\text{central term with coefficient}} \tag{27.16}$$

This is the answer, we get a central extension this way. Well, do we? Prove it! Check it is a Lie Algebra, check the Jacobi identity:

$$[[az^m, bz^n]_{\text{new}}, rz^s]_{\text{new}} = [[a,b]_{\text{old}} z^{m+n}, rz^s]_{\text{new}} \tag{27.17a}$$

$$= [[a,b]_{\text{old}}, r]_{\text{old}} z^{m+n+s} + (m+n)\langle [a,b]_{\text{old}}, r\rangle \delta_{m+n+s,0} c \tag{27.17b}$$

By taking cyclic permutations, the first term vanishes identically (we borrow the Jacobi identity from the old Lie bracket). What about the second term? Well, we use the invariant inner product, i.e. for a representation φ we have

$$\langle y, \varphi(z)\rangle + \langle \varphi(y), z\rangle = 0; \tag{27.18}$$

the adjoint representation has

$$\langle [x,y], z\rangle + \langle y, [x,z]\rangle = 0 \tag{27.19}$$

So really the cyclic permutations of $\langle [a,b], r\rangle$ are equal up to a sign (or more closely examined, we see it is identical). We can repeat more or less everything from finite-dimensional Lie algebras.

Remark 27.2. This is useful in physics, if symmetry coincides with Lie algebra, we need projective representations, thus we need central extensions; the cohomology of Lie algebras gives us the central extension being unique.

A Bibliography and Further References

The required text for the course was

- V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representation.* Graduate Texts in Mathematics v. 102, Springer-Verlag (1984).

There was a supplementary text that was recommended informally beforehand by another professor:

- William Fulton, Joe Harris, *Representation Theory: A First Course.* Graduate Texts in Mathematics, Springer-Verlag (1991).
- Brian C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Springer (2003).
- Daniel Bump, *Lie Groups.* Graduate Texts in Mathematics, Springer (2004).
- Anthony W. Knapp, *Lie Groups: Beyond an Introduction.* Birkhäuser, 2nd edition (2002).
- Robert Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications.* Dover Publications (2006).

With regards to further studies of infinite-dimensional Lie Algebras, there are a variety of different texts to read. Kac notes there are roughly 4 different meanings of "infinite-dimensional Lie Algebras":

1. The Lie algebras of vector fields and the corresponding groups of diffeomorphisms of a manifold.
2. Lie groups (resp. Lie algebras) of smooth mappings of a given manifold into a finite-dimensional Lie group (resp. Lie algebra). So this is a group (Lie algebra) of matrices over some function algebra but viewed over the base field. (Physicists refer to certain central extensions of these Lie algebras as current algebras.) The main subject of study in this case has been certain special families of representations.
3. Classical Lie groups and algebras of operators in a Hilbert or Banach space. There is apparantely a rather large number of scattered results in this area, which study the structure of these Lie groups and algebras and their representations. A representation which plays an important role in quantum field theory is the Segal-Shale-Weil (or metaplectic) representation of an infinite-dimensional symplectic group.
4. Kac-Moody algebras, which Kac investigates in his book.

The references to investigate this subject are:

1. Minorus Wakimoto, *Infinite-Dimensional Lie Algebras.* Translations of Mathematical Monographs, vol 195. American Mathematical Society (2000).
2. Victor Kac, *Infinite Dimensional Lie Algebras.* Cambrdige University Press (1990).
3. Yu. A. Neretin, *Categories of Symmetries and Infinite-Dimensional Groups.* London Mathematical society Monographs, New Series vol 16. Oxford University Press (1996).

Wakimoto describes his book as an "hors d'oeuvres" to Kac's book and the "great feast" of infinite-dimensional Lie algebras.

B Solutions to Exercises

B.1 Problem Set 1

▸ **EXERCISE 20**

Check:

1. that the vector space $\mathbb{R}^3$ is a Lie algebra with respect to the cross product of vectors;
2. this Lie algebra is simple (i.e. does not have any non-trivial ideals);
3. all derivations of this Lie algebra are inner derivations.

Answer to 20:
For the matter of $\mathbb{R}^3$ being a Lie algebra, we have the following proof:

Proof. We have a vector space $\mathbb{R}^3$ over $\mathbb{R}$. We need to show that when we equip it with the cross product operation, we obtain a Lie algebra. That is, we induce a Lie Bracket

$$[\vec{v}, \vec{w}] := \vec{v} \times \vec{w} \tag{B.1}$$

where $\vec{v}$, $\vec{w} \in \mathbb{R}^3$. We need to check that it obeys the properties of the Lie bracket, and that the property of distributivity holds.

For the properties of the bracket, we see that antisymmetry holds:

$$\begin{aligned} [\vec{v}, \vec{w}] &= \vec{v} \times \vec{w} && \text{(B.2a)} \\ &= -\vec{w} \times \vec{v} && \text{(B.2b)} \\ &= -[\vec{w}, \vec{v}]. && \text{(B.2c)} \end{aligned}$$

We see that it is linear in the second slot (and by antisymmetry, the first slot too):

$$\begin{aligned} [\vec{u}, \vec{v} + \vec{w}] &= \vec{u} \times (\vec{v} + \vec{w}) && \text{(B.3a)} \\ &= \vec{u} \times \vec{v} + \vec{u} \times \vec{w} && \text{(B.3b)} \\ &= [\vec{u}, \vec{v}] + [\vec{u}, \vec{w}]. && \text{(B.3c)} \end{aligned}$$

Lastly we see that the Jacobi identity holds. We first observe, by Lagrange's identity

$$\begin{aligned} \vec{u} \times (\vec{v} \times \vec{w}) &= \vec{v}(\vec{u} \cdot \vec{w}) - \vec{w}(\vec{u} \cdot \vec{v}) && \text{(B.4a)} \\ \vec{v} \times (\vec{w} \times \vec{u}) &= \vec{w}(\vec{v} \cdot \vec{u}) - \vec{u}(\vec{v} \cdot \vec{w}) && \text{(B.4b)} \\ \vec{w} \times (\vec{u} \times \vec{v}) &= \vec{u}(\vec{w} \cdot \vec{v}) - \vec{v}(\vec{w} \cdot \vec{u}) && \text{(B.4c)} \end{aligned}$$

Then we consider the Jacobi identity by plugging in our results from Eq (B.4):

$$\begin{aligned} [\vec{u}, [\vec{v}, \vec{w}]] + [\vec{v}, [\vec{w}, \vec{u}]] + [\vec{w}, [\vec{u}, \vec{v}]] &= \vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) && \text{(B.5a)} \\ &= \vec{v}(\vec{u} \cdot \vec{w}) - \vec{w}(\vec{u} \cdot \vec{v}) + \vec{w}(\vec{v} \cdot \vec{u}) - \vec{u}(\vec{v} \cdot \vec{w}) \\ &\quad + \vec{u}(\vec{w} \cdot \vec{v}) - \vec{v}(\vec{w} \cdot \vec{u}) && \text{(B.5b)} \\ &= [\vec{v}(\vec{u} \cdot \vec{w}) - \vec{v}(\vec{w} \cdot \vec{u})] + [\vec{w}(\vec{v} \cdot \vec{u}) - \vec{w}(\vec{u} \cdot \vec{v})] \\ &\quad + [\vec{u}(\vec{w} \cdot \vec{v}) - \vec{u}(\vec{v} \cdot \vec{w})] && \text{(B.5c)} \\ &= [0] + [0] + [0] = 0 && \text{(B.5d)} \end{aligned}$$

which holds. Thus the cross product satisfies the properties of the Lie bracket, implying $\mathbb{R}^3$ equipped with the cross product is a Lia algebra. □

Answer 1.2: For the matter of this Lie algebra being simple, we have another proof.

Proof. Assume for contradiction there is an ideal $I \subseteq \mathbb{R}^3$ which is nontrivial. Then there is a nontrivial center for the Lie algebra. That is, we have some $\vec{x} \in I$ such that

$$[\vec{x}, \vec{y}] = 0 \tag{B.6}$$

for all $\vec{y} \in \mathbb{R}$. However, this happens if and only if

$$\vec{x} = \lambda\vec{y} \quad \text{for some nonzero } \lambda, \text{ or} \quad \vec{x} = 0. \tag{B.7}$$

The second case is trivial, the first case implies $I = \mathbb{R}^3$. In either case, $\mathbb{R}^3$ does not have a nontrivial center, so it doesn't have a nontrivial ideal. □

Answer 1.3: Last part of the first exercise, we need to show that all derivations of this Lie algebra are inner derivations. We thus produce the following proof.

Proof. We find that a derivation $\alpha\colon \mathbb{R}^3 \to \mathbb{R}^3$ would be of the form

$$\alpha([\vec{u}, \vec{v}]) = [\alpha(\vec{u}), \vec{v}] + [\vec{u}, \alpha(\vec{v})]. \tag{B.8}$$

which occurs whenever $\alpha = [\vec{w}, -]$ (for some $\vec{w} \in \mathbb{R}^3$) by the Jacobi identity. We want to show that there are no other derivations. We see that α is represented by an antisymmetric matrix $X + X^T = 0$. But we also recall any matrix B can be written as

$$B = A + S \tag{B.9}$$

where A is antisymmetric, and S is symmetric. Then if B were a derivation we see that

$$B[\vec{u}, \vec{v}] = [B\vec{u}, \vec{v}] + [\vec{u}, B\vec{v}] \tag{B.10}$$

but this would have

$$S[\vec{u}, \vec{v}] = [S\vec{u}, \vec{v}] + [\vec{u}, S\vec{v}] \tag{B.11}$$

which is not true. This means that a derivation is of the form of an antisymmetric matrix, which is the same as being of the form $\alpha[\vec{w}, -]$. □

▸ **EXERCISE 21**

Check the Lie algebra in problem 20 is:

1. isomorphic to the Lie algebra $\mathfrak{so}(3)$ of real antisymmetric 3×3 matrices; and
2. isomorphic to the Lie algebra $\mathfrak{su}(2)$ of complex anti-Hermitian traceless 2×2 matrices.

Answer 2.1: For the first matter of $\mathfrak{so}(3)$ we have the following proof:

Proof. The linear map φ basically maps bijectively

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \mapsto \varphi(\vec{e}_1) = \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix} \tag{B.12a}$$

$$\vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \mapsto \varphi(\vec{e}_2) = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\-1 & 0 & 0\end{bmatrix} \tag{B.12b}$$

$$\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \mapsto \varphi(\vec{e}_3) = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix} \tag{B.12c}$$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \mapsto \varphi(\vec{x}) = \begin{bmatrix}0 & -x_3 & x_2\\x_3 & 0 & -x_1\\-x_2 & x_1 & 0\end{bmatrix} \tag{B.12d}$$

which we will prove is an isomorphism. By direct computation, we find the commutator $[\varphi(\vec{x}), \varphi(\vec{y})]$ is:

$$\begin{bmatrix}0 & -x_3 & x_2\\x_3 & 0 & -x_1\\-x_2 & x_1 & 0\end{bmatrix}\begin{bmatrix}0 & -y_3 & y_2\\y_3 & 0 & -y_1\\-y_2 & y_1 & 0\end{bmatrix} - \begin{bmatrix}0 & -y_3 & y_2\\y_3 & 0 & -y_1\\-y_2 & y_1 & 0\end{bmatrix}\begin{bmatrix}0 & -x_3 & x_2\\x_3 & 0 & -x_1\\-x_2 & x_1 & 0\end{bmatrix}$$

$$= \begin{bmatrix} -x_2y_2 - x_3y_3 & x_2y_1 & x_3y_1 \\ x_1y_2 & -x_1y_1 - x_3y_3 & x_3y_2 \\ x_1y_3 & x_2y_3 & -x_1y_1 - x_2y_2 \end{bmatrix} - \begin{bmatrix} -x_2y_2 - x_3y_3 & x_1y_2 & x_1y_3 \\ x_2y_1 & -x_1y_1 - x_3y_3 & x_2y_3 \\ x_3y_1 & x_3y_2 & -x_1y_1 - x_2y_2 \end{bmatrix} \tag{B.13a}$$

$$= \begin{bmatrix} 0 & x_2y_1 - x_1y_2 & x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 & 0 & x_3y_2 - x_2y_3 \\ x_1y_3 - x_3y_1 & x_2y_3 - x_3y_2 & 0 \end{bmatrix} = \varphi(\vec{x} \times \vec{y}) \tag{B.13b}$$

so it preserves the Lie bracket. We see by inspection

$$\varphi^{-1}\left([\varphi(\vec{x}), \varphi(\vec{y})]\right) = \vec{x} \times \vec{y} \tag{B.14}$$

the inverse map also preserves the Lie bracket. This implies that this linear mapping is a Lie algebra isomorphism. □

Answer 2.2: The isomorphism with $\mathfrak{su}(2)$ is contained in the proof:

Proof. We have another mapping ψ which is an isomorphism of vector spaces which behave on basis vectors and an arbitrary vector by:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \psi(\vec{e}_1) = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{B.15a}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \psi(\vec{e}_2) = i \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{B.15b}$$

$$\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \psi(\vec{e}_3) = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{B.15c}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \psi(\vec{x}) = i \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix}. \tag{B.15d}$$

To show that ψ is an isomorphism of Lie algebras, we need to show that the Lie bracket is preserved. We see that the commutator of basis elements of $\mathfrak{su}(2)$ are

$$[\psi(\vec{e}_3), \psi(\vec{e}_1)] = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{B.16a}$$

$$= -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{B.16b}$$

$$= 2\psi(\vec{e}_2) \tag{B.16c}$$

$$[\psi(\vec{e}_1), \psi(\vec{e}_2)] = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{B.16d}$$

$$= i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - i\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.16e}$$

$$= 2\psi(\vec{e}_3) \tag{B.16f}$$

$$[\psi(\vec{e}_2), \psi(\vec{e}_3)] = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{B.16g}$$

$$= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \tag{B.16h}$$

$$= 2\psi(\vec{e}_1) \tag{B.16i}$$

We see that this is isomorphic to the Lie algebra on $\mathbb{R}^3$ equipped with the Lie bracket incuced by

$$[\vec{u}, \vec{v}] = \vec{u} \times \vec{v} - \vec{v} \times \vec{u} \tag{B.17}$$

the commutator of the cross products. □

B.2 Problem Set 2

▸ **EXERCISE 22**

Find the Lie algebras for the following matrix groups:

1. The group of real upper triangular matrices.
2. The group of real upper triangular matrices with diagonal entries equal to 1.
3. The group T_k of real $n \times n$ matrices obeying $a_{ii} = 1$, $a_{ij} = 0$ if $j - i < k$ and $j \neq i$.

Answer 22.1: We see that a curve $\gamma\colon [0,1] \to G$ in the group of real upper triangular matrices such that $\gamma(0) = I$ has components of the form

$$\gamma(t) = I + c(t) \tag{B.18}$$

where $c(t)$ has zero lower triangular components. This implies that the Lie algebra consists of matrices $c'(0)$ which are upper triangular.

Answer 22.2: We see that curves $\gamma\colon [0,1] \to G$ in the group of real upper triangular matrices with the diagonal components equal to 1 (such that $\gamma(0) = I$) is of the form

$$\gamma(t) = I + \begin{bmatrix} 0 & c_{12}(t) & \cdots & c_{1n}(t) \\ 0 & 0 & \cdots & c_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{B.19}$$

The Lie algebra is then consisting of matrices of the form

$$\gamma'(0) = \begin{bmatrix} 0 & c'_{12}(0) & \cdots & c'_{1n}(0) \\ 0 & 0 & \cdots & c'_{2n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{B.20}$$

where $c'_{ij}(0) \in \mathbb{R}$ for $0 < i < j \leq n$. These are strictly upper triangular matrices with real entries.

Answer 22.3: We have a matrix T_k with components $a_{ij} = 0$ if both $j < i + k$ **and** $j \neq i$. For example, consider 3×3 matrices. We see that

$$T_1 = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{B.21}$$

where a_{12}, a_{13}, and a_{23} are real numbers not necessarily zero (we use the well known fact that $2 \not< 1 + 1$, $3 \not< 1 + 1$, and $3 \not< 2 + 1$ respectively). Similarly, we see that

$$T_2 = \begin{bmatrix} 1 & 0 & b_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.22}$$

where $b_{13} \in \mathbb{R}$, since here $b_{12} = b_{23} = 0$. Lastly we see that

$$T_3 = I \tag{B.23}$$

is the identity element. We see then that the product of two matrices $T_a T_b = \widetilde{T}_{\min\{a,b\}}$ produce another matrix in the group.

We see, however, that this group's elements of the form T_1 form a subgroup which is isomorphic to the Lie group described in Exercise 22.2. We also see that $\{T_{i+1}\} \subseteq \{T_i\}$ are subgroups, for all $i \in \mathbb{N}$. The Lie algebra for the group described in Answer 22.2 is thus completely isomorphic to the Lie algebra we are interested in.

▸ **EXERCISE 23**

Check the groups of Exercise 22 and corresponding Lie algebras are solvable.

Answer 23: Recall that if $\mathfrak{g}$ is the Lie Algebra for the group G, then we use the notation from Knapp's *Lie Groups: Beyond An Introduction* (Second Ed.) that

$$\mathfrak{g}^0 = \mathfrak{g}, \qquad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]. \tag{B.24}$$

We say that $\mathfrak{g}$ is "**Solvable**" if $\mathfrak{g}^j = 0$ for some $j \in \mathbb{Z}$. Here we are using notation that

$$[\mathfrak{a}, \mathfrak{b}] = \operatorname{span}\{[X, Y] \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\} \tag{B.25}$$

wbere $\mathfrak{a}$, $\mathfrak{b}$ are subsets of a Lie algebra $\mathfrak{g}$ and we take the span over a field $\mathbb{F}$ (in our case $\mathbb{R}$). A group G is solvable iff it is connected and its Lie algebra is solvable. So we need to check for each group that: 1) it is connected, and 2) its Lie algebra is solvable.

Answer 23.1: We consider $\mathfrak{g}$ the Lie algebra for the group of $n \times n$ upper triangular matrices with real value entries. Let $X, Y \in \mathfrak{g}$, write

$$X = D_X + \widetilde{X}, \qquad Y = D_Y + \widetilde{Y} \tag{B.26}$$

where D_X, D_Y are diagonal matrices, and $\widetilde{X}$, $\widetilde{Y}$ are off-diagonal matrices. We see that the commutator of these two elements are then

$$[X, Y] = [D_X + \widetilde{X}, D_Y + \widetilde{Y}] \tag{B.27a}$$

$$= [D_X, D_Y] + [D_X, \widetilde{Y}] + [\widetilde{X}, D_Y] + [\widetilde{X}, \widetilde{Y}] \tag{B.27b}$$

$$= 0 + 0 + 0 + [\widetilde{X}, \widetilde{Y}] \tag{B.27c}$$

and we can write in block form that

$$[\widetilde{X}, \widetilde{Y}] = a_{ij} = \begin{cases} \neq 0 & \text{if } i + 1 \leq j \\ 0 & \text{otherwise.} \end{cases} \tag{B.28}$$

By induction, we see that elements of $\mathfrak{g}^m$ are matrices of the form

$$a_{ij} = \begin{cases} \neq 0 & \text{if } i + m \leq j \\ 0 & \text{otherwise} \end{cases} \tag{B.29}$$

which is zero for all $m \geq n$.

We need to prove that the group is connected in order to conclude that it is solvable. We can see that given any two matrices X, $Y \in G$ there is a path $\gamma\colon [0, 1] \to G$ connecting them defined by

$$\gamma(t) = tX + (1 - t)Y \tag{B.30}$$

which is always in the group $\forall t \in [0, 1]$. Thus the group is path-connected. This implies that G is connected.

Answer 23.2: We see that the Lie algebra we are working with is actually isomorphic to $\mathfrak{g}^1$ from Answer 2.1, which we saw is solvable. We need to show that the group is connected to prove that it is solvable. Let X, $Y \in G$ be arbitrary group elements. We construct a path $\gamma\colon [0, 1] \to G$ from X to Y defined by

$$\gamma(t) = tY + (1 - t)X. \tag{B.31}$$

We see that $\gamma(t) \in G$ for all $t \in [0, 1]$, which means that the group is path-connected. This implies that the group is connected and, moreover, solvable.

Answer 23.3: We see that the Lie algebra we are working with is, again, isomorphic to $\mathfrak{g}^1$ from Answer 2.1, which we saw is solvable. We need to show that the group is connected, which we will do by proving it is path-connected (a stronger notion!). For T_i, T_j be arbitrary matrices in our group, we construct a path $\gamma\colon [0,1] \to G$ by

$$\gamma(t) = tT_j + (1-t)T_i. \tag{B.32}$$

We see that this path $\gamma(t) \in G$ for all $t \in [0,1]$. This implies path-connectedness and, more importantly, solvability of the group.

Definition B.1. A group is "**Solvable**" if

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\} \tag{B.33}$$

is a tower of groups such that $G_{n-1} \supseteq G_n$ and G_{n-1}/G_n is Abelian.

Definition B.2. A Lie Algebra $\mathfrak{g}$ is "**Solvable**" iff we can find ideals $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1$, ..., $\mathfrak{g}_n = 0$ such that $\mathfrak{g}_i \supseteq \mathfrak{g}_{i+1}$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is Abelian.

N.B.: Recall that an ideal $\mathfrak{h}$ for a Lie algebra $\mathfrak{g}$ satisfies the property that

$$[\mathfrak{h},\mathfrak{g}] = \operatorname{span}\{[X,Y] \mid X \in \mathfrak{h}, Y \in \mathfrak{g}\} \subseteq \mathfrak{h}. \tag{B.34}$$

Proposition B.3. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$, and inductively $\mathfrak{g}^{i+1} = [\mathfrak{g}^i,\mathfrak{g}^i]$. Then $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ is Abelian.*

Proof. It is obvious. We mod out all commutation relations to vanish, which is the necessary and sufficient conditions for a Lie algebra to be Abelian. □

Proposition B.4. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal. Then $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$ is an ideal of $\mathfrak{h}$.*

Proof. It is obvious. □

Proposition B.5. *Let $N \subseteq G$ be a normal Lie subgroup. Then $\mathfrak{n} \subseteq \mathfrak{g}$ is an ideal of the Lie algebra.*

Proof. We see that since N is normal, for any $g \in G$ that $gNg^{-1} \subseteq N$. Consider a curve $\gamma\colon [0,1] \to N$ such that $\gamma(0) = I$ is the identity element. □

Theorem B.6. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If $\mathfrak{g}$ is solvable, then G is solvable.*

Proof. We recall that $\exp\colon \mathfrak{g} \to G$ recovers the Lie group. We see that if we have a tower of ideals

$$\mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_n = \{0\}, \tag{B.35}$$

then by exponentiation we get a tower of normal Lie subgroups

$$G \supseteq G_1 \supseteq \cdots \supseteq G_n = \{\exp(0)\}. \tag{B.36}$$

We also see that proposition B.3 gives us a method to construct a tower of ideals to demonstrate solvability for a Lie algebra. Additionally, if $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ is Abelian, by exponentiation G^i/G^{i+1} is Abelian. This is sufficient to stating that if $\mathfrak{g}$ is solvable, then G is solvable too. □

Remark B.7. The Lie algebra $\mathfrak{sl}(n)$ (also denoted by the symbol A_{n-1}) consists of traceless $n \times n$ complex matrices. The symbol $E_{i,j}$ denotes a matrix with only one nonzero entry that is equal to 1 and located in the i-th row and j-th column.

B.3 Exercises

▸ **EXERCISE 24**

1. Check that the matrices $E_{i,j}$ (for $i \neq j$) and the matrices $h_i = E_{i,i} - E_{i+1,i+1}$ form a basis of $\mathfrak{sl}(n)$.
2. Find the structure constants in this basis.

Answer to 24:
1 We need to check that any matrix $X \in \mathfrak{sl}(n)$ can be written as a linear combination of $E_{i,j}$ and h_i. We see that if we work in the canonical basis of $\mathbb{C}^n$, then we can write out X with components

$$X = D + \widetilde{X} = \delta_{ij}\lambda_i + [\widetilde{x}_{ij}] \tag{B.37}$$

where D is diagonal, $\widetilde{X}$ has the diagonal components identically zero. We see then that we can trivially write out

$$\widetilde{X} = \sum_{i,j} \widetilde{x}_{ij} E_{i,j}. \tag{B.38}$$

So we need to show that we can write out the diagonal part D in terms of h_i.

We see that we can write

$$D = \lambda_1 h_1 + (\lambda_1 + \lambda_2)h_2 + \cdots + (\sum_{j=1}^{k} \lambda_j)h_k + \cdots + (\sum_{j=1}^{n} \lambda_j)h_n \tag{B.39}$$

which permits us to verify that D is indeed traceless, and a linear combination of the basis vectors h_i.

Answer to 24:
2 We see first of all that

$$[h_i, h_j] = 0 \tag{B.40}$$

for all $i, j = 1, ..., n$. It is obvious, since h is diagonal.

We observe that

$$E_{i,j}E_{j,k} = E_{i,k} \quad \Rightarrow \quad E_{i,j}E_{k,l} = \delta_{j,k}E_{i,\,l} \tag{B.41}$$

where $\delta_{j,k}$ is the Kronecker delta, which implies

$$[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{i,l}E_{j,k}. \tag{B.42}$$

This permits us to observe that

$$[E_{i,j}, h_k] = [E_{i,j}, E_{k,k} - E_{k+1,k+1}] \tag{B.43a}$$
$$= [E_{i,j}, E_{k,k}] - [E_{i,j}, E_{k+1,k+1}] \tag{B.43b}$$
$$= (\delta_{j,k}E_{i,k} - \delta_{i,k}E_{j,k}) - (\delta_{j,k+1}E_{i,k+1} - \delta_{i,k+1}E_{j,k+1}) \tag{B.43c}$$

Thus we have the commutation relations for the generators of $\mathfrak{sl}(n)$, which permits us to deduce the structure constants. Observe if we omit the commas in the indices, we can write

$$[E_{ij}, E_{kl}] = f_{ijkl}{}^{ab}E_{ab} \tag{B.44a}$$
$$= (\delta^a_i \delta^b_l \delta_{jk} - \delta^a_j \delta^b_k \delta_{il})E_{ab} \tag{B.44b}$$

which permits us to deduce some of the structure constants. We also have

$$[E_{ij}, h_k] = f_{ijlm}{}^{ab}(\delta^l_k \delta^m_k - \delta^l_{k+1}\delta^m_{k+1})E_{ab} \tag{B.45}$$

and

$$[h_i, h_j] = 0. \tag{B.46}$$

▸ **EXERCISE 25**

1. Check that subalgebra $\mathfrak{h}$ of all diagonal matrices is a maximal commutative subalgebra.
2. Prove that there exists a basis of $\mathfrak{sl}(n)$ consisting of eigenvectors for elements of $\mathfrak{h}$. (This means that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n)$.)

Answer to 25:
1 Let $\mathfrak{h}$ consist of diagonal matrices in $\mathfrak{sl}(n)$. We need to show (a) it is Abelian, (b) it is maximal. We see that

$$[\mathfrak{h},\mathfrak{h}] = \mathrm{Span}\{[X,Y] \mid X,Y \in \mathfrak{h}\} \tag{B.47a}$$
$$= \mathrm{Span}\{0\} \tag{B.47b}$$
$$= 0. \tag{B.47c}$$

Thus it is Abelian and moreover a nilpotent Lie algebra.
We see that

$$N_{\mathfrak{sl}(n)}(\mathfrak{h}) = \{X \in \mathfrak{sl}(n) \mid [X,Y] \in \mathfrak{h} \quad \forall Y \in \mathfrak{h}\} \tag{B.48a}$$
$$= \{E_{ij} \in \mathfrak{sl}(n) \mid [E_{ij},Y] \in \mathfrak{h} \quad \forall Y \in \mathfrak{h}\} \cup \{h_k \in \mathfrak{sl}(n) \mid [h_k,Y] \in \mathfrak{h} \quad \forall Y \in \mathfrak{h}\} \tag{B.48b}$$
$$= \emptyset \cup \{h_k \in \mathfrak{sl}(n) \mid [h_k,Y] \in \mathfrak{h} \quad \forall Y \in \mathfrak{h}\} \tag{B.48c}$$
$$= \emptyset \cup \mathfrak{h} = \mathfrak{h}. \tag{B.48d}$$

That is to say that $\mathfrak{h}$ is self-normalising, so $\mathfrak{h}$ is a Cartan subalgebra.
Since $\mathfrak{h}$ is self-normalising, if $\mathfrak{h}'$ is another Abelian subalgebra, then by definition for each $x \in \mathfrak{h}'$

$$[x,y] \in \mathfrak{h} \tag{B.49}$$

for every $y \in \mathfrak{h}$. This implies that $x \in \mathfrak{h}$ and more importantly $\mathfrak{h}' \subseteq \mathfrak{h}$.

Answer to 25:
2 We observe that

$$[a^i E_{i,i}, E_{j,k}] = (a^j - a^k)E_{j,k}. \tag{B.50}$$

It implies that

$$[a^i h_i, E_{j,k}] = [\widetilde{a}^i E_{i,i}, E_{j,k}] = (\widetilde{a}^j - \widetilde{a}^k)E_{j,k} \tag{B.51}$$

or that $E_{j,k}$ is an eigenvector for $\mathrm{ad}(a^i h_i)$.

▸ **EXERCISE 26**
Check that $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ form a system of multiplicative generators of $\mathfrak{sl}(n)$. Prove relations

$$[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0, \tag{B.52a}$$
$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \tag{B.52b}$$
$$(\mathrm{ad}\, e_i)^{-a_{ij}+1} e_j = 0, \qquad (\mathrm{ad}\, f_i)^{-a_{ij}+1} f_j = 0 \tag{B.52c}$$

for some choice of matrix a_{ij}.
We use here the notation $\mathrm{ad}(x)$ for the operator transforming y into $[x,y]$.

Answer to 26:
Observe that

$$E_{j,k} = \prod_{p=j}^{k} e_p \tag{B.53}$$

assuming that $k > j$ and

$$E_{j,k} = \prod_{p=k}^{j} f_p \tag{B.54}$$

otherwise. So e_i and f_j generate the algebra.

With regards to the commutation relations, we see first that

$$[h_i, h_j] = 0 \tag{B.55}$$

for all i, j since h_i is diagonal and thus commutes with other diagonal matrices.

We also see that

$$e_i f_j = E_{i,i+1} E_{j+1,j} \tag{B.56a}$$
$$= \delta_{i,j} E_{i,i} \tag{B.56b}$$

which implies

$$[e_i, f_j] = \delta_{i,j} E_{i,i} - \delta_{i,j} E_{i+1,i+1} = \delta_{i,j} h_i. \tag{B.57}$$

It follows then from the last homework problem that

$$[h_i, e_j] = 2\delta_{ij} e_j - \delta_{i,j+1} e_j. \tag{B.58}$$

Similarly

$$[h_i, f_j] = -2\delta_{ij} f_j + \delta_{i,j+1} f_j. \tag{B.59}$$

If we write

$$[e_i, e_j] = {c_{ij}}^k e_k, \qquad [f_i, f_j] = {\widetilde{c}_{ij}}^{\,k} f_k \tag{B.60}$$

then we see that if $|j - i| > 1$ then

$$\mathrm{ad}(e_i)(e_j) = 0, \qquad \mathrm{ad}(f_i)(f_j) = 0. \tag{B.61}$$

If $i = j$, then

$$\mathrm{ad}(e_j) e_j = 0, \qquad \mathrm{ad}(f_j) f_j = 0 \tag{B.62}$$

and for $i = j + 1$ we have

$$\mathrm{ad}(e_{j+1}) e_j = 0, \qquad \mathrm{ad}(f_{j+1}) f_j = 0. \tag{B.63}$$

This is precisely as desired.

B.4 Exercises

B.4.1 Algebra D_n

The Lie algebra D_n consists of $2n \times 2n$ complex matrices L obeying

$$(FL)^T + FL = 0 \tag{B.64}$$

where, in block form,

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{B.65}$$

▸ **EXERCISE 27**

Check that D_n is isomorphic to the complexification of the Lie algebra of the orthogonal group $O(2n)$.

Answer to 27:

We see first that we can diagonalize F. Observe that

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \tag{B.66}$$

So we have

$$F = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \tag{B.67}$$

We observe that since we are working with complex matrices, we can write

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} I \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} \tag{B.68}$$

where I is the $2n \times 2n$ identity matrix. Thus we have a mapping

$$\varphi(X) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} X \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{B.69}$$

which maps the condition

$$\varphi\left((FX)^T + FX\right) = \varphi(X)^T + \varphi(X) \tag{B.70}$$

to the condition for $\varphi(X) \in \mathbb{C}\mathfrak{o}(2n)$. We see that this mapping is invertible trivially.

▸ **EXERCISE 28**
Check that the matrices

$$e_{ij} := \begin{bmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{bmatrix} \tag{B.71}$$

together with the matrices

$$f_{pq} := \begin{bmatrix}0 & E_{pq} - E_{qp}\\ 0 & 0\end{bmatrix}, \qquad g_{pq} := \begin{bmatrix}0 & 0\\ E_{pq} - E_{qp} & 0\end{bmatrix} \tag{B.72}$$

form a basis of D_n.

Here $i, j = 1, \ldots, n$, $1 \le p < q \le n$, and $E_{i,j}$ has only one nonzero entry that is equal to unity located in the i^{th} row and j^{th} column.

Answer to 28:
We see that, when written in block form, the condition for D_n implies that

$$F\begin{bmatrix}A & B\\ C & D\end{bmatrix} = \begin{bmatrix}C & D\\ A & B\end{bmatrix} \tag{B.73a}$$

$$\begin{bmatrix}A^T & C^T\\ B^T & D^T\end{bmatrix} F = \begin{bmatrix}C^T & A^T\\ D^T & B^T\end{bmatrix} \tag{B.73b}$$

$$\Rightarrow \quad \begin{bmatrix}C & D\\ A & B\end{bmatrix} = -\begin{bmatrix}C^T & A^T\\ D^T & B^T\end{bmatrix} \tag{B.73c}$$

Thus we see that if $L \in \mathsf{D}_n$, it can be written in block form as

$$L = \begin{bmatrix}A & B\\ C & -A^T\end{bmatrix} \tag{B.74}$$

where A is any $n \times n$ matrix, B, C are antisymmetric $n \times n$ matrices. We see we can write this as

$$L = a^{ij} e_{ij} + b^{pq} f_{pq} + c^{pq} g_{pq} \tag{B.75}$$

where we sum over i, j, p, q.

▸ **EXERCISE 29**
Check that the subalgebra $\mathfrak{h}$ of all matrices of the form

$$\begin{bmatrix}A & 0\\ 0 & -A\end{bmatrix} \tag{B.76}$$

(where A is a diagonal matrix) is a maximal commutative subalgebra, and prove that there exists a basis of D_n consisting of eigenvectors for elements of $\mathfrak{h}$ acting on D_n by means of adjoint representation. (This means that $\mathfrak{h}$ is a Cartan subalgebra of D_n.)

Answer to 29:

Well, we see that $h_i = e_{i,i}$ forms a basis of $\mathfrak{h}$. We know from homework 1 that if we include nonzero off-diagonal components, the algebra is non-Abelian. So every Abelian subalgebra must be generated by some subset of h_i. This means that

$\mathfrak{h}$ is Cartan Subalgebra

$$\mathfrak{h} = \operatorname{span}\{a^i h_i \colon a^i \in \mathbb{C}\} \tag{B.77}$$

is a Maximal Abelian subalgebra.

We see that

Eigenbasis

$$\begin{aligned}
[h_i, e_{jk}] &= h_i e_{jk} - e_{jk} h_i && \text{(B.78a)}\\
&= \begin{bmatrix} E_{i,i} & 0 \\ 0 & -E_{i,i}\end{bmatrix}\begin{bmatrix} E_{jk} & 0 \\ 0 & -E_{kj}\end{bmatrix} - \begin{bmatrix} E_{jk} & 0 \\ 0 & -E_{kj}\end{bmatrix}\begin{bmatrix} E_{i,i} & 0 \\ 0 & -E_{i,i}\end{bmatrix} && \text{(B.78b)}\\
&= \begin{bmatrix} E_{i,i}E_{j,k} - E_{j,k}E_{i,i} & 0 \\ 0 & E_{i,i}E_{k,j} - E_{k,j}E_{i,i}\end{bmatrix} && \text{(B.78c)}\\
&= \delta_{i,j} e_{i,k} - \delta_{k,i} e_{j,i} && \text{(B.78d)}\\
&= (\delta_{i,j} - \delta_{k,i}) e_{j,k}. && \text{(B.78e)}
\end{aligned}$$

Similarly

$$\begin{aligned}
[h_i, f_{pq}] &= \begin{bmatrix} E_{i,i} & 0 \\ 0 & -E_{i,i}\end{bmatrix}\begin{bmatrix} 0 & E_{pq} - E_{qp} \\ 0 & 0\end{bmatrix} - \begin{bmatrix} 0 & E_{pq} - E_{qp} \\ 0 & 0\end{bmatrix}\begin{bmatrix} E_{i,i} & 0 \\ 0 & -E_{i,i}\end{bmatrix} && \text{(B.79a)}\\
&= \begin{bmatrix} 0 & E_{i,i}E_{p,q} - E_{i,i}E_{q,p} \\ 0 & 0\end{bmatrix} - \begin{bmatrix} 0 & -(E_{p,q}E_{i,i} - E_{q,p}E_{i,i}) \\ 0 & 0\end{bmatrix} && \text{(B.79b)}\\
&= \begin{bmatrix} 0 & E_{i,i}E_{p,q} + E_{p,q}E_{i,i} - E_{i,i}E_{q,p} - E_{q,p}E_{i,i} \\ 0 & 0\end{bmatrix} && \text{(B.79c)}\\
&= (\delta_{i,p} + \delta_{i,q}) f_{pq}. && \text{(B.79d)}
\end{aligned}$$

Lastly

$$\begin{aligned}
[h_i, g_{pq}] &= \begin{bmatrix} E_{i,i} & 0 \\ 0 & -E_{i,i}\end{bmatrix}\begin{bmatrix} 0 & 0 \\ E_{pq} - E_{qp} & 0\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ E_{pq} - E_{qp} & 0\end{bmatrix}\begin{bmatrix} E_{i,i} & 0 \\ 0 & -E_{i,i}\end{bmatrix} && \text{(B.80a)}\\
&= \begin{bmatrix} 0 & 0 \\ -E_{i,i}E_{p,q} + E_{i,i}E_{q,p} & 0\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ E_{p,q}E_{i,i} - E_{q,p}E_{i,i} & 0\end{bmatrix} && \text{(B.80b)}\\
&= -(\delta_{i,p} + \delta_{i,q}) g_{p,q}. && \text{(B.80c)}
\end{aligned}$$

Thus e_{jk}, f_{pq}, and g_{pq} form a basis for the Lie algebra, and are weight vectors.

▸ **EXERCISE 30**

Check that $e_i = e_{i,i+1}$ for $i = 1, \ldots, n-1$ and $e_n = f_{n-1,n}$; $f_i = e_{i+1,i}$ for $i = 1, \ldots, n-1$ and $f_n = g_{n-1,n}$ form a system of multiplicative generators of D_n. Prove the relations

$$\begin{aligned}
[e_i, f_j] &= \delta_{ij} h_i && \text{(B.81a)}\\
[h_i, h_j] &= 0 && \text{(B.81b)}\\
[h_i, e_j] &= a_{ij} e_j && \text{(B.81c)}\\
[h_i, f_j] &= -a_{ij} f_j && \text{(B.81d)}\\
(\operatorname{ad} e_i)^{1-a_{ij}} e_j &= 0 \qquad \text{when } i \neq j && \text{(B.81e)}\\
(\operatorname{ad} f_i)^{1-a_{ij}} f_j &= 0 \qquad \text{when } i \neq j && \text{(B.81f)}
\end{aligned}$$

Answer to 30:

It follows immediately from the calculations performed in the answer to 3.

B.4.2 Algebra C_n

Consider the Lie algebra C_n consisting of $2n \times 2n$ complex matrices obeying

$$(FL)^T + FL = 0 \tag{B.82}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{B.83}$$

▸ **EXERCISE 31**

Check that C_n is isomorphic to the complexification of the Lie algebra of the compact group $\mathrm{Sp}(2n) \cap \mathrm{U}(2n)$ where $\mathrm{Sp}(2n)$ stands for the group of linear transformations of $\mathbb{C}^{2n}$ preserving non-degerate anti-symmetric bilinear form and $\mathrm{U}(2n)$ denotes unitary group.

Answer to 31:

We see first of all that we may diagonalize F when we write it as

$$\frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} = F. \tag{B.84}$$

We see that we can construct a morphism

$$\varphi(X) = \frac{1}{2}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} X \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \tag{B.85}$$

which is invertible, whose domain is C_n and whose codomain is precisely the complexified Lie algebra for $\mathrm{U}(2n) \cap \mathrm{Sp}(2n)$.

▸ **EXERCISE 32**

Check that the matrices

$$e_{ij} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix} \tag{B.86a}$$

$$f_{pq} = \begin{bmatrix} 0 & E_{pq} + E_{qp} \\ 0 & 0 \end{bmatrix} \tag{B.86b}$$

$$g_{pq} = \begin{bmatrix} 0 & 0 \\ E_{pq} + E_{qp} & 0 \end{bmatrix} \tag{B.86c}$$

form a basis of C_n, where $i, j = 1, ..., n$ and $1 \leq p \leq q \leq n$.

Answer to 32:

If we write an element of our algebra in block form as

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{B.87}$$

then by our conditions, we deduce

$$FL = \begin{bmatrix} C & D \\ -A & -B \end{bmatrix} \tag{B.88a}$$

$$L^T F = \begin{bmatrix} C^T & -A^T \\ D^T & -B^T \end{bmatrix} \tag{B.88b}$$

which implies that B and C are symmetric, and $-A^T = D$. So in other words we can write

$$L = \begin{bmatrix} A & \frac{1}{2}(B + B^T) \\ \frac{1}{2}(C + C^T) & -A^T \end{bmatrix}. \tag{B.89}$$

However, this permits us to write

$$L = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}(B + B^T) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{2}(C + C^T) & 0 \end{bmatrix} \tag{B.90a}$$

$$= a^{ij} e_{ij} + b^{pq} f_{pq} + c^{pq} g_{pq} \tag{B.90b}$$

which is precisely what we desired to demonstrate.

▸ **EXERCISE 33**
Check that the subalgebra $\mathfrak{h}$ of all matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \tag{B.91}$$

where A is a diagonal matrix, is a maximal commutative subalgebra. Prove there exists a basis of C_n consisting of eigenvectors for elements of $\mathfrak{h}$ acting on C_n by means of adjoint representation.

Answer to 33:
The reasoning for the maximal commutative subalgebra is the same as for the D_n case.

▸ **EXERCISE 34**
Check that $e_i = e_{i,i+1}$ $(i = 1, ..., n-1)$ and $e_n = f_{n,n}$; $f_i = e_{i+1,i}$ $(i = 1, ..., n-1)$ and $f_n = g_{n,n}$ form a system of generators of C_n. Prove

$$[e_i, f_j] = \delta_{ij} h_i, \tag{B.92a}$$
$$[h_i, h_j] = 0 \tag{B.92b}$$
$$[h_i, e_j] = a_{ij} e_j, \tag{B.92c}$$
$$[h_i, f_j] = -a_{ij} f_j, \tag{B.92d}$$
$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad i \neq j \tag{B.92e}$$
$$(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0, \qquad i \neq j \tag{B.92f}$$

Answer to 34:
We see that it follows from the calculations performed in the previous answer to exercise 32.

B.4.3 Algebra B_n

The algebra B_n consists of $(2n+1) \times (2n+1)$ complex matrices obeying

$$L^T F + F L = 0 \tag{B.93}$$

where

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{B.94}$$

is written in block form.

▸ **EXERCISE 35**
Show that B_n is isomorphic to the complexified Lie algebra of $\mathrm{O}(2n+1)$.

Answer to 35:
We construct an isomorphism by diagonalizing F:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{B.95}$$

which allows us to write

$$\varphi(X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{B.96}$$

which clearly is an isomorphism as desired.

▸ **EXERCISE 36**
Check that the subalgebra $\mathfrak{h}$ of all matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -A \end{bmatrix} \tag{B.97}$$

(where A is a diagonal matrix) is a maximal Abelian subalgebra, and prove there is a basis of B_n consisting of eigenvectors for elements of $\mathfrak{h}$ acting on B_n by the adjoint representation.

Answer to 36:
The proof for $\mathfrak{h}$ being a Cartan subalgebra is *ALMOST the same* as for the D_n case. The basic reasoning is the same, we have our beloved isomorphism

$$\varphi\colon \mathsf{B}_n \to \mathbb{C}\,\mathrm{Lie}\big(\mathrm{O}(2n+1)\big) \tag{B.98}$$

which is defined by

$$\varphi(X) = PXP^T \tag{B.99}$$

where P is the orthogonal matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{I}{\sqrt{2}} & \frac{-I}{\sqrt{2}} \\ 0 & \frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{bmatrix} \tag{B.100}$$

where I here is the $n \times n$ identity matrix. The inverse to this morphism would be

$$\varphi^{-1}(Y) = P^T Y P \tag{B.101}$$

and we know the Cartan subalgebra is spanned by $h_i = E_{i,i+1} - E_{i+1,i}$, by applying φ^{-1} to it we deduce that

$$\varphi^{-1}(h_i) = E_{1+i,1+i} - E_{1+n+i,1+n+i} \tag{B.102}$$

up to some change of coordinates by multiplication by i.

▸ **EXERCISE 37**
Find a system e_i, f_j of multiplicative generators of B_n obeying

$$[e_i, f_j] = \delta_{ij} h_i \tag{B.103a}$$
$$[h_i, h_j] = 0 \tag{B.103b}$$
$$[h_i, e_j] = a_{ij} e_j \tag{B.103c}$$
$$[h_i, f_j] = -a_{ij} f_j \tag{B.103d}$$
$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad i \neq j \tag{B.103e}$$
$$(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0, \qquad i \neq j \tag{B.103f}$$

for “some” matrix a_{ij}.

Answer to 37:
We know that $\mathbb{C}\,\mathrm{Lie}\big(\mathrm{O}(2n+1)\big) \cong \mathsf{B}_n$, and that the basis for $\mathbb{C}\,\mathrm{Lie}\big(\mathrm{O}(2n+1)\big)$ consists of $n(2n+1)$ antisymmetric matrices. We observe

$$\varphi^{-1}\left(\begin{bmatrix} 0 & -\vec{u}^T & -\vec{v}^T \\ \vec{u} & 0 & 0 \\ \vec{v} & 0 & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i(\vec{v}-\vec{u})^T & -(\vec{v}+\vec{u})^T \\ i(\vec{v}-\vec{u}) & 0 & 0 \\ (\vec{v}+\vec{u}) & 0 & 0 \end{bmatrix} \tag{B.104}$$

which permits us to deduce how these particular basis vectors transform. The others are remarkably similar to D_n, which calculations have been performed or given.

If we let

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & -D \end{bmatrix} \in \mathfrak{h} \tag{B.105}$$

(for some diagonal $n \times n$ matrix D) and

$$L = \begin{bmatrix} 0 & -\vec{u}^T & -\vec{v}^T \\ \vec{u} & A & \frac{1}{2}(B - B^T) \\ \vec{v} & \frac{1}{2}(C - C^T) & -A^T \end{bmatrix} \in B_n \tag{B.106}$$

(for arbitrary $n \times n$ matrices A, B, C, and n-vectors $\vec{u}$, $\vec{v}$) be arbitrary, then we find

$$[H, L] = \begin{bmatrix} 0 & -\vec{u}^T D & \vec{v}^T D \\ D\vec{u} & [D, A] & \frac{1}{2}\{B - B^T, D\} \\ -D\vec{v} & -\frac{1}{2}\{C - C^T, D\} & [D, A^T] \end{bmatrix} \tag{B.107}$$

where $\{a, b\} = ab + ba$ is the anticommutator. We see that in addition to the basis root vectors given by D_n, we have the additional root vectors

$$\widetilde{e}_i = \begin{bmatrix} 0 & -\vec{u}_i^T & 0 \\ \vec{u}_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{B.108}$$

and

$$\widetilde{f}_i = \begin{bmatrix} 0 & 0 & -\vec{u}_i^T \\ 0 & 0 & 0 \\ \vec{u}_i & 0 & 0 \end{bmatrix} \tag{B.109}$$

where $\{\vec{u}_i\}$ is the canonical basis for $\mathbb{R}^n$. We see that, if

$$h_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_{i,i} & 0 \\ 0 & 0 & -E_{i,i} \end{bmatrix} \in \mathfrak{h} \tag{B.110}$$

then

$$[h_i, \widetilde{e}_j] = \delta_{ij}\widetilde{e}_j \tag{B.111}$$

and

$$[h_i, \widetilde{f}_j] = -\delta_{ij}\widetilde{f}_j \tag{B.112}$$

which tell us the roots corresponding to $\widetilde{e}_j$ and $\widetilde{f}_j$.

▸ **EXERCISE 38**

Describe the roots and root vectors of A_n, B_n, C_n, D_n.

Answer to 38:

We find the roots and root vectors described by the previous exercises for algebras D_n, C_n, and B_n respectively. We also examined the root system for A_n in a previous homework.

We can describe the roots by examining the Cartan matrices of these algebras. These are obtained by the commutation relations, the coefficients a_{ij} are the components of the Cartan matrix. Additionally we have by definition $a_{ij} \leq 0$ for non-diagonal components. With these conditions in mind, we can write the Cartan matrices merely from the results we have already computed. For A_n we have

$$a_{ij} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}. \tag{B.113}$$

For B_n we have

$$a_{ij} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -2 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}. \tag{B.114}$$

For C_n

$$a_{ij} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 2 \end{bmatrix}. \tag{B.115}$$

For D_n

$$a_{ij} = \begin{bmatrix} 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{bmatrix}. \tag{B.116}$$

B.5 Additional Exercises

▸ **EXERCISE 39**

Let V be the fundamental representation of $\mathfrak{sl}(n)$. Let V^* be the representation dual to V. Find the decomposition of $V \otimes V$ and $V \otimes V^*$ into the direct sum of irreducible representations.

Answer to 39:

Well, we first see that if

$$\rho\colon \mathfrak{sl}(n) \to \mathfrak{gl}(V) \tag{B.117}$$

is our representation morphism and

$$\rho^*\colon \mathfrak{sl}(n) \to \mathfrak{gl}(V^*) \tag{B.118}$$

is the dual representation, then we have

$$\rho \otimes \rho\colon \mathfrak{sl}(n) \to \mathfrak{gl}(V) \otimes \mathfrak{gl}(V) \tag{B.119}$$

and

$$\rho \otimes \rho^*\colon \mathfrak{sl}(n) \to \mathfrak{gl}(V) \otimes \mathfrak{gl}(V^*). \tag{B.120}$$

We know that we can decompose

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \Lambda^2(V) \tag{B.121}$$

which intuitively corresponds to writing an arbitrary matrix as the sum of an antisymmetric matrix and a symmetric matrix. We would like to show that this decomposition corresponds to a direct sum of irreps.

We first of all see that if

$$V = \mathrm{span}\{u_1, u_2, ..., u_n\} \tag{B.122}$$

where u_1 is the highest weight vector, $e_1 u_2 = u_1$, and so on, then we can write

$$V \otimes V = \mathrm{Span}\{u_i \otimes u_j \mid i, j = 1, ..., n\}. \tag{B.123}$$

However we note that the decomposition in eq (B.121) amounts to

$$V \otimes V = \underbrace{\mathrm{Span}\left\{\frac{1}{2}\left(u_i \otimes u_j + u_j \otimes u_i\right) \mid i, j = 1, ..., n\right\}}_{=\mathrm{Sym}^2(V)} \oplus \underbrace{\mathrm{Span}\left\{\frac{1}{2}\left(u_i \otimes u_j - u_j \otimes u_i\right) \mid i \neq j, \quad i, j = 1, ..., n\right\}}_{=\Lambda^2(V)}.$$

To show that the representation of the group acting on $V \otimes V$ is the direct sum of two irreducible representations, we will show that the representation has exactly one highest weight vector in $\mathrm{Sym}^2(V)$ and exactly one highest weight vector in $\Lambda^2(V)$. We will also show that the group acts on all basis vectors, which is sufficient to demand that the representations is irreducible (thus implying $\rho \otimes \rho$ is the direct sum of two irreps).

Claim 1: There is exactly one highest weight vector in $\mathrm{Sym}^2(V)$.

Proof. We see that

$$\rho \otimes \rho(h_i)(u_i \otimes u_j) = (\lambda_i + \lambda_j)(u_i \otimes u_j) \tag{B.124}$$

and thus

$$\rho \otimes \rho(h_i)(u_1 \otimes u_1) = 2\lambda_1(u_1 \otimes u_1). \tag{B.125}$$

We see that

$$\rho \otimes \rho(e_i)(u_1 \otimes u_1) = 0 \tag{B.126}$$

for all i, implying that $u_1 \otimes u_1$ is a highest weight vector. There are no others since

$$\rho \otimes \rho(e_i)(u_j \otimes u_k) = \delta_{i,j} u_{j-1} \otimes u_k + \delta_{i,k} u_j \otimes u_{k-1} \tag{B.127}$$

up to some constant, which implies there are no other possibilities for a highest weight vector. □

Claim 2: There is exactly one highest weight vector in $\Lambda^2(V)$.

Proof. We see that similar reasoning holds. More explicitly

$$\rho \otimes \rho(e_i)(u_j \otimes u_k - u_k \otimes u_j) = (\delta_{i,j} u_{j-1} \otimes u_k + \delta_{i,k} u_j \otimes u_{k-1}) - (\delta_{i,j} u_k \otimes u_{j-1} + \delta_{i,k} u_{k-1} \otimes u_j) \tag{B.128}$$

but only

$$\rho \otimes \rho(e_i)(u_1 \otimes u_2 - u_2 \otimes u_1) = 0 \tag{B.129}$$

for all i identically. This is precisely uniqueness of a vector that vanishes when acted upon by e_i for any i, which is the condition for the highest weight vector to be unique. □

Lemma B.8. *If $\rho \colon G \to GL(V)$ is a representation of a Lie group, its induced dual representation for the Lie Algebra is*

$$\rho^*(x) := -\rho(x)^T. \tag{B.130}$$

Proof. We observe that, from Lecture 15, the dual representation of a Lie group is defined to be

$$\rho^*(g) = \rho(g^{-1})^T \tag{B.131}$$

for any $g \in G$. If we take $g = 1 + \varepsilon X$ for an "infinitesimal" ε, then we can write

$$(1 + \varepsilon X)^{-1} = 1 - \varepsilon X \tag{B.132}$$

and

$$\rho^*(1+\varepsilon X) = \rho(1-\varepsilon X)^T. \tag{B.133}$$

By Taylor expanding about 1, we deduce

$$\rho^*(1+\varepsilon X) = \rho(1)^T + \varepsilon\left[-\rho(X)^T\right] \tag{B.134}$$

which permits us to induce a "dual" representation for the Lie Algebra defined by

$$\rho^*(X) = -\rho(X)^T \tag{B.135}$$

precisely as desired. □

Since we are working with the fundamental representation ρ of $\mathfrak{sl}(n)$ we have

$$\rho(e_i)^T = \rho(f_i). \tag{B.136}$$

Additionally, the Cartan subalgebra elements are such that

$$\rho(h_i)^T = \rho(h_i) \tag{B.137}$$

since they are represented by diagonal matrices. So we have

$$(\rho\otimes\rho^*)(e_i)(u_j\otimes u^k) = (\rho(e_i)u_j)\otimes u^k - u_j\otimes(\rho(f_i)u^k) \tag{B.138}$$

where superscript indices here indicate that it is a covector, a linear functional, an element of the dual vector space. We observe

$$\rho(f_i)u_i = u_{i+1}, \qquad \text{and} \quad \rho(e_i)u_i = u_{i-1} \tag{B.139}$$

since we noted in Lecture 16 the weight vectors for the fundamental representation is nothing more than the canonical basis. This permits us to deduce

$$(\rho\otimes\rho^*)(e_i)(u_1\otimes u^n) = 0 \tag{B.140}$$

for all i, which means $u_1\otimes u^n$ is a weight vector for this representation. We also see that

$$(\rho\otimes\rho^*)(e_i)(u_j\otimes u^j) = 0 \tag{B.141}$$

identically, since

$$(\rho\otimes\rho^*)(e_i)(u_j\otimes u^k) = (\lambda_j-\lambda_k)(u_j\otimes u^k). \tag{B.142}$$

Setting $j=k$ we find the right hand side vanishes *identically.*

▸ **EXERCISE 40**

Prove the fundamental represenation of $\mathfrak{so}(n)$ is equivalent to dual representation. Find the decomposition of the tensor square of this representation into direct sum of irreducible representations.

Claim 1: The fundamental representation of $\mathfrak{so}(n)$ is equivalent to its dual representation.

Proof. Well, we know the dual representation for $\rho\colon G\to GL(V)$ is

$$\rho^*(g) = \rho(g^{-1})^T. \tag{B.143}$$

By making $g = 1+\varepsilon X$ where ε is "infinitesimal," we get by Taylor expanding about 1

$$\rho(1-\varepsilon X)^T = \rho(1) + \varepsilon\left(-\rho(X)^T\right) \tag{B.144}$$

which permits us to deduce for a Lie algebra, the dual representation for an element X is

$$\rho^*(X) = -\rho(X)^T. \tag{B.145}$$

We know for the fundamental representation of $\mathfrak{so}(n)$ we are working with $n\times n$ antisymmetric matrices. However, this means that for the fundamental representation ρ of $\mathfrak{so}(n)$

$$\rho(X)^T = -\rho(X) \tag{B.146}$$

for all $X \in \mathfrak{so}(n)$, which implies

$$\rho(X) = -\rho(X)^T = \rho^*(X) \tag{B.147}$$

precisely as desired. □

Proposition B.9. *The Cartan subalgebra is*

$$\mathfrak{h} = \mathrm{span}\{\sum_{j=1}^{n} x_j(E_{2j,2j-1} - E_{2j-1,2j})\} \tag{B.148}$$

and the weight vectors are, for the fundamental representation

$$\vec{u}_j = \vec{e}_{2j} + i\vec{e}_{2j-1} \tag{B.149}$$

for $j = 1, ..., n$.

Proof. It follows from the definition of the maximal torus that eq (B.148) holds. The eigenvalues for the 2×2 matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{B.150}$$

is ± 1 with eigenvectors $(i, 1)$ and $(-i, 1)$ for $-i$, i respectively. Thus we deduce the weight vectors to be as desired. □

Proposition B.10. *We deduce that*

$$[h_j, \lambda^k(E_{k,2j} - E_{2j,k}) + \mu^\ell(E_{\ell,2j-1} - E_{2j-1,\ell})] = -\lambda^k(E_{k,2j-1} - E_{2j-1,k}) + \mu^\ell(E_{\ell,2j} - E_{2j,\ell}). \tag{B.151}$$

Proof. We find this by direct calculation. □

We can further compute how these terms act on the weight vectors. We observe

$$\lambda^k(E_{k,2j} - E_{2j,k})\vec{u}_j = \lambda^k\vec{e}_k - i\lambda^k\delta_{k,2j-1}\vec{e}_{2j} \tag{B.152a}$$
$$\mu^\ell(E_{\ell,2j-1} - E_{2j-1,\ell})\vec{u}_j = i\mu^\ell\vec{e}_\ell - \mu^\ell\delta_{\ell,2j}\vec{e}_\ell. \tag{B.152b}$$

Thus we observe that when $\ell = 2j + 1$ and $k = 2j + 2$, we get

$$\lambda^k(E_{k,2j} - E_{2j,k})\vec{u}_j + \mu^\ell(E_{\ell,2j-1} - E_{2j-1,\ell})\vec{u}_j = \vec{u}_{j+1}. \tag{B.153}$$

We also observe that when $\ell = 2j - 3$ and $k = 2j - 2$, we get

$$\lambda^k(E_{k,2j} - E_{2j,k})\vec{u}_j + \mu^\ell(E_{\ell,2j-1} - E_{2j-1,\ell})\vec{u}_j = \vec{u}_{j-1}. \tag{B.154}$$

Thus we have found our raising and lowering operators e_j and f_j respectively.

But notice since the representation is "self-dual", we have the decomposition of

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \Lambda^2(V). \tag{B.155}$$

Thus we have two irreps, one acting on $\mathrm{Sym}^2(V)$ and the other acting on $\Lambda^2(V)$. We can observe the following thing: a symmetric matrix X is such that

$$\mathrm{Tr}(PXP^T) = \mathrm{Tr}(X) \tag{B.156}$$

for $P \in \mathrm{O}(n)$. So, we can consider traceless symmetric matrices X which forms an invariant subspace of $\mathrm{Sym}^2(V)$. Additionally, matrices of the form cI where $c \in \mathbb{R}$ is some constant, also obey

$$P(cI)P^T = c(PIP^T) = c(PP^T) = cI \tag{B.157}$$

for $P \in O(n)$. Note these are group elements and the group representation acting on $V \otimes V$, which permit us to find invariant subspaces and thus subrepresentations for the group which correspond to subrepresentations of the Lie algebra.

So the invariant subspaces are three:

1. $\Lambda^2 V$ the antisymmetric part;
2. $\mathrm{Span}\{I\}$ the scalar matrices; and
3. $\{X \in \mathrm{Sym}^2 V \mid \mathrm{Tr}(X) = 0\}$ traceless matrices.

Each of these are irreducible, since there is precisely one highest weight vector for each of them.

▸ **EXERCISE 41**

(Schur's lemma) Let us consider a complex irreducible representation φ of Lie algebra $\mathscr{G}$. Let us assume that the operator A in the representation space commutes with all operators of the form $\varphi(x)$ where $x \in \mathscr{G}$. Prove that $A = \lambda \cdot 1$ where $\lambda \in \mathbb{C}$ and 1 stands for the identity operator.

Hint: Consider $\mathrm{Ker}(A - \mu \cdot 1)$.

Answer to 41:

We will first make a small lemma.

Lemma B.11. *Let $\varphi\colon \mathscr{G} \to V$ be an irreducible representation of the Lie algebra $\mathscr{G}$, and $L\colon V \to V$ be a linear mapping. Then either L is an isomorphism, or it is zero.*

Proof. Consider $\mathrm{Ker}(L)$. It would be an invariant subspace of V, but since φ is irreducible this subspace is either V or 0. For nonzero L, we have $\mathrm{Ker}(L) = 0$. This implies that L is injective. But since we have an injective endomorphism, thus it is surjective and moreover bijective. We have L be an isomorphism or, alternatively, 0. □

Consider A. It has at least 1 nonzero eigenvalue λ, or it is the zero mapping necessarily (in which case, $A = 0 \cdot I$). We see that $A - \lambda \cdot 1$ is not an isomorphism, by our lemma since its kernel is all of V it follows that $A = \lambda \cdot 1$ for some $\lambda \in \mathbb{C}$.

▸ **EXERCISE 42**

Let us consider a Lie algebra $\mathscr{G}$ with basis e_1, ..., e_n and structure constants ${c^k}_{ij}$:

$$[e_i, e_j] = {c^k}_{ij} e_k. \tag{B.158}$$

We define universal enveloping algebra $U(\mathscr{G})$ as a unital associative algebra with generators e_i and relations

$$e_i e_j - e_j e_i = {c^k}_{ij} e_k. \tag{B.159}$$

Let us assume that there exists an invariant inner product on $\mathscr{G}$ and that the basis e_1, ..., e_n is orthonormal with respect to this product. Prove that that the element

$$\omega = \sum_i e_i e_i \tag{B.160}$$

(Casimir element) belongs to the center of enveloping algebra.

Hint. It is sufficient to check that Casimir element commutes with all generators. Write the condition for $e_k \omega = \omega e_k$ in terms of structure constants and check that the same condition guarantees invariance of inner product.

Answer to 42:
Let B be a nondegenerate symmetric invariant bilinear form on $\mathscr{G}$. Let x_1, ..., x_n be the basis for $\mathscr{G}$, x^1, ..., x^n be the dual basis so

$$B(x_i, x^j) = \delta_i{}^j. \tag{B.161}$$

Let $z \in \mathscr{G}$, we have

$$[z, x_j] = \sum_i a^i{}_j x_i = a^i{}_j x_i \tag{B.162}$$

and

$$[z, x^j] = \sum_i b_i{}^j x^i = b_i{}^j x^i \tag{B.163}$$

where $a_i{}^j$ and $b_i{}^j$ are "some constants." Since B is invariant we have

$$0 = B([z, x_i], x^j) + B(x_i, [z, x^j]) = a_i{}^j + b_i{}^j. \tag{B.164}$$

We see

$$z(x_i x^i) = [z, x_i]x^i + x_i z x^i \tag{B.165a}$$
$$= \sum_j a_i{}^j x_j x^i + x_i z x^i \tag{B.165b}$$

and

$$(x_i x^i)z = (x_i[x^i, z]) + x_i z x^i = -\sum_j b_i{}^j x_i x^j + x_i z x^i \tag{B.166}$$

Thus we find

$$[z, x_i x^i] = \sum_j a_i{}^j x_j x^i + b^i{}_j x_i x^j = \sum_j (a_j{}^i + b^i{}_j) x_i x^j \tag{B.167}$$

but due to the invariance of B, we see that the parenthetic term must be zero (we sum over dummy indices, which we may rewrite as we like). This implies

$$[z, x_i x^i] = 0 \tag{B.168}$$

for any $z \in \mathscr{G}$.

▸ **EXERCISE 43**
A representation $\varphi\colon \mathscr{G} \to \mathfrak{gl}(n)$ can be extended to a homomorphism of universal enveloping algebra to the algebra of $n \times n$ matrices. If the representation is irreducible the image of the Casimir element has the form $\lambda \cdot 1$. Show that this follows from Schur's lemma.

Answer to 43:
We have that the Casimir element be denoted by ω. In the universal enveloping algebra, it is an $n \times n$ matrix. It has at least one nonzero eigenvalue $\lambda \in \mathbb{C}$ or it is the zero matrix. By Lemma B.11 we have for $\omega - \lambda \cdot 1$ its kernel is either 0 or all of $\mathbb{C}^n$. If

$$\mathrm{Ker}(\omega - \lambda \cdot 1) = \mathbb{C}^n \tag{B.169}$$

then

$$\omega = \lambda \cdot 1 \tag{B.170}$$

by Schur's lemma. Otherwise it is the zero matrix.

▸ **EXERCISE 44**
Calculate the image of Casimir element for an irreducible representation of the Lie algebra $\mathfrak{sl}(2)$.

Answer to 44:

Let

$$\rho\colon \mathfrak{sl}(2) \to \mathfrak{gl}(V) \tag{B.171}$$

be an irrep. We have the dual basis for e, f, h in $\mathfrak{sl}(2)$ be $2f$, $2e$ and h respectively. By our previous calculations, it follows that

$$\omega = H^2 + 2EF + 2FE \tag{B.172}$$

where $F = \rho(f)$, $E = \rho(e)$, $H = \rho(h)$.

Observe if $\vec{v}$ is the highest weight vector for the irrep in eq (B.171), and λ is the corresponding highest weight, we have

$$\begin{aligned} \rho(h^2)\vec{v} &= \rho(h)^2\vec{v} && \text{(B.173a)} \\ &= \lambda(h)^2\vec{v}. && \text{(B.173b)} \end{aligned}$$

Similarly, we see

$$\begin{aligned} \rho(ef + fe)\vec{v} &= (\rho(e)\rho(f) + \rho(f)\rho(e))\vec{v} && \text{(B.174a)} \\ &= \rho(e)\rho(f)\vec{v} + \rho(f)\left(\rho(e)\vec{v}\right) && \text{(B.174b)} \\ &= \rho(e)\rho(f)\vec{v} + 0 && \text{(B.174c)} \\ &= \rho(e)\rho(f)\vec{v} - 0 && \text{(B.174d)} \\ &= \rho(e)\rho(f)\vec{v} - \rho(f)\left(\rho(e)\vec{v}\right) && \text{(B.174e)} \\ &= (\rho(e)\rho(f) - \rho(f)\rho(e))\vec{v} && \text{(B.174f)} \\ &= [\rho(e), \rho(f)]\vec{v} && \text{(B.174g)} \\ &= \rho(h)\vec{v}. && \text{(B.174h)} \end{aligned}$$

Thus the Casimir acting on the highest weight vector is

$$\omega\vec{v} = (\rho(h)^2 + \rho(h))\vec{v} = \lambda(h)\left(1 + \lambda(h)\right)\vec{v}. \tag{B.175}$$