Algebraic Topology Notes: Homotopy Theory

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Abstract

Notes on homotopy theory, the first part of a trilogy on algebraic topology. Any typos, errors, mistakes, gaffes, etc., are entirely my folly.

Contents

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Part I
Homotopy Theory

Lecture 1.

The goals today are to explain: what is a topology, and why is it useful to know topology?

What is topology? The first notion is topological equivalence, or equivalence of two spaces, i.e., a homeomorphism. If we have X, Y be topological spaces, then a topological equivalence is a map

$$f\colon X \to Y \tag{1.1}$$

that is a one-to-one correspondence, f and f^{-1} are both continuous. But what kind of spaces are these? The most general situation is when X is a topological space, where we have a notion of an open set.

One of the main questions of topology is: given two spaces, are they equivalent? If we can construct a homeomorphism, we got it. What about if they're nonequivalent? We need to use properties called "**Topological Invariants**". If two spaces are topologically equivalent, then the topological invariants of the two spaces are the same.

Topological Invariant = Property Invariant under Homeomorphisms

The basic topological invariant for a space is connectedness. We will use the notion of pathwise connected. A space is pathwise connected if and only if any two points are connected by a path. What is a path? Well, it is a parametrized curve

$$\gamma\colon [0,1] \to X \tag{1.2}$$

such that $\gamma(0) = x$ and $\gamma(1) = y$, and γ is continuous.

We can consider the connected component of X. We can consider $x \sim y$ iff there is a path connecting the points. We see that the letter "A" is connected whereas "i" is disconnected (it has two components), so they are not topologically equivalent.

There is a simple idea regarding ho to construct a topological invariant given some topological invariant. We consider a functor F which is such that

$$X \sim Y \quad\implies\quad F(X) \sim F(Y) \tag{1.3}$$

We can consider, for example, the construction

$$F(-) = \operatorname{Hom}(A, -) \tag{1.4}$$

for some fixed topological space A. We may construct topological invariants this way. For example the number of connected components of $\operatorname{Hom}(A, X)$ gives an invariant of X.

We may consider topological spaces with a marked point (X, x_0) where $x_0 \in X$. We may consider

$$f\colon (X, x_0) \to (Y, y_0) \tag{1.5}$$

such that $f(x_0) = y_0$ preserves the marked point. The most interesting object of this kind is obtained in the following way: take a sphere with a marked point, take maps of this sphere to some other pointed space

$$\operatorname{Hom}\left((S^n, s_0), (X, x_0)\right) \tag{1.6}$$

which is a topological space with marked point (X, x_0). We consider the components of this space, this set has a group structure which we will call $\pi_n(X)$ called the "**Homotopy Group**". For $n = 1$ we get the fundamental group.

Some other invariants we will consider later are the Euler characteristic. We may decompose X into the disjoint union of open balls

$$X = \bigsqcup (\text{open balls}) \tag{1.7}$$

For example the sphere is equivalent to a point and the remainder is homologically equivalent to an open disc. So the Euler characteristic is then

$$\chi(X) = \sum (-1)^n \alpha_n \tag{1.8}$$

where α_n is the number of open n-balls. So

$$\chi(X \sqcup Y) = \chi(X) + \chi(Y) \tag{1.9}$$

This sum should not depend on how we decompose the space. To give a proper definition of the Euler characteristic, we need to use homology.

Topology may be applied to more-or-less everywhere and everything. That doesn't mean it answers every question. But we should ask ourselves "What is topology saying about this?"

Historically the first application was to study integrals. For example, Stoke's formula, Green's formula, etc., are of the form

$$\int_{\partial S} \omega = \int_S \mathrm{d}\omega, \tag{1.10}$$

so the notion of an integral is closely related to the notion of the boundary of some surface.

The homology is closed surfaces mod boundaries, meh we are sloppy here.

At any rate, integers are relevant to topology, viz. in $\mathbb{C}$. Another thing we'd like to mention is the application of topology to the study of $\vec{f}(\vec{x}) = \vec{0}$. The first question is how many solutions do we have? Topology cannot say the number of solutions, but it can tell us the *algebraic number* of solutions. For example, when considering $y = f(x)$ where

$$\lim_{x\to-\infty} f(x) < 0 \tag{1.11a}$$

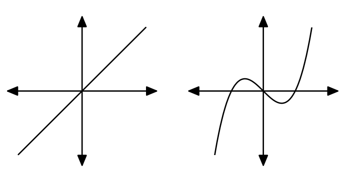
and

$$\lim_{x\to\infty} f(x) > 0 \tag{1.11b}$$

then there are an odd number of points $x_1, \ldots, x_{2n+1}$ such that

$$f(x_i) = 0 \tag{1.12}$$

This is topological, and looks like:



etc.

The last thing to mention is the calculation of the index of A, a Fredholm operator. We consider $\mathrm{Ker}(A)$ and assume $\dim(\mathrm{Ker}(A))$ is finite. We can consider

$$A\colon E \to E \tag{1.13}$$

then

$$\mathrm{Coker}(A) = \mathrm{Im}(A)/\mathrm{Ker}(A) \tag{1.14}$$

and $\dim(\mathrm{Coker}(A))$ is finite. The difference between these finite numbers is precisely the "**Index**" of the operator.

Topology has very important applications in physics. Namely, one of the ways is when we work with fields. We can consider the space of all fields (possibly with some restrictions, e.g. with finite energy). It is possibly disconnected. Foe example, in classical mechanics

$$V(x) = x^4 \tag{1.15}$$

the space of solutions is disconnected. But to show this, we need the homotopic group of the space. There are other applications of topology in physics, e.g., TQFT.

There are other topological applications, e.g., in the calculus of variation we may compute the number of critical points topologically.

Lecture 2.

We will follow Hatcher's book, and Schwarz's *Topology for Physicists*.

The first thing to discuss is topological spaces. We have a set E and a notion of an open set of E. We have a collection of open subsets of E, $\{U\}$ where

$$\bigcup U = E \tag{2.1a}$$

and

$$\bigcap_{\text{finite}} U \text{ is open} \tag{2.1b}$$

It is not enough to say

$$\bigcup (\text{open}) = (\text{open}) \tag{2.2a}$$

and

$$\bigcap_{\text{finite}} (\text{open}) = (\text{open}) \tag{2.2b}$$

We have closed sets be the complement of open sets. There is a requirement that E is both open and closed, which then implies that $\emptyset$ is both open and closed.

We can define a continuous function $f\colon E \to E'$ such that the preimage of open sets is open, i.e.,

$$f^{-1}(\text{open}) = \text{open}. \tag{2.3}$$

Box 1. Functorial view of Topology, Continuous Functions

This may seem odd at first why continuous functions obey this pre-image condition. There are a variety of explanations out there, but I prefer this explanation. Consider the category **Set**. Let

$$\mathrm{Hom}(-, \mathbf{2})\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set} \tag{2.4}$$

be the contravariant power set functor. So in other words, we have

$$\mathrm{Hom}(X, \mathbf{2}) = \begin{pmatrix}\text{set of indicator}\\ \text{functions for subsets}\\ \text{of the set } X\end{pmatrix} \tag{2.5}$$

We construct a topology by picking a subset of this collection of subsets $\mathrm{Hom}(X, \mathbf{2})$ which obey the axioms for a topology. That is, we have $T \subseteq \mathrm{Hom}(X, \mathbf{2})$ be a topology of X. That is to say, T consists of the indicator functions for open subsets of X. It is a structure-type. A topological space is then (X, T).

But note that we functor, so we have the immediate question:

Question. How does $\mathrm{Hom}(X \xrightarrow{f} Y, \mathbf{2})$ behave?

If we can answer this question, then we will have some idea of what a "topological-space morphism" would be like. Why? Because we just restrict focus to the functions preserving the "topological structure" $T \subseteq \mathrm{Hom}(X, \mathbf{2})$.

We should recall from our knowledge of category theory that the functor $\mathrm{Hom}(-, B)$ behaves on morphisms in the following manner: $\mathrm{Hom}(-, B)$ maps each morphism $h\colon X \to Y$ to the function $\mathrm{Hom}(h, B)\colon \mathrm{Hom}(Y, B) \to \mathrm{Hom}(X, B)$ given by $g \mapsto g \circ h$ for each g in $\mathrm{Hom}(Y, B)$.

What does this mean for our situation? Well, for each $f\colon X \to Y$ it is mapped to the function

$$\mathrm{Hom}(f,\mathbf{2})\colon\ \mathrm{Hom}(Y,\mathbf{2}) \to \mathrm{Hom}(X,\mathbf{2}) \tag{2.6}$$

given by $f \mapsto h\circ f$ where $h \in \mathrm{Hom}(Y,\mathbf{2})$. So h is really an indicator function of an open subset of Y. This is precisely the same condition as saying the preimage $f^{-1}(\text{open})$ is open. For a brief introduction to topology using this approach, see Nelson [7].

Usually we use the Hausdorff condition that two distinct points are contained in two disjoint neighborhoods.

There is a topological property of "**Compactness**" where every open covering has a finite subcovering.

There is one more thing that is relevant. What can we do with equivalence relations on topological spaces? We can consider equivalence classes $E/\sim$. There is a natural map

$$\pi\colon E \to E/\sim \tag{2.7}$$

What happens if E is a topological space, then we would like to have $E/\sim$ be a topological space and the map π to be continuous, i.e., the preimage $\pi^{-1}(\text{open})$ is open. We are saying $U \subset E/\sim$ is open iff the preimage $\pi^{-1}(U)$ is open in E. If the preimage of an open set is open, then the preimage of a closed set is closed. We see for a singleton $a \in E/\sim$ then the preimage $\pi^{-1}(a)$ is an equivalence class. We have the singletons be closed, so we require these equivalence classes be closed to avoid pathology.

List of topological spaces

Now why are we so interested in this construction? Because we want to have a construction of interesting topological spaces. We have some simple interesting topological spaces. What are they? First of all, $\mathbb{R}^3$ the space that surrounds us. More generally $\mathbb{R}^n$. Another interesting space is a ball

$\mathbb{R}^n$

$$\overline{D}^n = \{\vec{x} \in \mathbb{R}^n \ \mid\ \|\vec{x}\| \leq 1\} \tag{2.8}$$

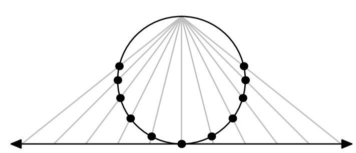
Closed ball $\bar{D}^n$

which is closed of radius 1. The radius doesn't change anything, balls of different radius are topologically equivalent. For example $x \mapsto \lambda x$ for $\lambda > 0$ is the topological equivalence. We won't repeat the definition of "topological equivalence" the curious reader may look it up. Another interesting space is the open ball

Open ball D^n

$$D^n = \{\vec{x} \in \mathbb{R}^n \ \mid\ \|\vec{x}\| < 1\} \tag{2.9}$$

We use the notation $\overline{D}^n$ to stress it is the closure of D^n. It's an interesting space, perhaps it is equivalent to $\overline{D}^n$? No! Why? Well, we see that D^n is not compact but $\overline{D}^n$ is compact, so they cannot be topologically equivalent.



But is $\mathbb{R}^n$ topologically equivalent to D^n? Yes, we can see this for $n = 1$, we use the stereographic projection which gives us a one-to-one correspondence between $S^1 - \{0\}$ and $\mathbb{R}^1$. We can take $n = 2$ and nothing conceptually changes. The same is true for $S^n - \{0\} \cong \mathbb{R}^n$. But we may say that $S^n - \{0\} \cong D^n$.

We would like to stress that the n-dimensional sphere is *not* a sphere in n-dimensional space. No, it is instead living in $\mathbb{R}^{n+1}$. In $\mathbb{R}^n$, the sphere is characterized by the points $\vec{x} \in \mathbb{R}^n$ satisfying

$$\|\vec{x}\| = 1 \tag{2.10}$$

which is an $(n-1)$-sphere. We have

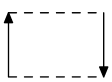
$$\overline{D}^n = S^{n-1} \cup D^n \tag{2.11}$$

where D^n are the interior points and S^{n-1} is the boundary.

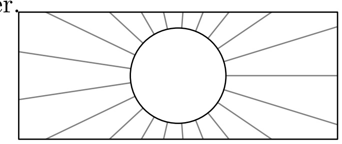
Now what we would like to say is that, more or less, all interesting spaces may be constructed from the simple spaces D^n, $\overline{D}^n$. We define a very general construction, namely given a topological space X, and a topological space Y, a closed subset $A \subset Y$, we'd like to paste together X and Y along A. What does this mean? We take any continuous map

$$f\colon A \to X \tag{2.12}$$

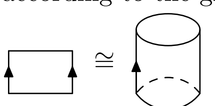
take the disjoint union $X \sqcup Y$, and then in this disjoint union introduce an equivalence relation that any $a \in A \subset Y \sim f(a) \in X$ and no other equivalences!



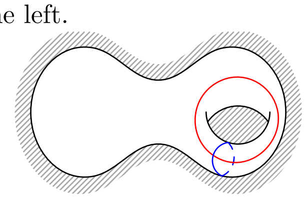
We require $a \in A \subset Y \sim f(a) \in X$ is the only nontrivial equivalence. Lets consider some examples. The Mobius band can be defined in this way: take a rectangle (which is topologically equivalent to D^2) and we consider the equivalence relation that the two arrows are pasted together.



We see that a rectangle is equivalent to a disc since both are convex and stretch the boundary to be a rectangle which permits us to formally write this equivalence but that won't be necessary. We can stretch according to the gray lines doodled to the right.



So more examples. We take the same rectangle, and paste together points at the same height. This topologically is equivalent to a cylinder. This is obvious, as we see in the doodle to the left.



But if we take our cylinder, and glue the two ends to each other without any twisting, what do we get? Well, we have a torus. This is doodled on the right hand side, very carefully, with colors to show where we glued the rectangle together. The red line indicates where we glued the rectangle to obtain a cylinder, and the blue line indicates where we glued the cylinder to obtain a torus. Do we really need all this information? Is there some easier diagram which yields the relevant data? Or are we forced to become artists to understand the topological properties of these exotic spaces?

There is a very general construction of something called a "**Cell Complex**", we will first describe it. Take a closed ball and some topological space X. Now we will take any continuous map

$$f\colon S^{n-1} \to X \tag{2.13a}$$

or in other words

$$f\colon \partial\overline{D}^n \to X \tag{2.13b}$$

and then we use the construction we just explained. That is, we glue a closed ball along its boundary to X. We get a new set

$$Y = X \cup \overline{D}^n \tag{2.14a}$$

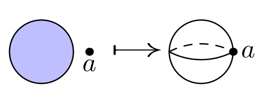
or as sets

$$Y = X \sqcup \overline{D}^n \tag{2.14b}$$

The simplest posssible case is when X is just a one point space

$$X = \{a\} \tag{2.15}$$

the boundary of the ball goes to a. This is a trivial map. We see in $n = 2$ what do we get with identifying the boundary to a? Look at the stereographic projection backwards.



This general construction, gluing the boundary of closed n-balls to a topological space (starting with $n = 0$, i.e., a set of vertices to begin with), gives us a cell complex.

Definition 2.1. The "n**-Dimensional Cell Complex**" can be done inductively by assuming we have the $(n-1)$-dimensional cell complex denoted X^{n-1} called the "$(n-1)$**-Skeleton**", now we have k-copies of n-discs and perform the same construction. We end up with a sequence of skeletons $X^0 \subset X^1 \subset X^2 \subset \ldots$, if we consider $X^n - x^{n-1} = \bigsqcup_k D^n_k$.

Lecture 3.

We have a sequence of sets X^k called a "k**-Dimensional Skeleton**". We get a $(k+1)$-dimensional skeleton by taking several closed balls $\overline{D}^{k+1}_i$ and take

$$f_i \colon S^k \to X^k \tag{3.1}$$

to paste $\partial\overline{D}^k_i$ to the k-dimensional skeleton. If we consider only the open balls, then

$$X^{k+1} = X^k \sqcup D^k_1 \sqcup \cdots \sqcup D^k_n \tag{3.2}$$

(NB: this is the *disjoint* union!) The skeleton is closed

$$\overline{X^k} = X^k \tag{3.3}$$

This is more or less the definition given, which is fine for the finite-dimensional case. In general, we should note exclude the case when $k \in \mathbb{N}$, then the cell complex is the union of all the skeletons:

$$X = \bigcup_{k\in\mathbb{N}} X^k \tag{3.4}$$

But we should think about the topology. We simply say that

$$U \subset X \text{ is open if } U \cap X^k \text{ is open } \forall k. \tag{3.5}$$

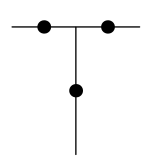
That's reasonable. We impose the condition that the embedding

$$X^k \hookrightarrow X \tag{3.6}$$

is continuous. The preimage of an open set $U \subset X$ is then such that $U \cap X^k$ is open too. That is, $U \subset X$ is open if and only if $U \cap X^k$ is open. This means we take the weakest possible topology in X... well, the weakest one satisfying the requirement $U \subset X$ is open $\iff U \cap X^k$ is open. Lets consider some examples.

Example 3.1. Zero dimensional cell complexes are just collections of vertices. One-dimensional cell complexes are called "**Graphs**". We have $\overline{D}^1$ be a line interval, the endpoints are the vertices in the X^0 skeleton. There are some cases when the graphs are topologically equivalent, e.g., E and T are topologically equivalent.

We can prove some guys are topologically nonequivalent. The number of components is a topological invariant, so the letter "i" is not equivalent to any uppercase letter. We introduce a notion of the "**Degree of a Vertex**" which is the number of edges going into the vertex, which is a bit ambiguous. But the degree of a vertex is a topological invariant. Why? Because we can give it a topological definition. The graph for T as doodled on the right, we can pick three points of the graph such that

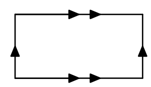


$$\text{graph} - (3 \text{ points}) = U \sqcup V \tag{3.7}$$

where U and V are open sets. Degree should be a local notion, defined by the behavior of the graph in a neighborhood.

We will say the degree of a vertex is less than or equal to k if in any neighborhood of the vertex we can find k points such that after deleting our points, the connected component of the vertex is smaller than the union of the edges coming into this vertex. In graph theory, this operation of removing vertices in this manner is called a "**Cut**" (or "*Vertex Cut*").

What is important is that degree of a vertex is defined in terms of connectedness. Although this notion of "connectedness" is a topological notion, Graph theorists use the confusingly obtuse term "**Vertex Connectivity**". So vertices of degree k must be mapped to vertices of degree k. Thus $A \ncong T$ is not a topological equivalence of graphs. If we cut any point from T, it's disconnected. However, if we cut a point in A while retaining connectedness, it becomes topologically equivalent to H — any other cut renders it disconnected.



Consider the cell complex for the torus, as doodled on the right. We see that the skeleton can be described by a rectangle as its only cell in 2-dimensions.

For a sphere we can consider it as many different cell complexes. For example, we can construct the sphere by

$$D^2/\partial D^2 \cong S^2 \tag{3.8}$$

which has a single 2-cell, and a single vertex. On the other hand, if we take two discs D_0^2 and D_1^2, then consider

$$(D_0^2 \sqcup D_1^2)/(\partial D_0^2 \sim \partial D_1^2) \cong S^2 \tag{3.9}$$

we have 2 vertices, 2 edges, and 2 faces.

Recall we discussed the Euler characteristic. We may define it as

$$\chi(X) = \sum (-1)^n \alpha_n \tag{3.10}$$

where α_n is the number of n-cells. This is a topological invariant, it is the simplest one. It obeys

$$\chi(A \sqcup B) = \chi(A) + \chi(B) \tag{3.11}$$

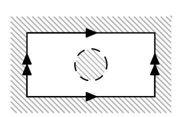
So

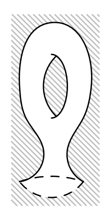
$$\begin{aligned}\chi(\text{torus}) &= (1 \text{ vertex}) - (2 \text{ edges}) + (1 \text{ face}) \\ &= 0.\end{aligned} \tag{3.12}$$

Observe for the sphere we have

$$\begin{aligned}\chi(\text{sphere}) &= (2 \text{ vertices}) - (2 \text{ edges}) + (2 \text{ faces}) \\ &= 2.\end{aligned} \tag{3.13}$$

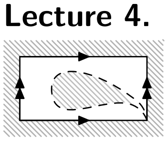
Quite simple!





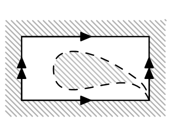
We consider a handle, obtained by taking a torus and deleting an open ball. It is an example of a manifold/surface with boundary. We may take a sphere and delete several discs. We paste in each cut-out disc a handle.

Lecture 4.



The last thing we did was consider something called a "handle". It is a torus, but we cut out a hole. We take our hole anywhere we'd like. So lets have it be touching a vertex, as doodled on the left.

Now what we claim is that we have an equality of cell complexes:



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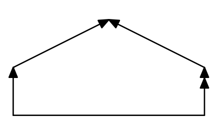
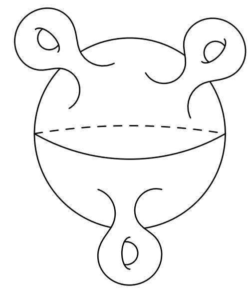


(4.1)

We obtain the right hand side through a cut. But is a cut allowed? Usually not, but since all the vertices are the same so it's okay here. We identify the base points to be the same. We also consider the Euler characteristic. Observe

$$\begin{aligned}\chi(\text{handle}) &= (1\text{ vertex}) - (3\text{ edges}) + (1\text{ face})\\ &= -1.\end{aligned} \tag{4.2}$$

This is just the first step in our considerations.



The next thing we may consider is a sphere with several holes, and we paste on each hole a handle, as doodled on the right. So this is a sphere with g-holes and each hole we glue a handle to it. We should draw its cell complex as a polygon with $4g$-edges. What is the Euler characteristic for this surface? We can calculate it quickly as:

$$\chi(g\text{-handled sphere}) = g\chi(\text{handles}) + \chi(\text{sphere with } g \text{ holes}) \tag{4.3a}$$

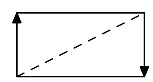
$$= g\chi(\text{handles}) + \chi\big((S^2 - g\text{ holes} - (g\text{ holes}\big) \tag{4.3b}$$

$$= g(-1) + (2 - g), \tag{4.3c}$$

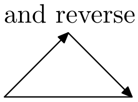
where we quickly compute

$$\chi(S^2) = \chi\big((S^2 - g\text{ discs}) \cup (g\text{ discs}\big) = 2 \tag{4.4}$$

for the sphere.



One more interesting thing, take a rectangle and identify opposite edges and reverse orientation. We doodle this on the right. We will cut this to get



two triangles along the dashed line. We erase the vertical line distinguishing the two triangles and we get the cell complex doodled to the left. This is the cell structure for the Mobius band. The boundary of the Mobius band is a circle. We may take a sphere with several punctures and paste Mobius bands instead of handles. We do not want to go into the theory of surfaces, so we leave it to the reader's imagination how this is done.

We will work with compact 2-dimensional manifolds.

Theorem 4.1. *All compact 2-dimensional manifolds are spheres with handles or Mobius bands.*

4.1 Homotopy

Now we would like to go to the definition of homotopy. Before going to this topic, we'd like to discuss operations on topological spaces. Let us take two sets A, B and we may construct their (direct) product

$$A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{4.5}$$

Now everything is very simple, if we have

$$f\colon X \times Y \to Z, \tag{4.6}$$

then this map is a function of two variables $f(x, y)$. That's obvious. Now there is a standard procedure. We may consider one of the variables as a *parameter*. Fix x, we get a map

$$f_x\colon Y \to Z \tag{4.7}$$

We say

$$\mathrm{Hom}(X \times Y, Z) = \mathrm{Hom}\big(X, \mathrm{Hom}(Y, Z)\big). \tag{4.8}$$

This is a completely trivial formula.

We now say that A, B are topological spaces. Then $A \times B$ may also be considered as a topological space. If $a \in A$ has a neighborhood $a \in U \subset A$, and similarly let $b \in V \subset B$ be a neighborhood, then

$$(a, b) \in U \times V \subset A \times B \tag{4.9}$$

is a neighborhood. And all other open sets of $A \times B$ are obtained by arbitrary unions and finite intersections of these guys.

Now, we have topological spaces X, Y. We may speak of continuous maps

$$f\colon X \times Y \to Z \tag{4.10}$$

and consider this construction $f(x, y) = f_x(y)$, obtaining a map

$$f_x\colon Y \to Z \tag{4.11}$$

fixing $x \in X$. What about our beloved formula (4.11)? Is it correct when the maps we take are continuous maps? It's a meaningless question, we don't know the topology of continuous maps $Y \to Z$. There is a meaningful question, namely what is the topology of the set of continuous maps $\mathrm{Hom}(Y, Z)$? This is not entirely honest, but not dishonest either! We define the topology on $\mathrm{Hom}(Y, Z)$ to satisfy

$$\mathrm{Hom}(X \times Y, Z) = \mathrm{Hom}\big(X, \mathrm{Hom}(Y, Z)\big). \tag{4.12}$$

We would like to construct topological invariants (for this space of continuous maps).

One is the number of connected components. Consider $\mathrm{Hom}(X, Y)$, if we fix X it's a topological invariant for Y; and if we fix Y, the number of connected components of $\mathrm{Hom}(X, Y)$ is a topological invariant for X. But still we may consider the number of components for $\mathrm{Hom}(X, Y)$, and we indicate this by

$$\mathrm{Hom}(X, Y) = \{X, Y\} \tag{4.13}$$

denoted with the brackets. Lets rephrase this in a more pedestrian way. Remember a component is connected if for any pair of points, there is a path connecting them. We say

$$f_0 \sim f_1 \text{ homotopic} \tag{4.14}$$

if and only if there is a path in the space of maps connecting these guys. What does it mean? Well, we have a path f_t where as $t \in [0, 1]$ varies and $f_t \in \mathrm{Hom}(X, Y)$ continuously varies. So really, it's a path in $\mathrm{Hom}(X, Y)$. But it is specifically such that

$$f_t|_{t=0} = f_0 \quad \text{and} \quad f_t|_{t=1} = f_1 \tag{4.15}$$

This may be seen as a deformation of the path from f_0 to f_1. We avoid difficulties by writing

$$f_t(x) = f(x, t) \tag{4.16}$$

as a continuous family of paths. We can now give another definition; let

$$f_0, f_1\colon X \to Y \tag{4.17}$$

be continuous, we say they are

$$f_0 \sim f_1 \text{ homotopic} \tag{4.18}$$

if there exists a map

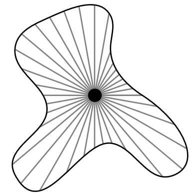
$$F\colon X \times I \to Y \tag{4.19}$$

such that

$$F(x, 0) = f_0(x) \quad \text{and} \quad F(x, 1) = f_1(x). \tag{4.20}$$

We did nothing new, we just gave a different definition of what we had[1].

[1] Alright, you got me, it's not even a different definition: it's just slightly different notation!



The first example is $S^1 \to \mathbb{R}^2$. It's a closed curve, as doodled on the right. We see every such map is homotopic to the zero map. How do we see this? Well, the light gray lines indicate the t value, with $t = 1$ being the outer most curve and $t = 0$ being the centered dot. This means that $S^1 \subset \mathbb{R}^2$ is contractible to a point. We may contract $\mathbb{R}^2$ to a point since

$$\mathrm{id}_{\mathbb{R}^2} \sim \text{trivial map} \tag{4.21}$$

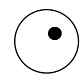
homotopic, which implies $\mathbb{R}^2$ is contractible. We can write

$$f_t(x) = tx \tag{4.22}$$

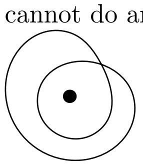
but that's a triviality.

Lets consider the simplest nontrivial case. Lets take

$$S^1 \to \mathbb{R}^2 - 0 \tag{4.23}$$



Here we have the following picture: if the point deleted is inside the circle, we cannot do anything. We cannot contract it to a point. We could, on the other hand, consider maps that go around the deleted point twice. This map is not homotopic to either zero or the map which goes around the deleted point once. Convince yourself this is the only topological invariant.



We have a space X and a space Y, we have maps

$$f\colon X \to Y \quad \text{and} \quad g\colon Y \to X \tag{4.24}$$

We say $X \sim Y$ homotopic if and only if

$$f \circ g \sim \mathrm{id}_Y \quad \text{and} \quad g \circ f \sim \mathrm{id}_X\,. \tag{4.25}$$

Theorem 4.2. *If we have homotopically equivalent spaces X and Y, then $\{A, X\} = \{A, Y\}$ for any space A, and $\{X, B\} = \{Y, B\}$ for any space B.*

We call $\{X, Y\}$ the "**Homotopic Classification of Maps**". We are going to prove that this relation $X \sim Y$ homotopic, sometimes called "homotopic equivalence", is really an equivalence relation.

EXERCISES

▸ **Exercise 1** (Reflexivity)**.** Prove or find a counter-example: for any X, $X \sim X$ homotopic.

▸ **Exercise 2** (Symmetry)**.** Prove or find a counter-example: for any X, Y topological spaces, $X \sim Y$ if and only if $Y \sim X$.

▸ **Exercise 3** (Transitivity)**.** Prove or find a counter-example: let X, Y, Z be topological spaces such that $X \sim Y$ and $Y \sim Z$ homotopic, both imply $X \sim Z$ homotopic.

Lecture 5.

Remember we consider $\mathrm{Hom}(A, X)$ the maps of topological spaces A, X. We will use the notation

$$\mathrm{Hom}(A, X) = \Omega_A(X), \tag{5.1}$$

we would like to stress that $\Omega_A(-)$ is a functor. So

$$\Omega_A\,(f\colon X \to X') \quad = \quad \Omega_A(f)\colon \Omega_A(X) \to \Omega_A(X') \tag{5.2}$$

which amounts to composing

$$A \xrightarrow{\varphi} X \xrightarrow{f} X', \tag{5.3}$$

i.e., $f \circ \varphi$ for all $\varphi \in \Omega_A(X)$. The identity morphism, and composition of morphisms, are preserved under action by a functor. So for every map $f \colon X \to X'$ we have a map of "spaces of maps"

$$\Omega_A(f) \colon \Omega_A(X) \to \Omega_A(X'). \tag{5.4}$$

What are the homotopy classes?

They are merely components of the function space. We may say the following: a set of homotopic classes

$$\{A, X\} = \begin{pmatrix} \text{set of} \\ \text{components} \\ \text{of } \Omega_A(X) \end{pmatrix} = \pi_0\big(\Omega_A(X)\big) \tag{5.5}$$

where π_0 is an assignment to each topological space its set of connected components, but it is also a functor! If

$$\psi \colon Z \to Z' \tag{5.6}$$

then

$$\pi_0(\psi) \colon \pi_0(Z) \to \pi_0(Z'). \tag{5.7}$$

That is trivial. What can we say? We can say if

$$f \colon X \to X' \tag{5.8}$$

and

$$\Omega_A(f) \colon \Omega_A(X) \to \Omega_A(X') \tag{5.9}$$

then

$$\pi_0\big(\Omega_A(f)\big) \colon \pi_0\big(\Omega_A(X)\big) \to \pi_0\big(\Omega_A(X')\big). \tag{5.10}$$

Of course, functoriality is completely irrelevant here.

If $\varphi_0 \sim \varphi_1$ homotopic, then $f \circ \varphi_0 \sim f \circ \varphi_1$ homotopic too. What did we get? A remark that if we have a map

$$f \colon X \to X' \tag{5.11}$$

then we obtain a mapping

$$\{A, X\} \to \{A, X'\} \tag{5.12}$$

but that's a triviality. Is this map a one-to-one correspondence, a bijection? We can construct a map

$$g \colon X' \to X \tag{5.13}$$

and we induce

$$\{A, X'\} \to \{A, X\}. \tag{5.14}$$

We would like it to compose with f to give the identity. We can require, of course,

$$f \circ g = \mathrm{id}_{X'} \quad \text{and} \quad g \circ f = \mathrm{id}_X \tag{5.15}$$

but that's too much. We instead require

$$f \circ g \sim \mathrm{id}_{X'} \quad \text{and} \quad g \circ f \sim \mathrm{id}_X \tag{5.16}$$

both homotopic. Thus we get

$$\{A, X\} = \{A, X'\}. \tag{5.17}$$

We have obtained a classification of homotopy equivalent maps.

We may do something a little bit different. We may take

$$\mathrm{Hom}(X, A) \stackrel{\text{def}}{=} \Omega^A(X) \tag{5.18}$$

which is a contravariant functor. If we have

$$X' \xrightarrow{f} X, \quad \text{and} \quad X \xrightarrow{\varphi} A \tag{5.19}$$

then we may consider

$$\varphi \circ f \colon X' \to A. \tag{5.20}$$

We have, if $X \sim X'$ homotopic, we may identify

$$\{X, A\} = \{X', A\}. \tag{5.21}$$

If $X \sim X'$ homotopic, and $Y \sim Y'$ homotopic, then $\{X, Y\} = \{X', Y'\}$.

One more definition. We would like to ask: when $A \subset X$ is a subset homotopically equivalent to the whole set? Definitely we have a map $\iota \colon A \hookrightarrow X$ but we should have a map in the opposite direction

$$f \colon X \to A. \tag{5.22}$$

We require two things. First that $\iota \circ f \sim \mathrm{id}_X$. Second that

$$f \circ \iota \sim \mathrm{id}_A \tag{5.23}$$

is homotopic. Did we say anything new? Nothing! Now we will require more and get a definition that implies homotopy equivalent. We require

$$f \circ \iota = \mathrm{id}_A\,. \tag{5.24}$$

Then f is called a "**Retraction**". By the way the statement that a retraction exists is a non-trivial statement. It is impossible to retract $[0, 1]$ to its boundary $\{0, 1\}$ without tearing. We should have a family of maps

$$f_t \colon X \to X \tag{5.25}$$

where $f_0 = \mathrm{id}_X$ and $f_1 = \iota \circ f = f$. We are a little bit sloppy here since

$$f \colon X \to A \quad \text{and} \quad f_0 \colon X \to X \tag{5.26}$$

so to be completely precise

$$f_1(x) = (\iota \circ f)(x). \tag{5.27}$$

So what does it mean? We require our retraction to be a "**Deformation Retraction**" which is a very typical case of homotopy equivalence. This means $A \subset X$ and $A \sim X$ homotopic. Now, examples!

Consider the letter "P". It is clear this guy is homotopically equivalent to "O", i.e. P $\sim$ O homotopic. Let us first note that

$$\mathrm{P} \sim \mathrm{D} \tag{5.28}$$

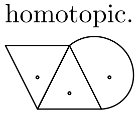
homotopic, and

$$\mathrm{D} \cong \mathrm{O} \tag{5.29}$$

homeomorphic, thus

$$\mathrm{P} \sim \mathrm{O} \tag{5.30}$$

homotopic. (Homotopic equivalence is weaker than topological equivalence.)



Let us suppose we have a cell complex X. First of all, we have $X^k \subset X^{k+1}$ for the k-dimensional and $(k+1)$-dimensional skeletons. Suppose we deleted one point from ever k-dimensional cell. Then X^k is a deformation retract of $X^{k+1} - \{\text{deleted points}\}$. We can work in every cell separately. We can stay in one of these points, blow it up into a larger hole, and the boundary remains in tact.

Gradually everything goes to the boundary. This is not a very rigorous explanation. A rigorous one is available. A cell complex comes from a ball of dimension $(k+1)$. If we remove a point inside this ball, it is the same as a k-dimensional sphere multiplied by an interval $\bar{D}^{k+1} - 0 = S^k \times [0, 1)$.

Theorem 5.1. *If $A \subset X$ closed and it is "good enough" (not pathological) and A is contractible (i.e., homotopically equivalent to a point), then $X \sim X/A$ homotopic.*

Lecture 6.

If X, Y are topological spaces, we may consider $\mathrm{Hom}(X,Y)$ and it is also a topological space. We may consider $\pi_0\big(\mathrm{Hom}(X,Y)\big)$ the connected components (homotopy classes) of the space. We say $X \sim X'$ homotopic if there exists maps

$$f\colon X \to X', \quad \text{and} \quad g\colon X' \to X \tag{6.1}$$

such that

$$g \circ f \sim \mathrm{id}_X \quad \text{and} \quad f \circ g \sim \mathrm{id}_{X'} \tag{6.2}$$

are homotopic.

Note that we are abusing language a little. We have the notion of homotopic maps $f \sim g$ and we construct from this a weak equivalence relation on topological spaces. We call this "homotopy equivalence of topological spaces" or simply "homotopic spaces".

For example, if $X \sim$ (point), then we say X is "**Contractible**". In particular every convex set C is contractible, it is obvious. We assume without loss of generality that $0 \in C$, we have

$$f_t(x) = tx \tag{6.3}$$

describe the contraction. Thus D^n, $\bar{D}^n$, $\mathbb{R}^n$ are contractible. But S^n is *not* contractible. If we delete a single point from a sphere, we get

$$S^n - \text{point} \cong \mathbb{R}^n \tag{6.4}$$

and that implies $S^n -$ (point) is contractible.

There is a notion of retraction and deformation retraction. If $A \subset X$, then it is a deformation retract if the inclusion is a homotopy equivalence. We have

$$f_0(x) = x \tag{6.5}$$

for all $x \in X$ and

$$f_1(x) \in A \tag{6.6}$$

for all $x \in X$ be a deformation retract. If $X \supset A$ (where A is a "good" subset, i.e. closed and we require if X is a cell complex that A be a subcomplex), is contractible, then $X \sim X/A$ homotopic.

We will consider homotopic classifications of graphs (we will consider connected graphs, this is not really a restriction).

Theorem 6.1. *Every connected graph is homotopically equivalent to a wedge sum of cirlces ("bouquet").*

If we are working with sets with marked points (X, x_0), then we may consider

$$f\colon (X, x_0) \to (Y, y_0) \tag{6.7}$$

which is a map $f\colon X \to Y$ such that $f(x_0) = y_0$.Then we may consider the set $\mathrm{Hom}\big((X,x_0),(Y,y_0)\big)$ which is again a topological space, or part of the topological space $\mathrm{Hom}(X,Y)$. We may generalize this to a pair (X, A) where $A \subset X$, and we may consider maps

$$f\colon (X, A) \to (Y, B) \tag{6.8}$$

which consists of a map $f\colon X \to Y$ such that $f(A) \subset B$. This is a straightforward generalization of marked points. Again, we consider $\mathrm{Hom}\big((X,A),(Y,B)\big)$ which is a topological space, and again we may speak of connected components.

If X, Y are topological spaces we may consider a disjoint union $X \sqcup Y$ which is a topological space with two components X and Y. We see

$$\mathrm{Hom}(X \sqcup Y, Z) = \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z). \tag{6.9}$$

Also note we have two notions of disjoint unions: a set theoretic sense, where we have

$$\bar{D}^n = S^{n-1} \sqcup D^n \tag{6.10a}$$

be true, whereas the topological version has

$$\bar{D}^n \neq S^{n-1} \sqcup D^n \tag{6.10b}$$

We may consider the wedge sum of X and Y to be

$$\begin{aligned}(X \vee Y, *) &= (X, x_0) \vee (Y, y_0) \\ &= (X \sqcup Y)/(x_0 \sim y_0)\end{aligned} \tag{6.11}$$

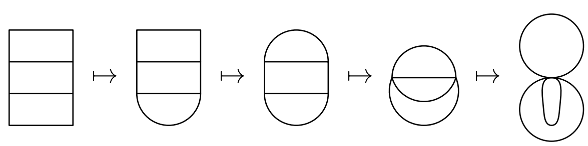
where we identify the marked points of the pointed spaces (X, x_0) and (Y, y_0). Thus

$$\mathrm{Hom}\big((X \vee Y, *), (Z, z_0)\big) = \mathrm{Hom}\big((X, *), (Z, z_0)\big) \times \mathrm{Hom}\big((Y, *), (Z, z_0)\big) \tag{6.12}$$

The marked point needs to be taken into account. The wedge sum is the coproduct in the category $\mathbf{Top}_*$ of topological spaces with marked point ("pointed topological spaces").

Proof (of Theorem 6.1). It is trivial. Suppose we have a connected graph. Every edge is an interval, but an interval is contractible. We can contract as long as the starting point of and edge is different from its ending point. We can contract any edge with two different vertices, which decreases the number of vertices. At some point we have only one vertex, and it's the wedge sum of circles. □

So to give an example of what this would look like, we doodle:



The second question: start with a completely general graph. How to say what is an equivalent graph? The answer is very simple if we know the Euler characteristic. The Euler characteristic of a compact set is a homotopy invariant. We will prove it next quarter. Suppose our graph is simply connected, then the Euler characteristic is

$$\alpha_0 - \alpha_1 = (\#\text{ vertices}) - (\#\text{ edges}), \tag{6.13}$$

and the wedge sum of k circles is

$$\chi(S^1 \vee \cdots \vee S^1) = 1 - (\#\text{ circles}) \tag{6.14}$$

as there is a single vertex. If we have homotopy equivalence, then we may compute the number of circles in a moment.

Graphs are one-dimensional guys, and reduce to circles. Now we should consider

$$\pi_0\big(\mathrm{Hom}(S^1, S^1)\big) = \mathbb{Z} \tag{6.15}$$

We should see this as obvious. We saw

$$S^1 \to \mathbb{R}^2 - (\text{point}) \sim S^1 \tag{6.16}$$

and the homotopy class of maps is characterized by the winding number.

Lecture 7.

Lemma 7.1 (Homotopy Extension Property)**.** *Let us suppose we have a pair $A \subset X$ closed and we have a map*

$$F\colon X \to Y. \tag{7.1}$$

We can restrict our F to A,

$$f = F|_A \tag{7.2}$$

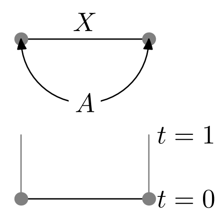
Suppose we have a deformation $f_t\colon A \to Y$ such that $f_0(a) = F(a)$ for all $a \in A$. What we would like to do is extend, construct a family of maps

$$F_t\colon X \to Y \tag{7.3}$$

*such that on A, we have $f_t = F_t|_A$. We want to extend this deformation from A to the whole space X. We say that this pair (X, A) has the "**Homotopy Extension Property**".*

Remark 7.2. We often abbreviate "Homotopy Extension Property" as HEP.

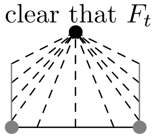
Lemma 7.3. *If A is a (closed) subcomplex of the cell complex X, then the pair (X, A) has the Homotopy Extension Property.*



Proof (Particular Case). Take X to be an interval and A consisting of boundary points. We have a map a map of the interval, and we have a deformation over A. The situation is doodled to the right, where the top picture shows X and in light gray A; the bottom picture is $A \times [0,1] \cup X \times \{0\}$. The parameter t of I is labeled as well. We would like $f_t\colon A \to Y$, which is $I \times A \to A$ a function intuitively taking the bottom diagram as the domain. We may construct a retraction

$$\varphi\colon X \times I \to X \times \{0\} \cup A \times I \tag{7.4}$$

which is precisely the dilapidated rectangle to the right. If we have such a map, we can construct $F_t = f \circ \varphi$ where $f = f_t$ on $A \times I$ and $f = F$ on $X \times \{0\}$. Now it is completely clear that F_t is an extension of f_t and that's it. If we have that retraction, we may apply



this to every case. Moreover, we may construct the deformation retraction. This is very easy. The procedure is doodled on the left, where we have $A \times I$ in light gray and the deformation retraction is dashed.

Case: Take X to be a disc, multiply the boundary by I and we get the cylinder. Again we should take the retraction. We repeat the procedure, taking a point above the cylinder and consider lines from $A \times I \cup D^2 \times \{0\}$ to the point. We may continue to generalize to higher dimension cases.

The general case we take $A \subset X$ and take a k-skeleton X^k and perform this induction on k. We can assume we took this for $(A \cup X^{k-1}) \subset (A \cup X^k)$. We extend from $(k-1)$-cells to k-cells, but this is precisely what we have done. □

Remember we had $A \subset X$ and A was contractible, i.e., we had a retraction from A to a point. We claim $X \sim X/A$ homotopic. We prove this when A is "nice", i.e., the pair (X, A) has the Homotopy Extension Property.

Proof. We have $\mathrm{id}_X\colon X \to X$ and on A we can deform this map—A is contractible, so we have $f_t\colon A \to A$ such that $f_0(a) = a$ and $f_1(a) = a_0$. But this is precisely the picture of the Homotopy Extension Principle, we have a deformation of the whole space. This permits us to extend from f_t to $F_t\colon X \to X$ which has the property $f_t = F_t|_A$ and in particular $f_1 = F_1|_A$ which means $F_1(a) = a_0$ for $a \in A$. We have a map $X/A \to X$, because all of A went to one point. This is the main step of the proof, we should prove it's a homotopy equivalence but we'll skip it. □

Recall we stated

$$\{S^1, S^1\} = \mathbb{Z}. \tag{7.5}$$

We will prove it, but it won't be absolutely rigorous (we'll be rigorous later in a more general setting). On a circle we have an angular coordinate $\alpha \in [0, 2\pi)$, or $\alpha \in \mathbb{R}$ subject to the equivalence $\alpha \simeq \alpha + 2\pi$. Therefore if we have

$$f\colon S^1 \to S^1 \tag{7.6}$$

we have two possibilities. One is, e.g.,

$$f(\alpha) = 5\alpha \tag{7.7}$$

which is discontinuous at $2\pi k/5$ for $k = 1, \dots, 4$. What to do? Well, we relax the map a little bit to be

$$f\colon [0, 2\pi] \to \mathbb{R} \tag{7.8}$$

such that

$$f(2\pi) \equiv f(0) \bmod 2\pi. \tag{7.9}$$

So in other words

$$f(2\pi) = f(0) + k2\pi \tag{7.10}$$

for some $k \in \mathbb{Z}$; then this f specifies a map of circles. But that is obvious. What is less obvious is any map of circles may be written in this way.

This $k \in \mathbb{Z}$ is called the "**Degree**" **of a Map**. It is a triviality that

$$\deg(f) = \deg(g) \implies f \sim g \tag{7.11}$$

homotopic. Maps of the same degree are homotopic. To see this triviality, take

$$h_t(x) = tg(x) + (1-t)f(x) \tag{7.12}$$

and since $\deg(g) = \deg(f)$ we see that $\deg(h) = \deg(f) = \deg(g)$ as well. There is no problem here. Let $k = \deg(f)$ and $k' = \deg(g)$, then

$$\begin{aligned} h_t(2\pi) &= t\big(k'2\pi + g(0)\big) + (1-t)\big(k2\pi + f(0)\big) \\ &= \big(tk' + (1-t)k\big)2\pi + h_t(0). \end{aligned} \tag{7.13}$$

But this works if and only if $k = k'$, otherwise we have an integer varying continuously while remaining integral... and then anything is possible! We have $f(\varphi) = k\varphi$ be a map of degree k, so all maps of degree k are homotopic to it.

EXERCISES

▸ **Exercise 4.** Classify first 15 letters of English alphabet up to topological equivalence and up to homotopy equivalence. (Consider capital letters only.)

▸ **Exercise 5.** Find a letter that is homotopy equivalent to a torus with one deleted point.

▸ **Exercise 6.** Calculate Euler characteristic of a sphere with g handles and h Moebius bands attached.

▸ **Exercise 7.** Calculate the Euler characteristic of a T-shirt. Find a graph that is homotopy equivalent to a T-shirt.

▸ **Exercise 8.** 5. Calculate Euler characteristic of projective plane. (We define projective plane as two-dimensional sphere $\|x\| = 1$ where the point x is identified with the point $-x$.)

Lecture 8.

We classified homotopy classes $\{S^1, S^1\}$ using angular coordinates. The requirement of continuity is that if $f\colon S^1 \to S^1$, then

$$f(2\pi) - f(0) = 2\pi k \tag{8.1}$$

where $k \in \mathbb{Z}$ is the degree of the map. It is an invariant of the homotopy class, i.e., a "homotopy invariant". Later we will see analogously

$$\deg\colon \{S^n, S^n\} \cong \mathbb{Z} \tag{8.2}$$

for $n \geq 1$, where this is a one-to-one correspondence. But more generically, what is the meaning of the "degree of a map"?

It tells us how many times the codomain is covered by the domain. Let us take for definiteness $f(0) = 0$. The simplest case is

$$f(\alpha) = k\alpha \tag{8.3}$$

What happens? Look, we see that the image raps around th circle k times, so f^{-1} is a one-to-k function. More precisely,

$$f^{-1}(0) = \left\{0, \frac{2\pi}{k}, 2\frac{2\pi}{k}, \dots, (k-1)\frac{2\pi}{k}\right\}. \tag{8.4}$$

But this is a very simple case. We should solve

$$f(\alpha) = 0 \bmod 2\pi, \tag{8.5}$$

we know

$$f'(\alpha) < 0 \quad \text{or} \quad f'(\alpha) > 0 \tag{8.6}$$

implies the curve grows with a positive/negative slope between the roots. We could say that the degree is the algebraic number of solutions to the equation

$$f(\alpha) \equiv \alpha_0 \bmod 2\pi \tag{8.7}$$

where $f'(\alpha_0) \neq 0$. We can only calculate the *algebraic* number of solutions for an equation.

We can prove the main (fundamental) theorem of algebra. One way to prove it is very simple (our proof will not be rigorous!). A trivial proof of the fact that every algebraic equation has a solution. What is an algebraic equation? We have a polynomial

$$p(x) = x^n + a_1 x^{n-1} + \dots + a_n, \tag{8.8}$$

which is a map. It is a map

$$p\colon S^2 \to S^2 \tag{8.9}$$

where we recall

$$S^2 = \mathbb{R}^2 \cup \{\infty\} \tag{8.10}$$

by Stereographic Projection. But

$$p(\infty) = \infty \tag{8.11}$$

so our claim in Eq (8.9) is correct. We can compute the degree of this map, and that is very easy since the degree is an invariant of the homotopy class. And look, we will write

$$p_t(x) = x^n + t\big(a_1 x^{n-1} + \dots + a_n\big) \tag{8.12}$$

what do we see? We see

$$p_1(x) = p(x) \tag{8.13a}$$

is our original polynomial, and

$$p_0(x) = x^n \tag{8.13b}$$

is the same degree as our original polynomial. But the degree of the map $x \mapsto x^n$ is very simple, since

$$p(x) \equiv \alpha \pmod{2}\pi \tag{8.14}$$

has $n > 0$ solutions (well, $n \neq 0$ solutions). Thus $p(x)$ is a map of degree n, or the algebraic number of solutions is n. We have no right in replacing t to be

$$\widetilde{p}_t(x) = a_{k-1}x^k + t\big(x^n + a_1 x^{n-1} + \dots\big) \tag{8.15}$$

otherwise we violate continuity.

There is a simple trick to prove $p(x)$ has a solution. We want to prove $p(x)$ is zero somewhere. Let us assume it is *nowhere zero*. We take

$$|x| = b \tag{8.16}$$

and consider

$$\varphi(x) = \frac{p(x)}{|p(x)|} \tag{8.17}$$

we can do this as we assumed $p(x) \neq 0$ for any x. It is a continuous map from a circle to a circle. Let us calculate the degree of this map. We take $b = \varepsilon > 0$ to be small, then $\varphi \sim 0$ homotopic. We can take $b \gg 0$, but then it is easy to see only the first term x^n dominates. We've seen that the degree of p is n in this case, so the degree of φ would be nonzero. A contradiction! Caused by what? By dividing by $p(x)$ since we cannot divide by 0.

Lecture 9.

Now,

$$\{S^1, S^1\} = \mathbb{Z} \tag{9.1}$$

the homotopy classification of maps $S^1 \to S^1$ are in one-to-one correspondence with $\mathbb{Z}$. So we are a little bit sloppy here. But the important question: what about $\{S^1, S^2\}$? Well, every map $f\colon S^1 \to S^2$ is homotopically equivalent to the constant map.

The proof is as follows. Suppose $f\colon S^1 \to S^2$, take $x \in S^2$ which is not covered by the image $x \notin f(S^1)$. Then we may delete this x. But we get

$$S^2 - \{x\} = \mathbb{R}^2 \tag{9.2}$$

which is contractible. We may contract the map to a point. For physicists, it's a good proof; but for mathematicians—no! We could have a Peano curve, where

$$f(S^1) = S^2 \tag{9.3}$$

i.e., a space filling curve. Differentiable, or even piecewise differentiable,maps of this kind are impossible. We get a trivial lemma:

Lemma 9.1 (Trivial)**.** *If $S^1 \to \mathbb{R}^n$, every such map may be approximated as good as you want by piecewise linear maps.*

Proof. But this is a triviality. Look, we have a circle, or an interval, we may decompose it into small pieces. If we have a map to $\mathbb{R}^n$, we may approximate the map along the subdivision by a linear map. If the subdivision is in "small pieces", then our approximation is "good." □

This lemma is pretty much universal, we may substitute the domain by S^2 or any cell complex. We may consider a rectangle, divide it up into "small triangles"—this is called a "**Triangulation**". If we consider

$$[0,1]\times[0,1]\to\mathbb{R}^2, \tag{9.4}$$

we may approximate it by piecewise linear functions. We extend it uniquely by linearity (perhaps "affine" is a better choice of words than "linear"). We may approximate the maps that are horrible by maps that are quite nice. But now, when we approximate, if we have two closed maps to a sphere—they are homotopic. We may say every map is homotopic to a good map.

A minor technicality with the codomain. We either generalize the notion of piecewise linear map to S^2, or (the easier choice) take a piece of S^2 homeomorphic to a square. We take the preimage of this to construct a piecewise linear map. Then we use the lemma on extension of homotopy, we get an extension homotopic to the approximation, and so on. Consider $\{S^k, S^n\}$ for $k < n$. We see that all maps are homotopically trivial.

9.1 Fundamental Group

A very important notion that may be explained as follows: we have a space X, and maps $S^1 \to X$. We want to consider their homotopy classes, but we want some algebraic structure. So we need some structure, some operation. These maps are loops. We will assume that our S^1 has a marked point, and X has a marked point; we will say that we would like to consider maps

$$(S^1,*)\to(X,x_0) \tag{9.5}$$

which take the marked point to the marked point. What do we have? We have two loops that start and finish at the same point. People use the word "concatenation". We get an operation of two loops. We write $f * g$ for the concatenation of g followed by f. Now we may define the fundamental group. We consider

$f * g$ = concatenation of paths

$$\{(S^1,*),(X,x_0)\} \tag{9.6}$$

so if $f\sim f'$ and $g\sim g'$, then

$$f*g\sim f'*g' \tag{9.7}$$

homotopic. This operation $f * g$ is associative and has inverses. We thus have a group, and we call it the "**Fundamental Group**" denoted $\pi_1(X, x_0)$. As usual, this is a sloppy definition. We will give a more precise one.

First: what is a loop? It is easier, formally, to work with intervals. So we will consider

$$f\colon[0,1]\to X \tag{9.8a}$$

and we require

$$f(0)=f(1)=x_0 \tag{9.8b}$$

both endpoints are mapped to the marked point. But this is the same as a map of a circle, which starts and stops at the marked point. Why did we take $[0,1]$? I don't know! We could take instead

$$f\colon[0,a]\to X$$

for any $a > 0$ with the condition $f(0) = f(a) = x_0$. But topology doesn't care about this: $[0,a]\cong[0,1]$ for topologists. Now we can consider the space of all these maps denoted by $\Omega(X,x_0)\subset\mathrm{Hom}\big([0,1],(X,x_0)\big)$.Now we can modify this if we we consider the space of maps

$$\widetilde{\Omega}(X,x_0)=\{f\colon[0,a]\to(X,x_0)\mid a>0, f(0)=f(a)=x_0\}. \tag{9.9}$$

Are these spaces the same? No, of course not. But

$$\Omega\cong\widetilde{\Omega} \tag{9.10}$$

which is a triviality since

$$\Omega \subset \widetilde{\Omega} \tag{9.11}$$

is embedded, we may retract $\widetilde{\Omega}$ to Ω. We may deform continuously $[0,a] \sim [0,1]$ which is why we may confidently state $\Omega \sim \widetilde{\Omega}$ homotopic.

We now want to define multiplication in these spaces. This is easy to do for $\widetilde{\Omega}$. If we have

$$f\colon [0,a] \to (X, x_0) \tag{9.12a}$$

and

$$g\colon [0,b] \to (X, x_0) \tag{9.12b}$$

then we may define their concatenation as

$$h = f * g\colon [0, a+b] \to (X, x_0) \tag{9.13a}$$

where

$$h(x) = \begin{cases} f(x) & 0 \leq x \leq a \\ g(x-a) & a \leq x \leq b+a. \end{cases} \tag{9.13b}$$

It is very clear in this picture this concatenation is an associative operation. We don't need to prove

$$(f * g) * h = f * (g * h). \tag{9.14}$$

So $\widetilde{\Omega}$ has an operation called "concatenation", and it is an associative operation. People usually work with $[0,1]$ but concatenation is still defined for

$$f, g\colon [0,1] \to (X, x_0). \tag{9.15}$$

We have

$$h(x) = \begin{cases} f(2x) & 0 \leq x \leq \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \leq x \leq 1 \end{cases} \tag{9.16}$$

for our concatenation operation. Is it associative? No, it will not be associative, because look

$$(f * g) * h = \begin{cases} f(4x) & 0 \leq x \leq 1/4 \\ g(4x-1) & 1/4 \leq x \leq 1/2 \\ h(2x-1) & 1/2 \leq x \leq 1 \end{cases} \tag{9.17}$$

but if we consider instead

$$f * (g * h) = \begin{cases} f(2x) & 0 \leq x \leq 1/2 \\ g(4x-1) & 1/2 \leq x \leq 3/4 \\ h(4x-2) & 3/4 \leq x \leq 1 \end{cases} \tag{9.18}$$

So we do not have associativity, strictly speaking. But if we care up to homotopy, we have full associativity for $\widetilde{\Omega}$.

Now we may define the fundamental group. What are we doing? Well, as a set $\pi_1(X, x_0)$ is the set of components

$$\pi_0\big(\Omega(X, x_0)\big) = \pi_0\big(\widetilde{\Omega}(X, x_0)\big) \tag{9.19}$$

but in Ω we have multiplication (and we have it in $\widetilde{\Omega}$ too), and — what a coincidence! — we have multiplication in $\pi_0(-)$. And, moreover, $\pi_0(\Omega(X, x_0))$ has multiplication be associative, which implies associativity in the others.

We have a unit element, namely

$$e(t) = x_0 \tag{9.20}$$

where we stay at the marked point. We may say

$$e * f \sim f. \tag{9.21}$$

It is very easy to say

$$f^{-1}(t) = f(1-t) \tag{9.22}$$

as the path that goes in the opposite direction. We end up with the fundamental group.

Lecture 10.

We considered a space with a marked point (X, x_0) and we constructed the loop space $\Omega(X, x_0)$ with loops that start and end at $x_0 \in X$. We reviewed two constructions which are homotopically equivalent. We also considered a binary operation called concatenation which is intuitively "multiplication". We can use this multiplication to define the group

$$\pi_1(X, x_0) = \pi_0\big(\Omega(X, x_0)\big) \tag{10.1}$$

which is just the set of components of the loop space. We proved this multiplication is associative, the unit element exists, and inversion exists. Well, we did not "prove" it, but it is obvious. We see

$$f^{-1}(t) = f(1-t) \tag{10.2}$$

and that $e(t) \sim f^{-1} * f$ homotopic.

The fundamental group is not, strictly speaking, an invariant of a topological space. It may depend on the marked point; the question is, can we (if we change the marked point) say

$$\pi_1(X, x_0) = \pi_1(X, \widetilde{x}_0)? \tag{10.3}$$

If X is pathwise-connected, yes. If X is disconnected, we have no chance. We will give two proofs of this. One is really short and (perhaps) the better, while the other is more pedestrian.

Proof. We have a connected space X, and a path

$$h\colon [0,1] \to X \tag{10.4}$$

where $h(0) = x_0$ and $h(1) = x_1$. What we can do, we can say we have a deformation from one marked point into the other. We have the homotopy extension property. So if we have a deformation of a subset, we may extend it to a deformation of the *whole* space. So

$$(X, x_0) \sim (X, x_1) \tag{10.5}$$

homotopic, and we may go in the other direction by using $h^{-1}(t)$. It is very easy to see these maps are homotopically equivalence. Then everything is fine. □

This proof has the disadvantage that the homotopy extension principles holds *almost* for every space. But if X is not among them, this proof does not hold.

Proof. We will give another proof that is perhaps a bit longer. Look we have these two points x_0, x_1 and a connecting path. We consider a path in $\Omega(X, x_0)$ but we want to get a path that lives in $\Omega(X, x_1)$. If $h\colon [0,1] \to X$ such that $h(0) = x_0$ and $h(1) = x_1$, then for any $f \in \Omega(X, x_0)$ we induce a map $h * f * h^{-1} \in \Omega(X, x_1)$. Thus if we are working with connected spaces, the fundamental group does not depend on the marked point. □

Remark 10.1. Is this true? Not really, because we cannot say that this isomorphism

$$\pi_1(X, x_0) \cong \pi_1(X, x_1) \tag{10.6}$$

is canonical. The isomorphism depends on the choice of the path connecting marked points. Why? Lets explain. These points x_0, x_1 can coincide, why not? No one said they had to be different! If they coincide, then the path connecting the marked points is a loop. Then our formula

$$f \mapsto h * f * h^{-1} \tag{10.7}$$

can be understood at the level of the fundamental group. At the level of the fundamental group, this gives us a nontrivial inner automorphism of $\pi_1(X, x_0)$. It is a canonical isomorphism when the group $\pi_1(X)$ is Abelian.

When working with the fundamental group, we will neglect the marked point. It is a little dangerous.

Now, let us consider $\{S^1, X\}$ the classification of maps from the circle to X. We assume X is connected. We see now that

$$\pi_1 \colon (X, x_0) \to \{S^1, X\} \tag{10.8}$$

we get a group. So lets say something that definitely is quite trivial, namely: we can say that the conjugacy classes of $\pi_1(X, x_0)$ is also a map to $\{S^1, X\}$. We are really saying if two guys are conjugate

$$h^{-1} g h = f \tag{10.9}$$

then they are mapped to the samje homotopy class. We can identify maps $S^1 \to X$ with conjugacy classes in the fundamental group.

But now we would like to calculate the fundamental group. Two fundamental groups are quite obvious. The first is

$$\pi_1(S^1) = \mathbb{Z}. \tag{10.10}$$

This is because we know

$$\{S^1, S^1\} \to \mathbb{Z} \tag{10.11}$$

is a one-to-one correspondence.

The other fundamental group that we know is

$$\pi_1(S^n) = 0 \tag{10.12}$$

for $n > 1$. It is the *trivial* group!

There are (at least) two ways to compute the fundamental group. One is by the van Kampen Theorem. We represent

$$X = A \cup B \tag{10.13}$$

and we will assume A, B, and $A \cap B$ are connected; we also assume A and B are open. (Pop quiz: is $A \cap B$ open?) We will consider the simplest case when $A \cap B$ is *simply* connected. (Here "**Simply Connected**" means connected and the fundamental group is trivial.) Without loss of generality, we will say that our marked point $\star \in A \cap B$. The typical case is when considering the wedge sum of two circles, the intersection is a single point, but then we work with closed sets. Well, okay, but that doesn't matter! In this case,

$$\pi_1(X, \star) = \text{free product of } \pi_1(A, \star) \text{ and } \pi_1(B, \star) \tag{10.14}$$

The free product is a general notion in group theory. Let G_1, G_2 be groups. Their free product can be deduced in terms of generators $a_1, \dots, a_m$ of G_1 and $b_1, \dots, b_n$ of G_2; we have some relations on G_1 as well as some on G_2. Then we combine generators and relations. This isn't a very good definition, since it depends on the choice of generators and relations.

We could define it in a slightly different way. Consider

$$\mathrm{Hom}(G_1 * G_2, G)$$

where $G_1 * G_2$ is the free product. We say

$$\mathrm{Hom}(G_1 * G_2, G) = \mathrm{Hom}(G_1, G) \times \mathrm{Hom}(G_2, G) \tag{10.15}$$

which doesn't explicitly depend on the generators and relations of G_1 and G_2. We have a pair of homomorphisms that act on elements of G_1 and G_2 respectively. Now we'd like to explain why we have this.

EXERCISES

▸ **Exercise 9.** Let f denote a differentiable map of a circle into another circle. In angular coordinates the map f can be represented by a multivalued function, but its derivative $f(\alpha)$ is well-defined. Prove that the degree of the map can be represented as an integral

$$\deg(f) = \frac{1}{2\pi}\int_0^{2\pi} f'(\alpha)\mathrm{d}\alpha. \tag{10.16}$$

▸ **Exercise 10.** Prove that a polynomial map $p\colon \mathbb{R} \to \mathbb{R}$ where p is a polynomial of degree n can be extended by continuity to a map of circles. (We use the fact that adding to $\mathbb{R}$ one point at infinity we get a circle). Calculate the degree of this map of circles.

Lecture 11.

So, recall we discussed van Kampen's theorem. The formulation involved a notion of "free product" of two groups $G_1 * G_2$. We gave two descriptions. One took the generators

$$\langle a_i \mid r_k \rangle \tag{11.1}$$

of G_1 and

$$\langle b_j \mid s_m \rangle \tag{11.2}$$

of G_2. Then we defined $G_1 * G_2$ to be

$$\langle a_i, b_j \mid r_k, s_m \rangle. \tag{11.3}$$

The other explanation looked at the coproduct in the category **Grp**. It is then

$$\mathrm{Hom}(G_1 * G_2, G) = \mathrm{Hom}(G_1, G) \times \mathrm{Hom}(G_2, G) \tag{11.4}$$

Sometimes it is useful to have a more explicit description, namely elements of the free product $a_1 b_1 a_2 b_2 (\dots)$ where $a_i \in G_1$ and $b_j \in G_2$ are arbitrary elements. The only problem is how do we find the product of two guys? It's easy,:

$$(a_1 b_1 \cdots a_n b_n)(a_1' b_1' \cdots a_m' b_m') = a_1 b_1 \cdots a_n b_n a_1' b_1' \cdots a_m' b_m' \tag{11.5}$$

but if $a_i = 1$ we don't write it. So, for example, $a_1 1 b_1 = a_1 b_1$.

The notion of the free product is closely related to the notion of free groups. Consider the free groups F_m, F_n with m and n generators respectively. We see

$$F_m * F_n = F_{m+n}. \tag{11.6}$$

Since

$$F_1 \cong \mathbb{Z} \tag{11.7}$$

we deduce

$$F_m \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{m \text{ times}}. \tag{11.8}$$

But enough humorless group theory!

We have a space X covered by two open sets A, B. We also have A, B, $A \cap B$ be connected. (Pop quiz: is X connected?) We take $\star \in A \cap B$. Then

$$\pi_1(X,\star) = \big(\pi_1(A,\star) * \pi_1(B,\star)\big)/N \tag{11.9}$$

where N is some normal subgroup. Our question is then regarding the description of the subgroup. First, a proof of our statement.

We should construct a morphism of the free product to $\pi_1(X,\star)$:

$$\pi_1(A,\star) \times \pi_1(B,\star) \to \pi_1(X,\star) \tag{11.10}$$

We should prove it is surjective. If we write an element of the free product as $a_1 b_1 \cdots a_k b_k$. We can easily construct a morphism

$$i_*\colon \pi_1(A,\star) \to \pi_1(X,\star), \tag{11.11}$$

and we have another morphism

$$j_*\colon \pi_1(B,\star) \to \pi_1(X,\star). \tag{11.12}$$

Then we have

$$a_1 b_1 \cdots a_k b_k \mapsto i_*(a_1) j_*(b_1)(\cdots) i_*(a_k) j_*(b_k) \tag{11.13}$$

It is a morphism, but we should prove it is surjective. We have basically done it, because look: we have a space X which is covered by A and B. We can take any loop that goes through A and B by inserting a loop in $A \cap B$. Then we really have two loops: one in A and the other in B that agree at the two points obtained from the loop in $A \cap B$.

We will commit a crime—it's not a felony but it is a misdemeanor. We have $a_1, \dots, a_n, b_1, \dots, b_m$ be the generators with relations r_i and s_j. We are saying we should impose new relations, that's what happens when we factorize by N. What are these relations? Consider

$$u \in \pi_1(A \cap B, \star), \tag{11.14}$$

it can be mapped by

$$\alpha\colon \pi_1(A \cap B, \star) \to \pi_1(A,\star) \tag{11.15a}$$

and

$$\beta\colon \pi_1(A \cap B, \star) \to \pi_1(B,\star). \tag{11.15b}$$

There is no doubt we should add the relation

$$\alpha(u) \sim \beta(u) \tag{11.16}$$

Why? This means they will give the same element in $\pi_1(X,\star)$. The relations are in N, the question is: do we have something else? No, N is generated by these relations.

We are a little bit sloppy here. IF we are being completely honest, we really mean $(i_* \circ \alpha)(u) = (j_* \circ \beta)(u)$. But this is nothing terrible, so we continue to write $\alpha(u) = \beta(u)$.

Theorem 11.1 (van Kampen)**.** *If a_i are generators of $\pi_1(A,\star)$, r_i are the relations for $\pi_1(A,\star)$, and if b_i are the generators of $\pi_1(B,\star)$ and s_j are its relations; THEN $\pi(X,\star)$ has generators a_i, b_j with relations r_i, s_j and $\alpha(u) \sim \beta(u)$ for any $u \in \pi_1(A \cap B, star)$.*

We need to prove that there are no other relations. A precise proof can be found in Hatcher [3, §1.2]. A more general version of the theorem may be found in May [4, §2.7].

Sketch of Proof. What should we do? We have the following picture: A, B, $A \cap B$; they're all open, but this is irrelevant; if we have a closed path in X, then we can divide our path into small pieces in such a way that every small piece is in either A, B, or $A \cap B$. We consider another such path. We consider the deformation of these paths. But a deformation is a function of a rectangle. We divide up the rectangle such that they are only in A, B, or $A \cap B$. Then each rectangle is a deformation only in A, B, or $A \cap B$. This is not a precise proof, which may be found in Hatcher. □

Now we may apply this theorem in many ways. We may consider this for connected cell complexes. Our cell complex may be described in the following way:

$$X = X^1 \cup (\text{cell}) \tag{11.17}$$

Really we may say it is the disjoint union of open sets. We take

$$A = X^1 \sqcup \big(\text{cell} - \text{center}\big), \tag{11.18}$$

so really we may say that

$$A = X - (\text{center of the cell}) \tag{11.19}$$

And

$$B = (\text{open cell}). \tag{11.20}$$

At this moment, let us say the dimension of the cell is 2. Let us try to get the answer for this particular case. First

$$\pi_1(X, *) = \pi_1(X^1) \tag{11.21}$$

since A is homotopically equivalent to its boundary. So we have $\pi_1(B, *)$ be trivial since it is contractible. What about $\pi_1(A \cap B)$? Well, we see that

$$A \cap B \sim S^1 \times (0, 1) \sim S^1 \tag{11.22}$$

homotopic, so

$$\pi_1(A \cap B) \cong \mathbb{Z}. \tag{11.23}$$

We should factorize $\pi_1(X^1)$ by relations from $A \cap B$. If $u \in A \cap B$, it gives us an element in X^1, so it gives us a relation in X^1. Observe, higher dimensional cells do nothing.

Lecture 12.

We really did the main job with computing the fundamental group. We have a set X, a 2-dimensional cell σ^2, we consider $X \cup (\sigma^2 - 0)$. The intersection is $\sigma^2 - 0 \sim S^1$ homotopic. Then apply van Kampen's theorem, we have generators from X and from σ^2, we apply relations from X, and relations induced from the intersection. We have

TODO: need to carefully reconsider notation used

$$\pi_1(S^1) \to \pi_1(X) = \pi_1\big(X \cup (\sigma - 0)\big) \tag{12.1}$$

Note σ^2 is open, so it has a trivial fundamental group. When we attach a 2-cell, it gives us a new operation.

Lets consider a k-cell: $X \cup \sigma^k$, $X \cup (\sigma^k - 0)$. If $k > 2$, we have the intersection give us a trivial fundamental group.

How do we calculate $\pi_1(X)$ for some cell complex X? Easy, take the fundamental group of the skeleton of the cell complex $\pi_1(X^1)$. We have a graph, homotopic to a bouquet with k circles. This is all if X is connected. Then

$$\pi_1(X^1) \cong F_k \tag{12.2}$$

is the free group with k generators. The Euler characteristic of the bouquet is

$$\chi(X^1) = 1 - k. \tag{12.3}$$

We then attach 2-cells, and each one gives us a relation. Then higher dimensional cells, and they do nothing! So

$$\pi_1(X) \cong \pi_1(X^2) \tag{12.4}$$

is a canonical isomorphism. Its generators are given by $\pi_1(X^1)$ with a relation for each 2-cell. We don't really know if the group is trivial or not, and we *can't* know either. For the fundamental group we can get *any* group. We can add the 2-cells in any particular order to get relations. Furthermore, we can continue adding 2-cells to get *any* relation.

What we can really calculate is the "**Abelianization**" of π_1 which is $\pi_1/[\pi_1, \pi_1]$. This means we have generators and we have relations. If a_i are our generators, then we have the additional relation

$$a_i a_j = a_j a_i \tag{12.5}$$

So we have an Abelian group. This is an invariant which can be calculated efficiently. It is the first homology group.

Example 12.1 (g-Handled Sphere)**.** A sphere with g-handles. We have the same picture for a single hand repeated several times. We have $2g$ edges and a single 2-cell. So if we want to look at the fundamental group, look at what happens. We have a_1, b_1, $\dots$, a_g, b_g and only 1 vertex. The fundamental group is a group with $2g$ generators. We should have 1 relation because we have a single 2-cell. We delete one point and consider a loop around the deleted point. It is a circle which cannot be contracted, and thus has the relation

TODO: draw pictures on pp. 33–34

$$a_1 b_1 (a_1^{-1})(b_1^{-1})(a_2 b_2)(a_2^{-1})(b_2^{-1}) = 1 \tag{12.6}$$

in the case $g = 2$. But we still do not know, maybe this group is trivial. We go to Abelianization—-all generators commute. But then this relation becomes trivial! So in this case, Abelianization gives us a free Abelian group with $2g$-generators. We accidentally have a theorem.

Theorem 12.2. *The number of handles on a surface is a topological invariant.*

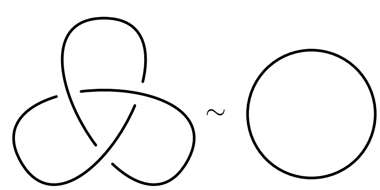
Example 12.3. Consider a sphere with g-handles and a hole. We have more generators. For $g = 2$, we have a_1, b_1, a_2, b_2, c. We have a relation of the form

$$a_1 b_1 (a_1^{-1})(b_1^{-1})(a_2 b_2)(a_2^{-1})(b_2^{-1}) c = 1 \tag{12.7}$$

We see we get a free group. We included a generator that is *not* necessary: we may express c in terms of a_i, b_j. So we end up with $2g$ generators, no relations, and thus a free group.

12.1 Knots

We would like to consider the topological classification of knots. We consider $\mathbb{R}^3$ (or S^3, for us it'd be equivalent). We consider a subset of $\mathbb{R}^3$ topologically equivalent to a circle. So a knot may be doodled on the right. But this means all knots are topologically equivalent to S^1. Consider two knots



$$K \subset S^3, \quad \text{and} \quad L \subset S^3 \tag{12.8}$$

If we can find a homeomorphism

$$\varphi\colon S^3 \to S^3 \tag{12.9}$$

such that

$$\varphi(K) = L \tag{12.10}$$

then two knots are "**Isotopic**".

We may consider $\pi_1(\mathbb{R}^3 - K)$, or $\pi_1(S^3 - K)$ since $S^3 = \mathbb{R}^3 \cup \infty$ through stereographic projection. From van Kampen's theorem, we have

$$\pi_1(\mathbb{R}^3 - K) \cong \pi_1(S^3 - K) \tag{12.11}$$

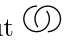
be a knot invariant. We see

$$S^3 - K \cong (\mathbb{R}^3 - K) \cup D^3 \tag{12.12}$$

where D^3 is an open ball "near infinity". By van Kampen's theorem, the intersection (D^3) is trivial, so we have

$$\pi_1(\mathbb{R}^3 - K) \cong \pi_1(S^3 - K) \tag{12.13}$$

which is an isomorphism. (Pop quiz: is it canonical?)

We would like to introduce the notion of a "**Link**" which is a disjoint union of knots (i.e., of topological circles). For example  is a link, but it is a trivial link. But  is two linked circles. These two pictures are topologically different. We can take

$$\pi_1(S^3 - \text{link})$$

and this will be the invariant of a link. If the invariant differs for two links, we have two topologically inequivalent links.

Lecture 13.

13.1 Projective Spaces

Recall the notion of a projective space $\mathbb{P}^n$. Consider the $(n+1)$-dimensional vector space $\mathbb{F}^{n+1}$ over $\mathbb{F}$. Consider all lines in $\mathbb{F}^{n+1}$ that contain the origin. We need to know only one point—then we know the line. (Why?) If $x \in \mathbb{F}^{n+1}$ and $x \neq 0$, then λx describes the lines, for arbitrary $\lambda \in \mathbb{F}$. The set of all such lines is the projective space over $\mathbb{F}$, denoted $\mathbb{FP}^n$. We can take

$$(\mathbb{F}^{n+1} - 0)/(x \sim \alpha x) \stackrel{\text{def}}{=} \mathbb{FP}^n \tag{13.1}$$

This is another description of it.

A point in projective space may be described by "**Homogeneous Coordinates**" denoted

$$(x_0 : x_1 : \cdots : x_n) \sim (\alpha x_0 : \alpha x_1 : \cdots : \alpha x_n). \tag{13.2}$$

(In physics, we have a similar picture where wave functions are defined up to a constant factor.) So this projective space contains an n-dimensional vector space

$$\mathbb{F}^n \subset \mathbb{FP}^n. \tag{13.3}$$

How? Consider $x_0 \neq 0$. Every point with this condition is equivalent to

$$(x_0 : x_1 : \cdots : x_n) = \left(1 : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right). \tag{13.4}$$

We take

$$y_i \stackrel{\text{def}}{=} x_i/x_0, \tag{13.5}$$

then

$$(x_0 : x_1 : \cdots : x_n) = (1 : y_1 : \cdots : y_n). \tag{13.6}$$

So $\mathbb{F}^n$ is sitting in projective space. The next thing we can do is take our projective space and delete this $\mathbb{F}^n$. What do we get? Well, it's quite simple: we get all the points where $x_0 = 0$. We thus get the picture that

$$\mathbb{FP}^n - \mathbb{F}^n = \mathbb{FP}^{n-1}. \tag{13.7}$$

So if you like, we may say that

$$\mathbb{FP}^n = \mathbb{F}^n \sqcup \mathbb{FP}^{n-1} \tag{13.8}$$

disjoint union of topological spaces.

Now we go to topology and consider two cases: $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$. Lets look at the simplest situations. What is $\mathbb{RP}^1$? It is very easy to see

$$\mathbb{RP}^1 = S^1 \tag{13.9}$$

We see

$$\mathbb{RP}^1 = \mathbb{R}^1 \sqcup \mathbb{RP}^0 \tag{13.10}$$

but $\mathbb{RP}^0$ consists of just a single point. What is $\mathbb{CP}^1$? Of course, we see

$$\begin{aligned}\mathbb{CP}^1 &= \mathbb{C} \cup \mathbb{CP}^0\\ &= \mathbb{C} \cup \{\text{point}\}\\ &= S^2\end{aligned} \tag{13.11}$$

as desired.

Let us look a little bit at

$$\mathbb{RP}^n = (R^{n+1} - 0)/(x \sim \lambda x) \tag{13.12}$$

But we may do something different. Namely every point on $\mathbb{R}^{n+1} - 0 \sim S^n$, why? We may divide x by $\|x\|$ so every point on $\mathbb{RP}^n$ may be represented by a point on a sphere. But still we should identify $x \sim \lambda x$ where both are on the sphere... but this happens when $|\lambda| = 1$. Therefore the only thing we should do is

$$\mathbb{RP}^n = S^n/(x \sim -x) \tag{13.13}$$

This may be represented by

$$\mathbb{RP}^n = \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}, \tag{13.14}$$

which is how we get a cellular decomposition (where we have a single k-cell in every dimension $k \leq n$).

For the complex case, we see

$$\begin{aligned}\mathbb{CP}^n &= (\mathbb{C}^{n+1} - 0)/(x \sim \lambda x)\\ &= S^{2n+1}/(x \sim \lambda x)\end{aligned} \tag{13.15}$$

where $|\lambda| = 1$. Is this true? Let first note

$$\mathbb{C}^{n+1} = \mathbb{R}^{2(n+1)} \tag{13.16}$$

but we demand $\|x\| = 1$ which eliminates a dimension, giving us

$$\mathbb{R}^{2(n+1)}/\sim = S^{2n+1} \tag{13.17}$$

This implies $|\lambda| = 1$. We can consider this set $S^1 = \{\lambda : |\lambda| = 1\}$ as a group. This group acts on S^{2n+1} simply by

$$x \mapsto \lambda x \tag{13.18}$$

One more definition of $\mathbb{CP}^n$. One more definition of $\mathbb{CP}^n$. This is

$$\mathbb{CP}^n = S^{2n+1}/S^1 \tag{13.19}$$

where we mod out by this action of S^1; we can write similarly

$$\mathbb{RP}^n = S^n/\mathbb{Z}_2 \tag{13.20}$$

since $\lambda \in \mathbb{R}$ and $|\lambda| = 1$ implies $\lambda = \pm 1$. Take $n - 1$ we get

$$\mathbb{CP}^1 = S^3/S^1 = S^2 \tag{13.21}$$

In a different way, we may say this as follows: there exists a mapping

$$h\colon S^3 \to S^2 \tag{13.22}$$

such that the preimage of a point $h^{-1}(\text{point}) = S^1$. We call h the "**Hopf Map**", later we will see h is not homotopic to 0, so it's very non-trivial.

13.2 Knot Rejoinder

How do we use h to analyze the structure of knots? We did say

$$h^{-1}(x) = S^1 \tag{13.23}$$

for any $x \in S^2$. We can see that

$$h^{-1}(\text{disc}) = \begin{pmatrix}\text{solid}\\ \text{torus}\end{pmatrix}. \tag{13.24}$$

How? Well, we could see the preimage $h^{-1}(S^1)$ is a torus, and S^1 is a boundary of a disc. So we fill in the disc "continuously" and we fill in the torus.

Lets be clear here. We are looking at S^2 by constructing it from two 2-discs

$$S^2 = (\bar{D}^2_0 \sqcup \bar{D}^2_1)/(\partial\bar{D}^2_0 \sim \partial\bar{D}^2_1) \tag{13.25}$$

We let

$$A = h^{-1}(\bar{D}^2_0) \tag{13.26a}$$

and

$$B = h^{-1}(\bar{D}^2_1) \tag{13.26b}$$

and we will use van Kampen's theorem.

So

$$h^{-1}(S^2) = A \cup B \tag{13.27}$$

where A, B are solid tori, and

$$A \cap B = T^2 \tag{13.28}$$

is a (non-solid) torus. We obtain this since

$$S^3 = \bar{D}^2 \cup \bar{D}^2. \tag{13.29}$$

We obtain this from the Hopf map.

Consider the solid torus in $\mathbb{R}^3$, it is bounded by the torus in $\mathbb{R}^3$. Consider this stuff in

$$S^3 = \mathbb{R}^3 \cup \infty. \tag{13.30}$$

We consider

$$S^3 - (\text{open solid torus}). \tag{13.31}$$

What do we get? A solid torus! We can see this result in our picture also. Of course, this is something with the solid torus as a boundary. The analog of the solid torus in higher dimension — we take a body with handle, it's called a "**Handle Body**". It's important to consider representations of them in "Heegaard Diagrams"… but we won't speak of it here.

We can take as a knot invariant $\pi_1(S^3 - K)$. Let us take for K a *trivial* knot, i.e., $K = S^1$. Then it's very simple, because we can take a small neighborhood of S^1, which is a solid torus, and

$$S^3 - K \sim \text{solid torus} \sim \text{circle} \tag{13.32}$$

are homotopy equivalences. Therefore

$$\pi_1(S^3 - K) \cong \pi_1(S^1) \cong \mathbb{Z} \tag{13.33}$$

But we would like to distinguish two unlinked circles from two linked circles:  vs. . We may take the first circle as K,

$$S^3 - K = \begin{pmatrix}\text{solid}\\ \text{torus}\end{pmatrix} \tag{13.34}$$

we then consider the second circle as an elemetn of $\pi_1(S^3 - K)$. If it's trivial, the knots are unlinked.

A knot is a topological circle in $\mathbb{R}^3$ or S^3. Now it is possible the circle lies on a torus. Then it's called a "**Torus Knot**". The next problem is to find

$$\pi_1\big(S^3 - (\text{solid torus})\big).$$

This will give us an invariant of the knot, we can classify them.

EXERCISES

- **Exercise 11.** Let $\mathbb{F}_p$ be the finite field with p elements, where p is prime. What is the cardinality of the set $\mathbb{FP}^n_p$?
- **Exercise 12.** Projective space $\mathbb{RP}^n$ can be obtained from the sphere S^n by means of identification of antipodes ($x \sim -x$). Describe the cell decomposition of $\mathbb{RP}^n$ and use it to calculate its fundamental group.
- **Exercise 13.** Let us consider an n-dimensional manifold X and its subspace $X = X - D^n$ (the space X with deleted open ball D^n). Express the fundamental group of X in terms of the fundamental group of X.
- **Exercise 14.** The connected sum of two n-dimensional manifolds X and Y is defined by means of deleting of open balls from X and Y and identification of boundaries of deleted balls. (In notations of Problem 13 we identify the boundary spheres in X and Y). Calculate the fundamental group of connected sum.

Remark 13.1. In Problems 13 and 14 we assume that $n > 1$. In the case $n = 2$ you can use the fact that every two-dimensional compact manifold is a sphere with attached handles and Moebius bands; if the manifold is not compact one should consider a sphere with holes (with deleted closed disks) instead of sphere.

Lecture 14. Toric Knots.

We would like to talk about toric knots. Recall last time we did the following: we considered

$$S^3 = \begin{pmatrix}\text{solid}\\ \text{torus}\end{pmatrix} \cup \begin{pmatrix}\text{solid}\\ \text{torus}\end{pmatrix} \tag{14.1}$$

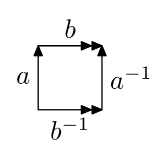
Remember that

$$\mathbb{R}^3 \cup \{\infty\} = S^3, \tag{14.2}$$

so

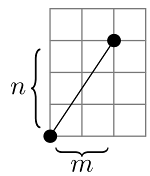
$$\mathbb{R}^3 - \begin{pmatrix}\text{solid}\\ \text{torus}\end{pmatrix} = \begin{pmatrix}\text{solid}\\ \text{torus}\end{pmatrix} \cup \{\infty\} \tag{14.3}$$

If we take a knot on the solid torus, we get a knot. Specifically when it is a non-self-intersecting curve on the torus. Recall a torus is identified from a square as doodled on the right, with the relation



$$aba^{-1}b^{-1} = 1 \iff ab = ba, \tag{14.4}$$

so we get the free Abelian group with 2 generators. We may consider instead $\mathbb{R}^2/\mathbb{Z}^2$ which is equivalent to what we have drawn.



But a vertex $(a, b) \sim (a+1, b) \sim (a, b+1)$. If we'd like to draw a closed curve on a torus we draw a line. Suppose we draw it on $\mathbb{R}^2$ with slope n/m for $m, n \in \mathbb{Z}$. Well, we can see if

$$\gcd(m, n) = 1 \tag{14.5}$$

then the curve on the torus (obtained by transporting the line using the quotient $\mathbb{R}^2/\mathbb{Z}^2$) is not self-intersecting. However, for

$$\gcd(m, n) \neq 1 \tag{14.6}$$

this curve covers the torus several times. There is another way to look at this picture. We may say when we factorize $\mathbb{R}^2/\mathbb{Z}^2$, we may choose the basis in whatever manner we want. The canonical choice is $(0, 1)$ and $(1, 0)$ ... but we may choose instead (a_1, b_1) and (a_2, b_2) but we require

$$\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \pm 1, \tag{14.7}$$

so we have

$$\begin{pmatrix} m \\ n \end{pmatrix} = x\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + y\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \tag{14.8}$$

for any integers m and n. This is a linear equation, but we need integer solutions. This demands the determinant conditions in Eq (14.7).

If we have $\gcd(a_1, b_1) = 1$, then we may always find a (a_2, b_2) such that

$$\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 1, \tag{14.9}$$

but that's trivial. To create a non-self-intersecting closed line, we can choose $\gcd(m, n)$.

What can we say about the complement of the curve? If we delete a single edge of the cell complex, we end up with a cylinder. How can we see this? Well, recall how we constructed the torus from the rectangle through first constructing a cylinder and then gluing the cylinder's top and bottom together. We simply undo this last step. For any line described by a relatively prime pair m and n, we can change the coordinates to get the same picture. What is the fundamental group generated by this guy? Well, if

$$u = (1, 0) \tag{14.10a}$$

and

$$v = (0, 1) \tag{14.10b}$$

then

$$u^m v^n = 1. \tag{14.10c}$$

Every solid torus has its fundamental group be $\mathbb{Z}$. But

$$(\text{solid torus}) \sim S^1 \tag{14.11}$$

homotopic, so we see why!

We take $m, n \in \mathbb{Z}$ such that $\gcd(m, n) = 1$, in the torus this gives us some information in the fundamental group, namely

$$\pi_1(\text{torus}) = u^m v^n. \tag{14.12}$$

We have two morphisms to fundamental groups of solid torus:

$$\begin{array}{ccc} \pi_1(\text{torus}) & = u^m v^n \longrightarrow & \pi_1\big(\text{solid-torus}\big) = v^n \\ \downarrow & & \\ \pi_1\big(\text{solid-torus}\big) & = u^m & \end{array} \tag{14.13}$$

We wil use van Kampen's theorem, and use the fact that

$$S^3 - K = \begin{pmatrix}\text{open}\\ \text{solid}\\ \text{torus}\end{pmatrix} \cup \begin{pmatrix}\text{open}\\ \text{solid}\\ \text{torus}\end{pmatrix} \cup (T^2 - K) \tag{14.14}$$

for some knot K, and $T^2 = S^1 \times S^1$ is the torus. We would like to apply van Kampen's theorem, and then we have a problem: the conditions of the theorem are not satisfied. Oru space

$$S^3 - K = A \cup B \tag{14.15}$$

should be represented as the union of open sets A, B. We considered morphisms

$$\pi_1(B) \leftarrow \pi_1(A \cap B) \rightarrow \pi_1(A) \tag{14.16}$$

but here we do not have this situation. The union of open sets *are not the whole space*. We also have this piece $(T^2 - K)$. If we consider the closures of the open tori, then $(T^2 - K)$ is contained in the other two. But this implies

$$\overline{(\text{solid torus})} \cap \overline{(\text{solid torus})} = (T^2 - K), \tag{14.17}$$

but *it is not open!* Well from the homotopy viewpoint, nothing happens.

First, this intersection has fundamental group

$$\pi_1(A \cap B) = \mathbb{Z} \tag{14.18}$$

But this generator may be written as $u^m v^n$, which is mapped to u^m for one of the solid torus' fundamental group, and v^n for the other's. The relation is simple:

$$u^m = v^n. \tag{14.19}$$

We computed the fundamental group of the toric know — but we have a question: can we say that toric knots are "the same"? It depends on $m, n \in \mathbb{Z}$. If we have the generators and relations, there is no algorithm to determine the group[2]. This is not an expression of our ignorance, but our knowledge: we *know* there exists no such algorithm.

The first tool is Abelianization. We may factorize with respect to our commutator. But if

$$\gcd(m, n) = 1, \tag{14.20}$$

then

$$\pi_1(S^3 - K)/[-,-] \cong \mathbb{Z}, \tag{14.21}$$

but this doesn't work in any case for knots — Abelianization of $\pi_1(S^3 - K)$ always results in $\mathbb{Z}$. We may take the fundamental group and factorize with respect to its center $\big(\pi_1/Z(\pi_1)\big)$ this is also an invariant of the knot. So for $\langle u, v \mid u^m = v^n\rangle$, what is the center?

First we see that u^m, v^n are both in the center, since the both commute with u and v. They're really generators for the center. If we consider $\pi_1/(u^m = v^n)$, we get a group $\langle u, v \mid u^m = v^n = 1\rangle$. What we get is the free product

$$\pi_1/(u^m = v^n) = \mathbb{Z}_m * \mathbb{Z}_n. \tag{14.22}$$

But what can we do? Again, we may take the Abelianization of this group

$$(\mathbb{Z}_m * \mathbb{Z}_n)/[-,-] = \mathbb{Z}_m \oplus \mathbb{Z}_n, \tag{14.23}$$

which becomes the direct sum. We have the same relation plus commutativity:

$$\langle u, v \mid u^m = v^n = 1, uv = vu\rangle.$$

[2] This is a "well known" result in group theory.

Since $\gcd(m,n) = 1$ we obtain

$$(\mathbb{Z}_m * \mathbb{Z}_n)/[-,-] = \mathbb{Z}_{mn}, \tag{14.24}$$

so we get mn is an invariant of the knot. But we still don't know if m, n *separately* are invariants of the knot.

We return to $\mathbb{Z}_m * \mathbb{Z}_n$, and consider torsion elements of this group. We may consider the maximal order of the torsion elements, and one can provide it is $\max(m,n)$. So we have complete information, assuming $m > 0$ and $n > 0$ (which we can always assume). It is sufficient to say both m and n are invariants. The lesson is *we have some tools to answer these questions.* If the group is finite, look at the order of elements, etc.

Lecture 15.

We covered van Kampen's theorem, which lets us calculate the fundamental group. There is another approach which is just as powerful.

Theorem 15.1. *Let X be simply connected, suppose a group G acts freely on X. IF we consider the space of orbits X/G, then $\pi_1(X/G) = G$.*

What does it mean "G acts freely"? Well, G acts on X means that for every element $g \in G$ we have a transformation

$$\varphi_g \colon X \to X \tag{15.1}$$

and we require

$$\varphi_{gh} = \varphi_g \varphi_h \tag{15.2a}$$

and

$$\mathrm{id}_X = \varphi_1. \tag{15.2b}$$

But we can write this as

$$\varphi_g(x) = gx, \tag{15.3}$$

where we have associativity

$$(gg')x = g(g'x). \tag{15.4}$$

This is the left action; there is a right action which is more or less the same stuff, but written on the right, e.g., $\varphi_{gh} = \varphi_h \varphi_g$. If we don't say explicitly otherwise, we use the left action.

We have a notion of an "**Orbit**" of every point. For one point, an orbit is

$$Gx = \{gx \mid g \in G\}. \tag{15.5}$$

We also have a "**Stabilizer**"

$$\mathrm{stab}(x) = \{g \in G \mid gx = x\} = H_x \tag{15.6}$$

It is a subgroup of G. A free action (if the group is finite) is very simple: all the stabilizers are trivial.

When X is a topological space, we require

$$\varphi_g \colon X \to X \tag{15.7}$$

to be continuous. We may say a little bit more, namely: what does it mean we have a free action on a topological space? We have an orbit, and each point in the orbit is distinct. So if $g, g' \in G$ and $g \neq g'$, then

$$gx \neq g'x. \tag{15.8}$$

What may we do? We may take a neighborhood of a point of the orbit. Then we get a neighborhood of *every* point of the orbit. Here we should note we assume X is at least Hausdorff. We may take U "sufficiently small" so that

$$U \cap gU = \emptyset, \tag{15.9}$$

we see that we may say that

$$g_iU \cap g_jU = \emptyset. \tag{15.10}$$

Now we may define the free group action.

Definition 15.2. A group G "**Acts Freely**" on X if for any $x \in X$ we may find a neighborhood $U \ni x$ such that $gU \cap g'U = \emptyset$ for distinct $g, g' \in G$.

Really we have already used this notion. Well, a particular case of it. We computed the fundamental group of S^1. But really we have

$$S^1 \cong \mathbb{R}/\mathbb{Z} \tag{15.11}$$

where the action is

$$x \mapsto x + n. \tag{15.12}$$

The orbit of $0 \in \mathbb{R}$ is precisely $\mathbb{Z}$. We have then

$$[0, 1]/(0 \sim 1)$$

we have a circle. So $\pi_1(S^1) = \mathbb{Z}$ by our theorem.

The torus is a more complicated example, since $\mathbb{R}^2/\mathbb{Z}^2$ describes the torus. The action is

$$(x, y) \mapsto (x + n_1, y + n_2) \tag{15.13}$$

So

$$\pi_1(T) = \mathbb{Z}^2. \tag{15.14}$$

More generally, the multidimensional torus is obtained from

$$T^n = \mathbb{R}^n/\mathbb{Z}^n \tag{15.15}$$

and this gives us

$$\pi_1(T^n) \cong \mathbb{Z}^n. \tag{15.16}$$

One more example. On the sphere we have the action of the group $\mathbb{Z}_2$. That is

$$S^2/\mathbb{Z}_2 \cong \mathbb{RP}^2, \tag{15.17}$$

so we have

$$\pi_1(S^2/\mathbb{Z}_2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2. \tag{15.18}$$

We can generalize to $\mathbb{RP}^n = S^n/\mathbb{Z}_2$ for $n \geq 2$. We see

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2. \tag{15.19}$$

But we cannot do this for S^1, since it is *not* simply connected!

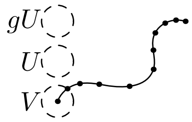
Let us suppose we have a path on X/G. It can be lifted to X. What does this mean? Recall we have a natural map

$$f\colon X \to X/G. \tag{15.20}$$

When we have a point $y \in X/G$, it may be considered by a set of points $\{x \in X \mid f(x) = y\}$. We may lift y to x. We may do this several different ways, and the number of different ways is precisely the number of elements of G. In other words

$$|G| = |f^{-1}(y)| \tag{15.21}$$

describes the cardinality. So this lifting of the path is not unique.



We divide our path into small pieces. This is doodled on the left. We lift the end point. Consider a neighborhood U such that

$$gU \cap U = \emptyset \tag{15.22}$$

for any $g \neq 1$ in the group G. But

$$f(U) = V = f(gU). \tag{15.23}$$

We may say that

$$f^{-1}(V) = \bigsqcup_{g \in G} gU. \tag{15.24}$$

We would like to lift our piece of path contained in V, but a path is continuous. Therefore if we lift it, the piece of path should be lifted and moreover the small piece is lifted uniquely. We want to lift the *whole* path. Now we use compactness to have a finite covering that may be lifted in a unique way.

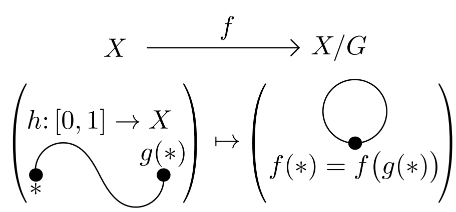
Lemma 15.3. *For every path, there is a unique lifting of the path for every lift of the starting point.*

So now we would like to say the following thing: suppose we have a family of paths $g_t(\tau)$ and everything varies continuously. Then the lifts also vary continuously. This follows almost immediately from uniqueness. We assumed that the starting points are lifted continuously. Now we can prove the theorem.

Proof (Theorem 15.1). This correspondence is very simple. Take any fixed (i.e., not varying) point $*$, we can consider its orbit $g(*)$. We may consider the path $h(t)$ where $h(0) = *$ and $h(1) = g(*)$. We apply our map f and we have

$$f\big(h(0)\big) = f\big(h(1)\big) \tag{15.25}$$

which is a loop! This defines an element of the fundamental group. Now there is a question whether we may say this element depends on the choice of g. But all paths are homotopic, by simple connectedness of X.



We then have a map $G \to \pi_1(X/G)$. We need to prove it is one-to-one, but we proved this: every point in the quotient may be lifted. This map is surjective. We should prove this map is injective (if we can lift the homotopy, then definitely g remains the same). We should prove it preserves multiplication, but we will do this next time.

Lecture 16. Fibrations.

I missed this lecture, but Steenrod's book [10] is the standard reference for fibre bundles. So instead, I will review the topological notion of fibre bundles.

In a sense it is a straightforward generalization of what we have: a group G acting freely on a topological space X. Then we have a surjective continuous map

$$p\colon X \to X/G. \tag{16.1}$$

We call the space X/G the "**Base Space**", and note that for any $x \in X/G$ we have

$$p^{-1}\{x\} \cong G \tag{16.2}$$

be a homeomorphism.

How do we generalize this? Well, the first steps is to let the base space B be any topological space. We have the "**Total Space**" be a topological space E. A "**Fibration**" is then a surjective continuous map

$$p\colon E \to B \tag{16.3}$$

with the extra condition that, for any $b, b' \in B$, we have $F_b = p^{-1}\{b\}$ and $F_{b'} = p^{-1}\{b'\}$ be the preimages such that

$$F_b \cong F_{b'} \tag{16.4}$$

are homeomorphic.

Example 16.1 (Trivial Fibration)**.** Example number zero is quite simple. Let F and B be topological spaces, and

$$E = F \times B \tag{16.5}$$

be the total space. It's just the product space. This is a basic fibre bundle; in fact, the fibre bundle *generalizes* the notion of a product space. But as a fibration, we call it the "**Trivial Fibration**" whenever the fibration is just the product space.

Example 16.2 (Tangent Bundle)**.** Consider a smooth n-dimensional manifold M, at each $x \in M$ we may consider the vector space T_xM of all tangent vectors with base point x. We can construct the total space

$$TM = \bigsqcup_{x \in M} T_xM \tag{16.6}$$

which is a fibration. Really? Well, we see that we have a continuous surjective function

$$p\colon TM \to M \tag{16.7}$$

which gives the base point of the tangent vector. What's the fibre? Well,

$$p^{-1}\{x\} = T_xM \cong \mathbb{R}^n \tag{16.8}$$

describes the fibre: it is the tangent vector space. The fibre is then just the vector space $\mathbb{R}^n$, where $n = \dim(M)$.

A "**Fibre Bundle**" is then a fibration (F, E, B, p) which is "**Locally Trivial**" in the sense that, $U \subset B$ open implies

$$p^{-1}(U) \cong U \times F \tag{16.9}$$

Another way to think of it is as pasting together a bunch of direct products.

EXERCISES

▸ **Exercise 15.** Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

▸ **Exercise 16.** Let X be the quotient space of S^2 obtained by identifying the north and south pole into a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

▸ **Exercise 17.** Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times x_0$ in one torus with the corresponding cycle $S^1 \times x_0$ in another torus.

Lecture 17.

We introduced a notion of fibration. If we have a map $p\colon E \to B$ which is surjective $p(E) = B$, we have fibres

$$F_b = f^{-1}\{b\} \tag{17.1}$$

for $b \in B$. If $F_b \cong F_{b'} \cong F$, where F is "some topological space", then we have a fibration. The space F is called the typical fibre.

There are very nice fibrations called "**Trivial Fibrations**" where $E = F \times B$ where all the fibres are canonically homeomorphic.

A mapping of fibrations $\varphi\colon E \to E'$ is such that such that

$$\varphi\colon F_b \to F'_{\varphi(b)} \tag{17.2}$$

We also demand that, if $p\colon E \to B$ and $p'\colon E' \to B'$, then

$$p' \circ \varphi = p \tag{17.3}$$

holds.

A "**Locally Trivial Fibration**" is precisely a fibre bundle. We take a fibration $p\colon E \to B$ and require for $U \subset B$ open that

$$p^{-1}(U) = U \times F. \tag{17.4}$$

Another way to think of it is as pasted together from direct products. We have a cover $\{U_\alpha\}$ of B, we consider $U_\alpha \times F$ pasted together, i.e., $(U_\alpha \cap U_\beta) \times F \subset U_\alpha \times F$ and $(U_\alpha \cap U_\beta) \times F \subset U_\beta \times F$, the intersection is embedded in both. We should have a map that identifies the overlap

$$(b, f) \sim \big(b, \varphi_{\alpha\beta}(b)f\big) \tag{17.5}$$

where

$$\varphi_{\alpha\beta}(b)\colon F \to F \tag{17.6}$$

which is defined for all $b \in U_\alpha \cap U_\beta$. We can introduce different notation

$$\widetilde{\varphi}_{\alpha\beta}(b, f) = (b, \varphi_{\alpha\beta}(b)f). \tag{17.7}$$

These guys are called the "**Transition Functions**" which obeys some properties of compatibility.

We may consider the case when the fibre is discrete. Then a locally trivial fibration is a covering. This is a definition. We have $p\colon E \to B$, consider

$$p^{-1}(U) = \begin{pmatrix} \text{disjoint union} \\ \text{of homeomorphic} \\ \text{components} \end{pmatrix} \tag{17.8}$$

where $U \subset B$ is "small." But $p^{-1}(U) = U \times F$, but F is a discrete space where every point is open.

Example 17.1. Let G be a discrete group that acts freely on X. We have the identification map

$$\pi\colon X \to X/G \tag{17.9}$$

and this is a covering. This is called a "**Regular Covering**"

Remember we assumed X is simply connected, so by definition it is a regular covering — but it has one more name: a "**Universal Covering**". Every connected covering may be obtained from the universal one. We will focus exclusively on connected coverings for the rest of this lecture.

Why is this universal? If we have a (regular) covering, we may obtain other coverings in the following way: let

$$G = \pi_1(X/G) = \pi_1(X^1) \tag{17.10}$$

We have $X^1 = X/G$ be connected. We may take *any* subgroup $H \subset G$ which acts freely on X. Therefore we may repeat the same construction, obtaining a map

$$X \xrightarrow{\alpha_H} (X/H) \xrightarrow{\pi_H} (X/G). \tag{17.11}$$

The first remark is that α_h, π_H both are coverings, moreover α_H is a regular covering. We take some $V \subset X/G$. We have its preimage

$$\pi^{-1}(V) = \sqcup U_i = \sqcup g_i U \tag{17.12}$$

where $g_i \in G$. This induces an open covering. When we consider $\pi_H^{-1}(V) = \sqcup U_i$, we recall $\pi_H^{-1}(V) \subset \pi^{-1}(V)$ is a proper subset. All connected coverings of X/H may be obtained in this way.

One more remark. Is this quotient X/H always a regular covering? Not always. But it is clear if $H \subset G$ is a normal subgroup, then π_H^{-1} is a regular covering. Because (G/H) acts on (X/H). The points of (X/H) are orbits Hx; and if $H \subset G$ is normal, and $\gamma \in G/H$, then it acts on (X/H) by taking $g \in \gamma$ and applying it to hx. We may represent it as

$$ghx = hgx \tag{17.13}$$

so $ghg^{-1} = h' \in H$ by virtue of H being a normal subgroup.

Theorem 17.2. *Connected coverings of a "good" space Y are in one-to-one correspondence with subgroups of $\pi_1(Y)$. Connected regular coverings correspond to normal subgroups of $\pi_1(Y)$.*

A "good" space is one with at least one universal covering, but in reality we need less. The idea is that it is a maximal covering — so a good space has small closed loops be contractible. It is not precisely clear where they are contractible. More precisely every point has a neighborhood such that closed loops are contractible in a bigger neighborhood.

Lecture 18.

Proposition 18.1. *Simply connected covering is unique.*

The "unique" means that two simply connected coverings are the same. Nothing is said of their *existence!*

Proof. Let $p\colon \widetilde{X} \to X$ be a simply connected covering. So X is connected. We may declare any point in $\widetilde{X}$ to correspond to the marked point in X, i.e.,

$$p(*) = *. \tag{18.1}$$

Now we take $x \in \widetilde{X}$ so $p(x) \in X$. We will take a path from $*$ to x in $\widetilde{X}$,

$$\alpha\colon [0,1] \to \widetilde{X} \tag{18.2}$$

such that $\alpha(0) = *$, $\alpha(1) = x$. Now this path may be projected to X. Every path is homotopic in $\widetilde{X}$, thus when projected to X they remain homotopic. So we have

$$x \in \widetilde{X} \rightleftharpoons \left(p(x) \in X, \quad \begin{matrix}\text{homotopy class of}\\ \text{paths from } * \text{ to } p(x)\end{matrix}\right) \tag{18.3}$$

If we are given only X, we may construct

$$\widetilde{X} = (z \in X, \text{homotopy class of paths from } * \text{ to } z). \tag{18.4}$$

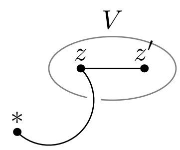
But this means that $\widetilde{X}$ is basically unique, so there exists a unique simply connected covering. Moreover this appears to be a construction that always gives something we call a "simply connected covering"—but this is wrong. It gives a set. We need to prove it is a topological space, and that it is a covering. Lets suppose a simply connected covering exists. If $V \ni *$ is a neighborhood, then it is covered by disjoint sets in $\widetilde{X}$

$$(U_1 \sqcup U_2 \sqcup \dots) = p^{-1}V, \tag{18.5}$$

every map $U_i \to V$ is a homeomorphism. But then let us take any closed path in V. It is a small path, it's covered by some path $\widetilde{\alpha} \in U_i$ such that

$$p(\widetilde{\alpha}) = \alpha. \tag{18.6}$$

But this covering path $\widetilde{\alpha}$ is contractible in $\widetilde{X}$. It follows that α is contractible in X. This isn't always true, there is a pathological counter-example.



A neighborhood U of a pair (z, path). Let V be a neighborhood of z. Then U has

$$(z', \text{path}') \in U \tag{18.7}$$

where $z' \in V$. We have

$$\text{path}' = \text{path} * \gamma \tag{18.8}$$

where $\gamma\colon [0,1] \to V$ has

$$\gamma(0) = z', \quad \gamma(1) = z. \tag{18.9}$$

This gives us a topology, and it is simply connected, etc. What should be required is really something a little bit stronger:

Requirement: for every point $z \in X$ and every neighborhood V of z, there exists a smaller neighborhood $U \subset V$ such that every closed path in U is contractible in V.

On $\widetilde{X}$ we can define an action of $\pi_1(X, *)$ such that (1) this action is free, (2) $X = \widetilde{X}/\pi_1(X, *)$. How to prove this?

We lift everything we did in the opposite order. Remember

$$\widetilde{X} = (z \in X, \text{path from } * \text{ to } z \text{ in } X) \tag{18.10}$$

We see

$$\beta\colon I \to X \tag{18.11}$$

with $\beta(0) = \beta(1) = *$, which specifies an element of $\pi_1(X)$. Further, α is a path from $*$ to z. We concatenate $\beta * \alpha$, we changed the path though not the end points. We remain in the same fibre. Thus it's an action of $\pi_1(X)$. So it justifies the statement $X = \widetilde{X}/\pi_1(X)$.

So we've shown: a simply connected covering is unique, and it is a regular covering. But what about other coverings? The simply connected covering is universal. It is easy to see without formal proofs. Let us take any connected covering of X, denote it by Y. But it also has a simply connected covering $\widetilde{Y}$ of Y. But now the covering of a covering is a covering. So it is the simply connected covering of X, i.e.,

$$\widetilde{X} = \widetilde{Y}. \tag{18.12}$$

We may therfore say

$$Y = \widetilde{X}/\pi_1(Y) \tag{18.13}$$

and that's the end of the story.

We should check the simply connected covering exists. But we may prov ethis. Instead, we'll do something different. Namely, look: we take some covering Y and we project

$$\pi\colon Y \to X. \tag{18.14}$$

But from this construction it follows we may construct a map $\widetilde{X} \to Y$ by definition of $\widetilde{X}$. It is almost obvious this map is a covering.

EXERCISES

▸ **Exercise 18.** Describe all connected coverings of Moebius band. Show that all compact covering spaces are homeomorphic either to Moebius band or to annulus.

▸ **Exercise 19.** Let us consider a group G of transformations of the plane $\mathbb{R}^2$ generated by transformations $(x, y) \to (x, y+2)$ and $(x, y) \to (x+1, -y+1)$.

1. Prove that G acts freely on $\mathbb{R}^2$.
2. Prove that the quotient space $\mathbb{R}^2/G$ is homeomorphic to Klein bottle (i.e. it can be be obtained from two Moebius bands by means of identifying their boundaries).
3. Use this construction to show that there exists a covering of Klein bottle homeomorphic to a torus.

▸ **Exercise 20.** One says that a map $p\colon X \to Y$ is a ramified covering of the surface Y if the map p is a covering of the set $Y - F$ where F is a finite subset of Y (i.e. the map $p\colon p^{-1}(Y) \to Y$ is a covering. Let us suppose that p is n-sheeted covering of $Y - F$ (i.e. every point of $Y - F$ has n preimages) and the ramification points (points of F) have μ_1, $\ldots$, μ_k preimages (here k stands for the number of points in F). Prove the following formula connecting Euler characteristics of X and Y:

$$\chi(X) = n\chi(Y) + \sum_{1 \le i \le k} (\mu_i - n) \tag{18.15}$$

(this is a version of Riemann-Hurwitz formula).

Hint. You can use additivity of Euler characteristic.

▸ **Exercise 21.** Check the Riemann-Hurwitz formula for the map $p\colon S^2 \to S^2$ defined by the formula $p(z) = z^n$ (we consider the sphere as the set of complex numbers with addition of a point at infinity: $S^2 = \mathbb{C} \cup \infty$).

▸ **Exercise 22.** Calculate the Euler characteristic of a surface defined by the equation $w^2 = q(z)$ where $q(z)$ is a polynomial of degree m. (Such a surface is called elliptic if $m = 3$ or $m = 4$ and hyperelliptic if $m > 4$). Here $w, z \in S^2 = \mathbb{C} \cup \infty$.

Hint. Consider the surface as a ramified covering of the sphere. Take into account that for $m = 2n$ the function $w = \pm\sqrt{q(z)}$ looks like $w = \pm z^n$ at infinity and therefore consists of two separate branches; this means that in the neighborhood of infinity we have two-sheeted covering. For $m = 2n + 1$ this function looks like $w = \pm z^n \sqrt{z}$, hence we have a ramification point at infinity.

Lecture 19. Homotopy Groups.

TODO: Write up the notess on homotopy groups from Schwarz's book [9, Ch. 8].

Iterative construction of homotopy groups

We will start with a reminder of the fundamental group. We took a space X with a marked point $* \in X$. We take $\Omega(X, *)$ to be the space which consists of loops in X that have base point $*$. We considered the connected components of the space Ω, i.e., $\pi_0(\Omega, *)$. We then just

$$\pi_1(X, *) = \pi_0(\Omega, *). \tag{19.1}$$

We use topological invariant of Ω to get a topological invariant of X. But we can use anything we want! We can keep iterating, and take

$$\pi_1(\Omega, *) = \pi_2(X, *) \tag{19.2}$$

is a topological invariant of X, we have functoriality. If $f\colon X \to Y$, then $\Omega_X \to \Omega_Y$. We can iterate this construction n-times to get π_n, but we will focus on π_2.

Different Construction

Look, what are the elements of Ω? They are maps of an itnerval to X:

$$\alpha\colon I \to X \tag{19.3}$$

which are closed paths, i.e.,

$$\alpha(0) = \alpha(1) = *. \tag{19.4}$$

We want to consider the fundamental group of Ω. A path of Ω is then of the form $\alpha_\tau(t)$ where τ is the parameter of the path in Ω. Perhaps it is better to write

$$\alpha_\tau(t) = \alpha(\tau, t). \tag{19.5}$$

We want to have a closed path that starts and ends at

$$\alpha_\tau(0) = \alpha_\tau(1) = *, \quad \text{and} \quad \alpha_0(t) = \alpha_1(t) = *. \tag{19.6}$$

We can look at $(\tau, t) \in I^2$ as a square. So in other words, our path α is a mapping

$$\alpha\colon I^2 \to X \tag{19.7}$$

such that

$$\alpha(\partial I^2) = * \tag{19.8}$$

it takes the boundary to the marked point. This mapping is called a "**Spheroid**". Why is this name a good one? Well, recall $I^2/\partial I^2 \cong (S^2, *)$ is a homeomorphism. So equivalently, we have it be

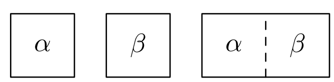
$$\alpha\colon (S^2, *) \to (X, *) \tag{19.9}$$

a mapping from the marked 2-sphere to the marked space $(X, *)$. Using this language, π_2 is the collection of homotopy classes of spheroids.

Remark 19.1 ("Spheroid" and Nationality). IT appears that only the Russian topologists use the word "spheroid" in their writing. A cursory google search would reveal that it is a relatively antiquated or esoteric word, and indeed only Russians employ it.

We will abuse language and notation to refer to the mapping and/or the cube as the "spheroid."

Suppose we have two spheroids α and β. We can construct a third spheroid using

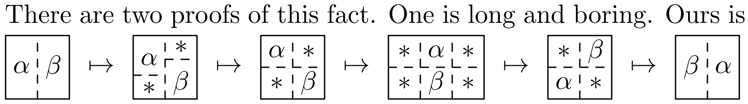


$$\gamma(\tau, t) = \begin{cases} \alpha(2\tau, t) & 0 \le \tau \le \frac{1}{2} \\ \beta(2\tau - 1, t) & \frac{1}{2} \le \tau \le 1 \end{cases} \tag{19.10}$$

This is doodled to the above right, taking the two spheroids labeled α and β and combines them into the third as we specified. This is precfisely a reformulation of the definition of π_2 in terms of π_1.

Theorem 19.2. *The group $\pi_2(X, *)$ is commutative.*

There are two proofs of this fact. One is long and boring. Ours is a sequence of pictures:



Now, let us introduce the ntoion of a topological group G which is a group object internal to **Top**. In other words, it consists of a topological space G equipped with "group structure": continuous maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$. We will prove that

$$\pi_1 \begin{pmatrix} \text{Topological} \\ \text{Group} \end{pmatrix} \in \mathbf{Ab} \tag{19.11}$$

We can define the multipication of paths as one of two ways: (1) concatenation $\alpha * \beta$; (2) $\gamma(t) = \alpha(t)\beta(t)$ pointwise using the group's multiplication.

These are the same up to homotopy. And that is obvious. Why? Lets write down

$$\alpha' = \alpha * e \tag{19.12}$$

where the constant path $e(t) = e$ is the multiplicative identity. Obviously $\alpha' \sim \alpha$ homotopic. Also take

$$\beta' = e * \beta \tag{19.13}$$

then we see

$$\alpha' * \beta' = \alpha'\beta' \tag{19.14}$$

pointwise.

We can consider the opposite group G^{op} where multiplication is defined as

$$x \circ^{\mathrm{op}} y = x \circ y \tag{19.15}$$

But

$$\alpha\beta \sim \beta\alpha \tag{19.16}$$

Thus we deduce commutativity up to homotopy. We want to prove π_2 is commutative. What to do? We apply this statement to Ω. But it is not a group. It has multiplication, and a unit. But this is up to homotopy! Nevertheless, everything goes through for Ω, so $\pi_1(\Omega)$, which is an H-space ("almost topological group"), is commutative. Thus $\pi_2(X)$ is commutative.

Lecture 20.

Last time we gave a definition of π_2, the second homotopy group of a space with a marked point. We can generalize to $\pi_n(X, *)$ by considering

$$f\colon I^n \to X \tag{20.1}$$

and requiring

$$f(\partial I^n) = \{*\}. \tag{20.2}$$

This guy is called an n-dimensional "**Spheroid**". We can define the concatenation of spheroids

$$h = f * g \tag{20.3}$$

by taking the domain and splitting it up (as we did for curves). So

$$h(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \le t_1 \le \frac{1}{2} \\ g(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \le t_1 \le 1. \end{cases} \tag{20.4}$$

Here $\pi_n(X, *)$ is the set of homotopy classes of spheroids — if we can deform one spheroid to another while remaining a spheroid throughout deformation.

There is another definition which may be given through induction. We have Ω be the sapce of paths beginning and ending at $*$. We inductively define $\pi_n(X, *) = \pi_{n-1}(\Omega, *)$. If $f \in \pi_{n-1}(\Omega, *)$, then we see it is

$$f\colon I^{n-1} \to \Omega \tag{20.5}$$

and therefore we write

$$f_\tau(t_1, t_2, \ldots, t_{n-1}) = f(t_1, \ldots, t_{n-1}, \tau)$$

This is an n-dimensional spheroid. We see this is a correspondence between $(n-1)$-dimensional spheroids in Ω and n-dimensional spheroids in X. We should check concatenation is preserved. We can take concatenation with any coordinate, so use τ and we're done.

We can give a third definition. Namely if we write the spheroid as a map

$$I^n/\partial I^n \to X \tag{20.6}$$

taking the boundary to the marked point $* \in X$. We may say $\pi_n(X, *)$ is the homotopy group of spheroids; it's clearer conceptually, but the operation is ambiguous.

The first thing to state is π_n is Abelian for $n \geq 2$. The second thing is that π_n is a functor. If $f\colon X \to Y$ is a map of topological spaces such that $f(*)*$ then

$$\pi_n\big(f\colon X \to Y\big) \quad = \quad f_*\colon \pi_n(X, *) \to \pi_n(Y, *) \tag{20.7}$$

and it has functorial properties

$$(f \circ g)_* = f_* \circ g_*, \quad \text{and} \quad \mathrm{id}_* = \mathrm{id} \tag{20.8}$$

as desired.

Theorem 20.1. *If $(X, *) \sim (Y, *)$ homotopy equivalent, then $\pi_n(X, *) \cong \pi_n(Y, *)$ is an isomorphism of groups for all n.*

Proof. We have $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$f \circ g \sim \mathrm{id}_Y \tag{20.9}$$

and

$$g \circ f \sim \mathrm{id}_X \tag{20.10}$$

are homotopic. Then we may write $g_* \circ f_* = \mathrm{id}_{\pi_n(X)}$ and $f_* \circ g_* = \mathrm{id}_{\pi_n(Y)}$. Thus we may speak of π_n as an invariant. □

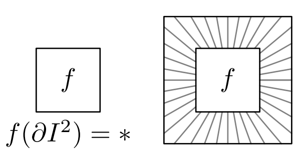
There is a repetition: everything we did for π_1 we do for π_n.

Suppose X is connected. Then

$$\pi_n(X, *) \cong \pi_n(X, \widetilde{*}) \tag{20.11}$$

is isomorphic, but it is not a canonical isomorphism: there are many. Lets consider the $n = 2$ proof, we may draw pictures. Everything is very simple. We have a spheroid

$$f\colon I^2 \to X, \quad f(\partial I^2) = *. \tag{20.12}$$



We want to create a spheroid which has the property that the boundary goes to another marked point $\widetilde{*}$. We take a larger square, as doodled on the left, and embed our spheroid into it. The "bonus space" is just concatenation with the trivial spheroid, mapping everything to the marked point. One could also think of this as mapping the boundary "thicker"[3]. This is a bunch of paths connecting $*$ to $\widetilde{*}$, which are doodled in light gray. Our extended version of f is constructed in a nonunique way. This α connects $*$ to $\widetilde{*}$, lets abuse notation to write

$$\alpha\colon \pi_n(X, *) \to \pi_n(X, \widetilde{*}). \tag{20.13}$$

We could consider α^{-1}, or better

$$\beta(\tau) = \alpha(1 - \tau) \tag{20.14}$$

which goes in the opposite direction. So we may use it to construct a map

$$\beta\colon \pi_n(X, \widetilde{*}) \to \pi_n(X, *). \tag{20.15}$$

[3] I suppose the technical term is "adding a collar" to our boundary region, if one prefers this exotic terminology.

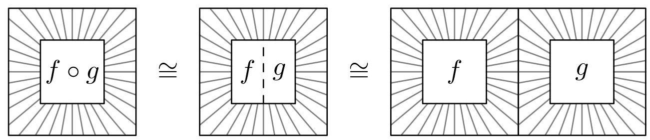
Now we may state, at the level of π_n, we have $\alpha \circ \beta = 1$ and $\beta \circ \alpha = 1$. Why? Because

$$\alpha \circ \beta \sim \begin{pmatrix}\text{trivial}\\ \text{path}\end{pmatrix} \tag{20.16a}$$

homotopic, and

$$\beta \circ \alpha \sim \begin{pmatrix}\text{trivial}\\ \text{path}\end{pmatrix} \tag{20.16b}$$

homotopic too.



We should note that we have $\alpha(f \circ g) = \alpha(f) \circ \alpha(g)$, which we prove with a series of pictures doodled to the right. These isomorphism are quite nontrivial. We've seen for $n = 1$ they're nontrivial. In particular we may take the marked points to be equal $* = \widetilde{*}$. By taking a nontrivial path, we get an isomorphism

$$\alpha\colon \pi_n(X, *) \to \pi_n(X, *). \tag{20.17}$$

We may say $\alpha(f) = \alpha \circ f \circ \alpha^{-1}$, so this is the case for $* = \widetilde{*}$, we may write this for any case. But then α^{-1} is not very well defined. We've shown that $\pi_1(X, *)$ acts on $\pi_n(X, *)$ by means of automophisms. So we have

$$\pi_1(X, *) \to \operatorname{Aut}\bigl(\pi_n(X, *)\bigr) \tag{20.18}$$

Did we prove this? Not really. We proved there exists a map, but we need to prove it is a morphism. The proof, however, is easy. So what did we prove? We proved if X is connected, then $\pi_n(X, *)$ does not depend on the marked point $*$.

Lets consider the homotopy class

$$\{S^n, X\} = \{\text{orbits of } \pi_1(X, *) \text{ in } \pi_n(X, *)\} \tag{20.19}$$

provided X is connected. This is merely a structureless set. The proof of this is very simple. First there is a natural map

$$\pi_n(X, *) \to \{S^n, X\}, \tag{20.20}$$

the only difference is we fix a point in the domain—this map is surjective (that's obvious). Because there is a map

$$f\colon S^n \to X \tag{20.21}$$

such that $* \mapsto f(*) = \widetilde{*}$. Is this a spheroid? No. But we may take another path α from $*$ to $*$, so we can consider $\alpha(f)$. We explained how to do this. The remark is $\alpha(f) \sim f$ homotopic, as a map of spheres. We should look at the kernel of

$$\pi_n(X, *) \to \{S^n, X\}. \tag{20.22}$$

It is clear $\alpha(f) \sim f$ homotopic as spheroids, or more precisely as a map of spheres.

Lecture 21.

We defined homotopy groups which generalize the fundamental group. We would like to compute the simplest example. Homotopy groups are Abelian, and Abelian groups are easy to describe — but hard to compute.

The first thing we will do is calculate the homotopy group of $\mathbb{R}^n$. It is immediately obvious: the homotopy group is trivial, since $\mathbb{R}^n$ is contractible. Alternatively: if we have a spheroid, then we may contract it.

What about $\pi_k(S^n)$? Well, for $k < n$, we have a spheroid

$$f\colon I^k \to S^n \tag{21.1}$$

such that $f(\partial I^k)$ is the north pole. Well, if $k < n$ and f is "not too bad", then it doesn't cover the whole sphere. This means there is a point x_0 which is not contained in $f(I^k)$. We recall

$$S^n - \{x_0\} = \mathbb{R}^n \tag{21.2}$$

and every spheroid is contractible. But what does it mean for f to be "not too bad"? We should approximate our map by a polynomial, or a C^r map, or a piecewise-linear map; then this approximation is homotopic to f.

What about $\pi_n(S^n)$? Well, we have

$$\pi_n(S^n) = \{S^n, S^n\}, \tag{21.3}$$

but why? This is normally not the case! In general we have

$$\{S^k, X\} = \begin{pmatrix}\text{orbit of}\\ \pi_1(X) \text{ in } \pi_k(X).\end{pmatrix} \tag{21.4}$$

For $\pi_1(X)$ Abelian, we see

$$\{S^1, X\} = \pi_1(X). \tag{21.5}$$

(Recall how $\pi_1(X)$ acts on $\pi_k(X)$ via inner automorphisms.) So what happens for $k > 1$? Well, we see

$$\pi_n(S^n) = \{S^n, S^n\} \xrightarrow{1:1} \mathbb{Z} \tag{21.6}$$

We even talked about this before: we proved it for $n = 1$, but for $n > 1$ it is also true. If $f\colon S^n \to S^n$, then $f^{-1}(x)$ is an algebraic number called the "**Degree**" of f.

What happens for $\pi_k(S^n)$ and $k > n$? Consider $\pi_3(S^2)$, we constructed a map $S^3 \to S^2$, the Hopf map and it is not contractible. The explanation for this, recall the preimages for points were circles. The preimages for two points are two linked circles. One may see this group is nontrivial. What about in general? This situation is very nontrivial. We have

$$\pi_k(S^k) \cong \mathbb{Z} \tag{21.7a}$$

and

$$\pi_{4k-1}(S^{2k}) \cong \mathbb{Z} + \begin{pmatrix}\text{finite}\\ \text{group}\end{pmatrix} \tag{21.7b}$$

All others are finite!

We will consider a covering $p\colon \widetilde{X} \to X$ where X, $\widetilde{X}$ are connceted. We would like to consider the situation with homotopy groups

$$\pi_k(\widetilde{X}, *) = \pi_k(X, *) \tag{21.8}$$

for $k \geq 2$. The proof is as follows: ifwe have a path in X, it may be uniquely lifted to a path in $\widetilde{X}$ — well, not uniquely, but if we lift the starting point then it's *unambiguous*.

So we now have a spheroid in X. The trick is to look at a spheroid as a family of paths in X, then we may lift it to $\widetilde{X}$. But this lifts the paths, does it lift the spheroid? Well, the initiall point of the paths are lifted to the marked point. When we lift $f(\partial I^k)$ to the marked point, then we may lfit the spheroid. In principle, if

$$f(\partial I^k) \subset p^{-1}(*) \tag{21.9}$$

so we have some freedom in lifting.

Remember we considered G acting on a simply connected space $\widetilde{X}$, we let

$$X = \widetilde{X}/G. \tag{21.10}$$

Assume G acts freely, we proved $G = \pi_1(X)$. We can get every universal covering this way. We now know $\pi_n(X) = \pi_n(\widetilde{X})$ for $n \geq 2$. We see therefore that

$$G \cong \pi_1(X) \tag{21.11}$$

acts on $\pi_n(X)$. But we kknow $\pi_1(X)$ acts on $\pi_n(X)$, we merely proved it in a different way. But this is geometrically obvious. Lets use this for an interesting calculation.

We know the universal covering of S^1 is $\mathbb{R}$, so

$$S^1 = \mathbb{R}/\mathbb{Z}. \tag{21.12}$$

The action of $\mathbb{Z}$ on $\mathbb{R}$ is $x \mapsto x + n$. We know

$$\pi_1(S) = \mathbb{Z}, \quad \text{and} \quad \pi_n(S^1) = 0 \tag{21.13}$$

for $n \geq 2$. We have $\{S^n, S^1\}$ be trivial, for $n \geq 2$.

A more interesting case: we have $\mathbb{Z}_2$ act on S^n by $x \mapsto -x$. We see

$$S^n/\mathbb{Z}_2 = \mathbb{RP}^n \tag{21.14}$$

We know

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2, \qquad \pi_k(\mathbb{RP}^n) \cong \pi_k(S^n) \tag{21.15}$$

otherwise. In particular

$$\pi_n(\mathbb{RP}^n) \cong \mathbb{Z} \tag{21.16}$$

Consider $\{S^n, \mathbb{RP}^n\}$, we see $\deg(\mathrm{id}_{S^n}) = 1$ and

$$\begin{aligned} \varepsilon\colon S^n &\to S^n \\ x &\mapsto -x \end{aligned} \tag{21.17}$$

has $\deg(\varepsilon) = \pm 1$. The conclusion is, for n even, we have

$$\{S^n, \mathbb{RP}^n\} = \mathbb{Z}/(n \sim -n) \tag{21.18}$$

whereas for n odd we have no identification.

Lecture 22.

Let Σ be a 2-dimensional manifold. Let $\widetilde{\Sigma}$ be the universal cover of Σ. But there are only two choices for $\widetilde{\Sigma}$: S^2 or $\mathbb{R}^2$. For simplicity we will suppose that Σ is compact. We know how to calculate $\pi_1(\Sigma)$, the only thing we need is the statement that $\pi_1(\Sigma)$ is finite in two cases: $\Sigma = S^2$ or $\mathbb{RP}^2$. This is easy looking at the Abelianization of the fundamental group. In both of these cases, $\widetilde{\Sigma} = S^2$. We have

$$\pi_k(\Sigma) = \pi_k(\widetilde{\Sigma}) \tag{22.1}$$

for $k \geq 2$. Now let us suppose $\pi_1(\Sigma)$ is infinite. Then $\widetilde{\Sigma}$ is not compact. Why? Because when we look at the covering

$$\widetilde{\Sigma} \to \Sigma$$

the number of sheets in this covering are the number of elements in $\pi_1(\Sigma)$, which is infinite. Over every disc, we have an infinite number of discs, which is definitely noncompact. We have only one chocie for $\widetilde{\Sigma}$. We see then that

$$\pi_k(\Sigma) = \pi_k(\mathbb{R}^2) = 0 \tag{22.2}$$

for $k \geq 2$.

We would like to show

$$\pi_k(X \times Y) \cong \pi_k(X) \times \pi_k(Y). \tag{22.3}$$

It's a one minute proof. If we have a mapping

$$f\colon Z \to X \times Y = \{(x, y)\} \tag{22.4}$$

this map means $(x, y) = f(z)$. This means

$$x = f_1(z), \quad \text{and} \quad y = f_2(z). \tag{22.5}$$

When we apply this to spheroids, everything follows. When we deform $f(z)$, we deform these two guys. We have, e.g., an n-torus be

$$T^n = (S^1)^n \tag{22.6}$$

so $\pi_k(T^n) \cong \pi_k(S^1)^n$.

22.1 Relative Homotopy Groups

We will have a pair of topological spaces X, A (so $A \subset X$), and consider $* \in A \subset X$. For simplicity, A and X are connected. We will define a "**Relative Homotopy Group**" $\pi_n(X, A, *)$ or sometimes $\pi_n(X, A)$. We will neglect something in X, namely, we neglect A — this is the notion of "relative".

Recall we defined π_n by means of spheroids

$$(S^n, *) \to (X, *).$$

The homotopy group $\pi_n(X, *)$ is then the homotopy classes of spheroids. This is nice but incomplete. We need to define an operation. We did this by considering a spheroid as a map on a cube, generalizing concatenation.

Lets consider something similar for relative homotopy groups. We introduce "**Relative Spheroids**" $(\bar{D}^n, S^{n-1}, *)$ where

$$S^{n-1} = \partial\bar{D}^n, \tag{22.7}$$

and relative spheroid is a map

$$(\bar{D}^n, S^{n-1}, *) \to (X, A, *). \tag{22.8}$$

This means we have a map

$$f\colon \bar{D}^n \to X \tag{22.9}$$

such that

$$f\colon \partial\bar{D}^n \to A \tag{22.10}$$

but $f(*) = *$. Such a map is a relative spheroid.

The relative homotopy group $\pi_n(X, A, *)$ is a set of homotopy classes of relative spheroids. It's exactly the same for ordinary homotopy group, the only difference is we use relative spheroids. We will discuss the operation later on.

One relation we'd like to note is we have a map

$$\pi_n(X, A, *) \to \pi_{n-1}(A, *) \tag{22.11}$$

from the relative homotopy group to the full homotopy group. Why? For a trivial reason that a relative spheroid

$$f\colon (\bar{D}^n, S^{n-1}, *) \to (X, A, *) \tag{22.12}$$

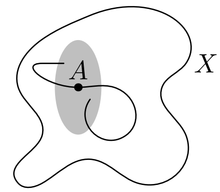
is really a full spheroid on A. Thus $f\colon (S^{n-1}, *) \to (A, *)$ is an $(n-1)$-spheroid.

A different formulation of the relative homotopy group. Recall we considered the space Ω of all closed loops starting and ending at $*$. We consider

$$\pi_n(X, *) = \pi_{n-1}(\Omega, *)$$

Now what about the relative groups? We have $A \subset X$, consider all paths that are closed modulo A. That is to say

$$\Omega(A) = \{f : I \to X \mid f(0) = *, f(1) \in A\} \tag{22.13}$$



This sort of path is doodled on the right, where it begins at the marked point and ends anywhere inside the gray region. We may consider its homotopy groups. Thus we may define $\pi_n(X, A, *) = \pi_{n-1}(\Omega(A), *)$ where $*(t) = *$ is the stationary path. Okay, this is a definition. We see this is a group. Also, for $n \geq 3$ we see $\pi_n(X, A, *)$ is Abelian. The only problem is that this is not a very good definition.

Let us decode this definition. Consider $\pi_{n-1}\big(\Omega(A), *\big)$. What is this? We define this in terms of spheroids as a map of a cube

$$f : I^{n-1} \to \Omega(A) \tag{22.14}$$

which sends $\partial I^{n-1} \to *$. What is $\Omega(A)$? It consists of paths. Our function $f(t_1, \dots, t_{n-1})$ itself is a path, so really

$$f = f(t_1, \dots, t_{n-1}, \tau) \in X. \tag{22.15}$$

What ar ethe conditions on f? First of all, the condition is

$$f_\tau : \partial I^{n-1} \to *(\tau), \tag{22.16}$$

so

$$f(\partial I^{n-1}, \tau) = *(\tau). \tag{22.17}$$

Another is that, for $\tau = 0$, we have

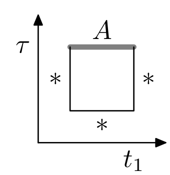
$$f(\dots, 0) = * \tag{22.18}$$

be our marked point, whereas for $\tau = 1$ we require

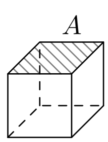
$$f(\dots, 1) \in A \tag{22.19}$$

and that's it!

We will try to reconcile everything. Consider $f : I^n \to X$. For $n = 2$, we will have a square and when $\tau = 1$ we go to A. This is doodled on the right. When $t = 0, 1$ we go to $*$ and when $\tau = 0$ we also go to $*$.



This needs to be rewritten for clarity



For the $n = 3$ case, we have the top face go to A (it is shaded grey). This is really what is defined by Hatcher as a relative spheroid. But this is already defined. We should prove it is the same. Here we have the map of a ball, which sends its boundary to A. But here we have some extra stuff, namely, the rest of the boundary. We identify it witha pouint. So really, this is equvialent to a ball with a marked point. Consider

$$\big(I^n, I^{n-1} \times \{1\}, (\partial I^{n-1} \times I) \cup (I^{n-1} \times \{0\})\big) \mapsto (\bar{D}^n, S^{n-1}, *) \tag{22.20}$$

since they both are mapped to $(X, A, *)$. So this identifies the notion of a relative spheroid with Hatcher's notion of a spheroid. Observe

$$(\partial I^{n-1} \times I) \cup (I^{n-1} \times \{0\})$$

is a contractible set. When we contract, the upperface is mapped to S^{n-1} the whole boundary. So our new sense of relative spheroid agrees with the relative spheroid in the old sense.

Lecture 23.

We gave a definition of homotopy groups $\pi_k(X,*)$ in various ways. Using spheroids

$$f\colon (S^k,*)\to(X,*) \tag{23.1}$$

such that $f(*)=*$, or as

$$f\colon I^k\to(X,*) \tag{23.2}$$

such that $f(\partial I^k)=*$. We introduced the notion of relative homotopy groups on $(X,A,*)$. We can have relative spheroids

$$f\colon \bar{D}^n\to X \tag{23.3}$$

such that

$$f(S^{n-1})\subseteq A,\quad\text{and}\quad f(*)=*. \tag{23.4}$$

This is very nice but it doesn't give a group structure. Therefore one can consider instead of maps of a ball, well, maps of a cube. That would be

$$f\colon I^n\to X \tag{23.5}$$

such that

$$f(I^{n-1})\subset A,\quad\text{and}\quad f(\partial I^n - I^{n-1})=*. \tag{23.6}$$

This is an equivalent picture, but we see how to form a binary operation now. This definition may now be formulated as $\pi_{n-1}\big(\Omega(A)\big)$.

The first thing to say is this definition is functorial. What does this mean? Well, we have

$$\alpha\colon (X,A,*)\to(Y,B,*) \tag{23.7}$$

such that

$$\alpha\colon X\to Y\quad\text{and}\quad \alpha(A)\subset B \tag{23.8}$$

and $\alpha(*)=*$. We then have "by functoriality" a map

$$\pi_n\big(\alpha\colon (X,A,*)\to(Y,B,*)\big)\quad=\quad\alpha_*\colon\pi_n(X,A,*)\to\pi_n(Y,B,*). \tag{23.9}$$

Moreover α_* is a morphism. We have the functoriality property that

$$(\alpha\circ\beta)_*=\alpha_*\circ\beta_*\quad\text{and}\quad(\mathrm{id})_*=\mathrm{id}_*\,. \tag{23.10}$$

These are the functorial properties.

23.1 Exact Homotopy Sequence of a Pair

We see that a spherodi in A is definitely a spheroid in X. In other words, functoriality acts on this inclusion

$$i\colon A\hookrightarrow X \tag{23.11}$$

and gives us a morphism

$$i_*\colon\pi_n(A)\to\pi_n(X). \tag{23.12}$$

Note that we do abuse notation slightly, we should write something like $i_{*,n}$ to indicate we have $\pi_n(i)$, i.e., it depends on the $n\in\mathbb{N}_0$.

Next we have absolute homotopy groups, we had absolute spheroids. Now we may consider relative spheroids and relative homotopy groups

$$\pi_n(A,*)\xrightarrow{i_*}\pi_n(X,*)\to\pi_n(X,A,*). \tag{23.13}$$

Why do we have this? Well, any absolute spheroid is-a relative spheroid. Why? Because the simple reason is $*\in A$. So a spheroid is just a relative spheroid "ending at $*$".

We have one more map, which we already described

$$\pi_n(X, A, *) \to \pi_{n-1}(A, *). \tag{23.14}$$

This comes from the fact a relative spheroid

$$f\colon I^n \to X \tag{23.15}$$

can be restricted to I^{n-1}, but $f(I^{n-1}) \subset A$. This is a spheroid in A! So we induce this morphism.

But we also have

$$i_*\colon \pi_{n-1}(A, *) \to \pi_{n-1}(X, *). \tag{23.16}$$

Why? Well, we saw this earlier using the inclusion and applying the functor π_{n-1}. So we have a sequence. We have at the end of this

$$\cdots \to \pi_1(A, *) \to \pi_1(X, *) \to \underbrace{\pi_1(X, A, *) \to \pi_0(A, *) \to \pi_0(X, *)} \tag{23.17}$$

where the underlined terms are not really defined. So what to do? Simple: *define them!* We really have a problem here: we have no group for $\pi_0(X)$, but we have a set. The stuff underlined in Eq (23.17) are not groups, but they are sets. If we consider $\pi_0(X)$, it may be considered as $S^0 \to X$ which maps marked point to marked point. Please note that S^0 consists of two points, one of them is marked. So what is

$$f\colon (S^0, *) \to (X, *)? \tag{23.18}$$

It is a map of one point, so naively we would expect $\pi_0(X, *)$ to be in one-to-one correspondence with the points of X, right? Wrong: π_0 is the *homotopy classes* of such mappings, and two points are homotopic if they are on the same components. So really,

$$\pi_0(X, *) = \begin{pmatrix}\text{components}\\ \text{of } X\end{pmatrix} \tag{23.19}$$

A deformation of elements of $\mathrm{Hom}\big((S^0, *), (X, *)\big)$ is a path. So we have really $\pi_0(X, *)$ be homotopy classes of these thngs, i.e., of the components of X. It is a set.

A relative spheroid

$$f\colon I \to (X, A, *) \tag{23.20}$$

is a path

$$\gamma\colon [0, 1] \to X \tag{23.21}$$

such that

$$\gamma(0) = *, \quad \text{and} \quad \gamma(1) \in A. \tag{23.22}$$

So a spheroid is determined where it ends. What is important is we have a sequence of groups that ends with a sequence of groups that ends with a sequence of sets.

Theorem 23.1. *This sequence*

$$\cdots \to \pi_n(A, *) \to \pi_n(X, *) \to \pi_n(X, A, *) \to \pi_{n-1}(A, *) \to \dots \tag{23.23}$$

is exact.

Definition 23.2. Consider a sequence of groups

$$\dots \xrightarrow{\mathrm{d}_n} A_n \xrightarrow{\mathrm{d}_{n-1}} A_{n-1} \xrightarrow{\mathrm{d}_{n-2}} A_{n-2} \to \dots \tag{23.24}$$

it is said to be "**Exact**" iff $\mathrm{Im}(\mathrm{d}_k) = \mathrm{Ker}(\mathrm{d}_{k-1})$.

Observe as a consequence in an exact sequence

$$\mathrm{Im}(\mathrm{d}_{k-1} \circ \mathrm{d}_k) = 0. \qquad (23.25)$$

Can we say $\mathrm{Im}(\mathrm{d}_k) = \mathrm{Ker}(\mathrm{d}_{k-1})$? Well, yes, but the statement, for any k,

$$\mathrm{Im}(\mathrm{d}_k) \subseteq \mathrm{Ker}(\mathrm{d}_{k-1}) \qquad (23.26)$$

is sufficient to say we have an exact sequence, and

$$\mathrm{Im}(\mathrm{d}_k) \supseteq \mathrm{Ker}(\mathrm{d}_{k-1}) \qquad (23.27)$$

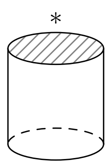
is necessary. We could write

$$\mathrm{Im}(\mathrm{d}_k) = \mathrm{d}_{k-1}^{-1}(0) \qquad (23.28)$$

for exactness conditions.

Remark 23.3. The definition of exactness remains meaningful if we wok with sets with marked points. We simply write $*$ for our marked point, and $\mathrm{d}_{k-1}^{-1}(*) = \mathrm{Im}(\mathrm{d}_k)$, etc.

Sketch of Proof. The full proof requires checking 6 terms showing an equality of sets. □ ***TODO:*** *write in the proof*



If we have $S^{n-1} \to A$ homotopic to the trivial path, then it may be extended to a union of nonintersecting S^{n-1}, i.e., to D^n. Consider $S^{n-1} \times I$, then on the upper part of the cylinder (if it is homotopic to 0) is $*$, we get what is doodled on the left.

Consider $\pi_{n-1}(A, *) \to \pi_{n-1}(X, *)$. We can extend the sphere to a ball. But that means we have $\pi_n(X, A, *) \to \pi_{n-1}(A, *)$, because this extension to a ball is a relative spheroid. So we have the kernel included in the image.

The best thing to do is to go home and think about it yourselves: there are some things so simple, it cannot be explained.

Lecture 24.

We will see how to apply our theorem. Recall we have a space X, a subset $A \subset X$, and a point $* \in A$. We considered homotopy groups $\pi_n(A, *)$ but every spheroid of A is a spheroid in X. We thus get a morphism

$$\pi_n(A, *) \to \pi_n(X, *). \qquad (24.1)$$

But we can consider an absolute spheroid in X as a relative spheroid, thus we get a morphism

$$\pi_n(X, *) \to \pi_{n-1}(X, A, *). \qquad (24.2)$$

Thus we get our exact sequence

$$\cdots \to \pi_n(A, *) \to \pi_n(X, *) \to \pi_{n-1}(X, A, *) \to \pi_{n-1}(A, *) \to \dots \qquad (24.3)$$

We have to check that

$$\pi_n(A, *) \to \pi_n(X, *) \to \pi_{n-1}(X, A, *) \qquad (24.4)$$

is exact. Lets first introduce some notation:

$$i_*\colon \pi_n(A, *) \to \pi_n(X, *) \qquad (24.5a)$$
$$j_*\colon \pi_n(X, *) \to \pi_n(X, A, *) \qquad (24.5b)$$
$$\partial\colon \pi_n(X, A, *) \to \pi_{n-1}(A) \qquad (24.5c)$$

Then we have

$$\partial \circ j_* = 0, \quad j_* \circ i_* = 0, \quad i_* \circ \partial = 0. \tag{24.6}$$

So we have

$$\mathrm{Im}(i_*) \subseteq \mathrm{Ker}(j_*), \quad \mathrm{Im}(j_*) \subseteq \mathrm{Ker}(\partial), \quad \mathrm{Im}(\partial) \subseteq \mathrm{Ker}(i_*). \tag{24.7}$$

In other words, this sequence is a "**Complex**". Last time we proved that

$$\mathrm{Im}(i_*) \supseteq \mathrm{Ker}(j_*), \quad \mathrm{Im}(j_*) \supseteq \mathrm{Ker}(\partial), \quad \mathrm{Im}(\partial) \supseteq \mathrm{Ker}(i_*). \tag{24.8}$$

which imply that these are equal and our sequence is exact.

Consider $X = \bar{D}^n$, $A = S^{n-1}$, and $* \in A$. We can write down an exact sequence:

$$\pi_k(S^{n-1}) \to \pi_k(\bar{D}^n) \to \pi_k(\bar{D}^n, S^{n-1}) \to \pi_{k-1}(S^{n-1}) \to \dots \tag{24.9}$$

We definitely know that $\bar{D}^n$ is contractible, which means they have trivial homotopy groups. We obtain

$$\pi_k(S^{n-1}) \to 0 \to \pi_k(\bar{D}^n, S^{n-1}) \to \pi_{k-1}(S^{n-1}) \to \dots \tag{24.10}$$

which implies an isomorphism. If $0 \to A \hookrightarrow B$ where we have $A \hookrightarrow B$ injective, then $\mathrm{Im}(A \hookrightarrow B) \subset B$. Similarly, we have $A \twoheadrightarrow B \to 0$ imply the kernel is contained in the image and, being sloppy with notation, have $B \subset A$. Thus $0 \to A \to B \to 0$ implies $A \cong B$.

Important: $0 \to A \hookrightarrow B$ and $A \twoheadrightarrow B \to 0$ for exact sequences

So what? Well, this implies

$$\pi_k(\bar{D}^n, S^{n-1}) \cong \pi_{k-1}(S^{n-1}) \tag{24.11}$$

That is the end of the story.

24.1 Homotopy Groups of Fibrations

We have a notion of fibration $p\colon E \to B$ surjective, and

$$p^{-1}\{b\} = F_b \cong F \tag{24.12}$$

for any $b \in B$, where F is the fibre. A "**Locally Trivial Fibration**" is one which locally behaves as a direct product. So for some neighborhood $U \subset B$, we have

$$p^{-1}(U) \cong U \times F. \tag{24.13}$$

We may consider a fibre

$$p^{-1}\{b\} = F_b \subset E \tag{24.14}$$

we may compute the relative homotopy groups of this pair: $\pi_k(E, F_b, *)$ where $* \in F_b$. So

$$p(*) = b = * \tag{24.15}$$

we mark a point in the base

$$b \in B. \tag{24.16}$$

Lets clarify notation a bit: We have $b = *$ in base B, and in F_b the marked point denoted $*$. We have F be the fibre over the marked point. *Immediately* relative homotopy groups of E relative to F is mapped to the absolute homotopy group of B:

$$p_*\colon \pi_n(E, F, *) \to \pi_n(B, *). \tag{24.17}$$

We have a fibration over the base.

Theorem 24.1. *We have this morphism p_* be an isomorphism if the fibration is locally trivial.*

There is a notion of a cell fibrationdefined in this way. Using this theorem for complexes, if

$$\pi_n(X, A) \cong \pi_n(B) \tag{24.18}$$

then it's locally trivial, etc. So in other words, our cell complex X corresponds to E, and the subcomplex A correspond to the "fibre". When we use this analogy on our cell complex, we get a cell fibration. The only thing we need is the ability to lift from a spheroid in B to a relative spheroid. We will elaborate later, now we will focus on examples.

If (E, F) are a pair, we may consider the exact sequence

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(E, F) \to \pi_{n-1}(F) \to \dots. \tag{24.19}$$

Topological results are invariant with respect to letters used for variables. But recall our theorem, that is

$$\pi_n(E, F) \cong \pi_n(B). \tag{24.20}$$

So we can write

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \dots \tag{24.21}$$

we have an "**Exact Homotopy Sequence of Fibrations**".This permits me to consider, to calculate homotopy groups and everyt fibration gives some information. So first of all, we need examples of fibrations. We have one! The trivial fibration, but that is a triviality.

But we have also the Hopf fibration, where we have $E = S^3$, $B = S^2$ and $F = S^1$. For a more general picture, we had the complex projective space which is defined in terms of $(n+1)$ complex numbers defined up to a factor $(z_0 : \cdots : z_n)$. It is a sphere of dimension S^{2n+1}, when we factorize by an action of S^1 we get

$$S^{2n+1}/S^1 = \mathbb{CP}^n. \tag{24.22}$$

We may consider a fibration $S^{2n+1} \to \mathbb{CP}^n$ with the fibre S^1. In the case $n = 1$, we have $\mathbb{CP}^1 = S^2$. It is clear this fibration is locally trivial. We see we get

$$\cdots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{CP}^n) \to \pi_{k-1}(S^1) \to \dots \tag{24.23}$$

describing the exact homotopy sequence.

Remember that we already mentioned

$$\pi_k(\text{space}) = \pi_k \begin{pmatrix} \text{covering} \\ \text{space} \end{pmatrix} \tag{24.24}$$

for $k \geq 2$. We thus have

$$\pi_k(B) \cong \pi_k(E) \tag{24.25}$$

since E is the covering space of the base B. We see F is discrete, so

$$\pi_k(F) = 0 \tag{24.26}$$

for $k \geq 1$. This situation we explained, every third term of our sequence vanishes. So

$$\pi_k(S^1) = 0 \tag{24.27}$$

for $k \geq 2$, and

$$\pi_1(S^1) \cong \mathbb{Z}. \tag{24.28}$$

We may write down this sequence, we are in this wonderful situation that lets us say

$$\pi_k(S^{2n+1}) \cong \pi_k(\mathbb{CP}^n). \tag{24.29}$$

But this is not always true. Why? Because

$$\pi_1(S^1) \neq 0. \tag{24.30}$$

We should consider this portion of the exact sequence

$$\big(\pi_2(S^1) = 0\big) \to \pi_2(S^{2n+1}) \to \pi_2(\mathbb{CP}^n) \to \big(\pi_1(S^1) \cong \mathbb{Z}\big) \to \big(\pi_1(S^{2n+1}) = 0\big) \tag{24.31}$$

But

$$\pi_2(S^{2n+1}) = 0 \tag{24.32}$$

which gives us

$$0 \to \pi_2(\mathbb{CP}^n) \to \big(\pi_1(S^1) \cong \mathbb{Z}\big) \to 0 \tag{24.33}$$

We get

$$0 \to \pi_2(\mathbb{CP}^n) \to \mathbb{Z} \to 0 \tag{24.34}$$

or in other words

$$\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}, \tag{24.35}$$

and for $k > 2$ we have

$$\pi_k(\mathbb{CP}^n) \cong \pi_k(S^{2n+1}) \tag{24.36}$$

as desired.

Lecture 25.

So, last time we discussed the "*Exact Homotopy Sequence Fibration*"

$$\pi_k(F) \xrightarrow{i_*} \pi_k(E) \xrightarrow{p_*} \pi_k(B) \xrightarrow{\partial} \pi_{k-1}(F) \tag{25.1}$$

where $p\colon E \to B$ forms the fibration, and

$$\varphi\colon I^k \to B \tag{25.2a}$$

is a spheroid,

$$\psi\colon I^k \to E \tag{25.2b}$$

is a spheroid such that

$$p \circ \psi = \varphi \tag{25.3}$$

holds.

Consider the Hopf fibration

$$S^3 \xrightarrow{S^1} S^2. \tag{25.4}$$

The sequence we had was

$$\pi_k(S^1) \to \pi_k(S^3) \to \pi_k(S^2) \to \pi_{k-1}(S^1). \tag{25.5}$$

Notice for $k = 2$ we get

$$\pi_2(S^1) = 0 \to \pi_2(S^3) \to \pi_2(S^2) \to \mathbb{Z} \cong \pi_1(S^1) \to 0 = \pi_1(S^3). \tag{25.6}$$

We should remember that

$$\pi_2(S^2) \cong \mathbb{Z} \tag{25.7}$$

thus we obtain

$$0 \to \pi_2(S^3) \to \mathbb{Z} \to \mathbb{Z} \to 0. \tag{25.8}$$

But we also have

$$\pi_k(S^n) = 0 \quad \text{for } k < n. \tag{25.9}$$

Thus we obtain

$$0 \to \pi_2(S^3) = 0 \to \mathbb{Z} \to \mathbb{Z} \to 0. \tag{25.10}$$

For $k > 2$ we find $\pi_k(S^1) = 0$, thus obtaining a sequence

$$\pi_k(S^1) = 0 \to \pi_k(S^3) \to \pi_k(S^2) \to 0 = \pi_{k-1}(S^1). \tag{25.11}$$

This implies

$$\pi_k(S^3) \cong \pi_k(S^2) \tag{25.12}$$

for all $k \geq 3$.

We would like to extract some interesting sequences. First we will introduce a "**Section**" of a fibre bundle $p\colon E \to B$. So we take a single point in every fibre F_b, and we do this continuously. In othr words, it is a mapping

$$q\colon B \to E \tag{25.13}$$

such that

$$q(b) \in F_b \quad \forall b \in B. \tag{25.14}$$

Thus

$$(p \circ q)(b) = b \tag{25.15}$$

or simply $p \circ q = \mathrm{id}_B$.

Box 2. Sections in Fibre Bundles

The intuition should really be guided by thinking of vector fields in $\mathbb{R}^3$. Here, our fibre is $F \cong \mathbb{R}^3$ *the vector space* and the base is $B = \mathbb{R}^3$ *the topological space*. What do we do? Well, it's quite simple: we have a continuous mapping

$$\begin{aligned} \mathbb{R}^3 &\to E \cong \mathbb{R}^3 \times \mathbb{R}^3 \\ \boldsymbol{p} &\mapsto (\boldsymbol{p}, \vec{v}\,) \end{aligned} \tag{25.16}$$

where $\boldsymbol{p} \in \mathbb{R}^3$ is a point in our topological space, and $\vec{v} \in \mathbb{R}^3$ is a vector in our vector space. The ordered pair $(\boldsymbol{p}, \vec{v})$ is a tangent vector with base point $\boldsymbol{p}$ and vector part $\vec{v}$.

Note that as a vector bundle, the space $E \cong \mathbb{R}^3 \times \mathbb{R}^3$ is a *trivial* vector bundle. This enables us to write guys living in E as an ordered pair. Also note, this is not the tangent bundle! It resembles it an awful lot, but I am being lazy and referring to something similar but different.

What if we want to assign something else to each point "continuously"? For example: assign to each point a differential operator? Or a group element? How do we handle these situations? The solution is obvious: work with a *topological gadget* (e.g., a topological group). What does this mean? Well, it means the gadget has topological structure. This enables us to *continuously* assign data to each point.

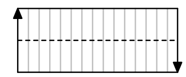
The problem is *not every bundle has sections*. If the bundle is "twisted" too much (i.e., not "trivial" enough), then the topology "obstructs" global sections existing. What's a global section? It is a section defined on B, not just a neighborhood $U \subset B$. Compare this to a local section, which is defined on some neighborhood $U \subset B$.

Global Section defined on every $U \subset B$; local section defined on some U

The trivial bundle $p\colon B \times F \to B$ has sections, of course. We just take $f = f(b)$ continuous. Then

$$\begin{aligned} q\colon B &\to B \times F \\ b &\mapsto \big(b, f(b)\big) \end{aligned} \tag{25.17}$$

We have a section map $B \to F$.



The Mobius band is a fibration if we take the dashed line, doodled to the right. We get a section, where the fibres are doodled in grey. What's the base space B for the Mobius band? Well, it's a circle. What's the fibre? Well, $F_b \cong I$ it's homeomorphic to the unit interval. So *locally*, i.e. for a sufficiently small neighborhood $U \subset B$, we have $p^{-1}(U) \cong U \times I$.

Lets consider a surface, say in $\mathbb{R}^3 \supseteq \Sigma$. Lets specifically consider the following picture: our surface is smooth, so we may speak meaningfully of tangent vectors. If we set $B = \Sigma$, then $E = T\Sigma$ is our "**Tangent Bundle**". For our surface Σ, a tangent vector has two components; thus our tangent space $T_x\Sigma = p^{-1}(x)$ is two-dimensional. What does this mean? Well, $T_x\Sigma \cong \mathbb{R}^2$ for all $x \in \Sigma$. Our tangent bundle is a locally trivial fibration, so at some $U \subset \Sigma$ "small neighborhood", we have $p^{-1}U \cong U \times \mathbb{R}^2$.

Do we have a section for $T\Sigma$? Well, there is a trivial one: $q(b) = 0$ for any $b \in B$. But generally, a section for a tangent bundle is a "**Tangent Vector Field**". Can we have a nontrivial tangent vector field? Well, just get rid of zero in the fibre:

$$E' = E - (B \times \{0\})$$

and consider the fibration

$$p' \colon E' \to B. \tag{25.18}$$

This is really the same guy as $p \colon E \to B$ with the demand of nontriviality. A section would be a mapping

$$q \colon B \to (E - B \times \{0\}). \tag{25.19}$$

Lets note that the fibre we are working with looks like

$$F'_b \cong \mathbb{R}^2 - \{0\}. \tag{25.20}$$

So what? Well,

$$F'_b \cong S^1 \tag{25.21}$$

describes the fibre's topology. The million-dollar question: does a section exist? How to prove that the section exists?

Lets first discuss some very trivial statements. If we have a fibration $p \colon E \to B$ and a section on our fibration, it is a map $q \colon B \to E$. Observe: these are maps. So what? Well, functoriality induces morphisms

$$q_* \colon \pi_n(B) \to \pi_n(E) \tag{25.22a}$$

and

$$p_* \colon \pi_n(E) \to \pi_n(B). \tag{25.22b}$$

We use functoriality and obtain

$$p \circ q = \mathrm{id}_B \quad\implies\quad p_* \circ q_* = \mathrm{id}_{\pi_n(B)}, \tag{25.23}$$

which implies p_* is surjective. Recall our exact homotopy sequence for a fibration has

$$\pi_k(E) \xrightarrow{p_*} \pi_k(B) \xrightarrow{\partial} \pi_{k-1}(F) \tag{25.24}$$

So p_* surjective gives us

$$\mathrm{Im}(p_*) = \pi_k(B) \tag{25.25}$$

and exactness gives us

$$\mathrm{Im}(p_*) = \mathrm{Ker}(\partial). \tag{25.26}$$

Together, these imply

$$\pi_k(B) = \mathrm{Ker}(\partial) \tag{25.27}$$

provided a section exists. Lets reiterate this:

$$\boxed{\text{If } \mathrm{Ker}(\partial) = \pi_k(B)\text{, then a section exists.}} \tag{25.28}$$

For example, we have our Hopf fibration, does it have a section? Well, $\pi_2(S^3) \to \pi_2(S^2)$ is not surjective, so it is impossible for a section to exist.

Lets introduce the notion of a "**Stiefel Manifold**" $V_{n,k}$. Here

$V_{n,k}$ consists of $k \times n$ matrices whose columns are orthonormal *column vectors in $\mathbb{R}^n$*

$$V_{n,k} = \{(e_1, \dots, e_k)\} \tag{25.29}$$

where e_i is a column vector living in $\mathbb{R}^n$. Morever, we demand orthonormality

$$\langle e_i, e_j \rangle = \delta_{ij}. \tag{25.30}$$

We may give a different definition: we require $e_1, \dots, e_k$ be linearly independent. Lets denote this different definition by $\widetilde{V}_{n,k}$. We see that $V_{n,k} \subset \widetilde{V}_{n,k}$. But this embedding is really a homotopy equivalence. How can we say this? Well, the Grahm-Schmidt procedure gives us a mapping

$\widetilde{V}_{n,k}$ consists of $k \times n$ matrices whose columns are linearly independent *column vectors in $\mathbb{R}^n$*

$$\widetilde{V}_{n,k} \to V_{n,k} \tag{25.31}$$

which is a homotopy equivalence.

We have a special case

$$V_{n,1} = S^{n-1} \tag{25.32}$$

and

$$\widetilde{V}_{n,1} = \mathbb{R}^n - \{0\}. \tag{25.33}$$

There are other fascinating examples,

$$V_{n,2} = \{\text{normalized tangent vectors to } S^{n-1}\}, \tag{25.34}$$

but we also have a fibration

$$V_{3,2} \to S^2 \tag{25.35a}$$

constructed by

$$(e_1, e_2) \mapsto e_1. \tag{25.35b}$$

The fibre is S^1. Writing down the exact sequence for this fibration is very easy, but it is a Hopf fibration if we replace S^3 with $V_{3,2}$.

First $V_{3,2} = \mathrm{SO}(3)$ where $\mathrm{O}(n)$ is the group of orthogonal matrices, and $\mathrm{SO}(n)$ is the subgroup with unit determinant. Why is this obvious? First, we have

$$V_{3,3} = \mathrm{O}(3) \tag{25.36}$$

trivially, since each column is orthonormal. We have, given $(e_1, e_2) \in V_{3,2}$, the third vector e_3 be orthogonal to e_1 and e_2. But we have two options $\pm e_3$ are both orthogonal to e_1 and e_2! We get a mapping

$$\begin{aligned} V_{3,2} &\to V_{3,3} \\ (e_1, e_2) &\mapsto X = (e_1, e_2, s e_3) \end{aligned} \tag{25.37}$$

where $s = \pm 1$ is such that $\det(X) = 1$. This gives us a one-to-one correspondence between $V_{3,2}$ and $\mathrm{SO}(3)$. But that's a triviality. More generally, we have

$$V_{n,n-1} = \mathrm{SO}(n) \tag{25.38}$$

but

$$V_{3,2} \cong \mathbb{RP}^2. \tag{25.39}$$

Then $\pi_1(V_{3,2}) \cong \mathbb{Z}_2$. We see our exact homotopy sequence is

$$\pi_2(V_{3,2}) \to \pi_2(S^2) \cong \mathbb{Z} \to \mathbb{Z}_2 \to 0 \tag{25.40}$$

but it then follows that

$$\pi_2(V_{3,2}) \to \mathbb{Z} \tag{25.41}$$

is *not* surjective. Thus this fibration has no section. There exists no nonzero tangent vector field to the sphere. The moral: *Don't bring hedehog to barber shop.*

Lecture 26. Principal Fibrations.

26.1 Aside on Principal Fibrations

Some particular cases of fibrations are called "**Principal Fibrations**". Let G be a topological group (usually a Lie group; for notes on Lie groups, see Nelson [8]). Lets consider a space E where G acts freely. What does it mean? Well, for any nontrivial $g \in G$ there are no fixed points, i.e.,

$$gx \neq x \quad \text{provided} \quad g \neq e. \tag{26.1}$$

The obvious exception is the identity transformation has fixed points, but that's a triviality. Although this is unsatisfactory, but sufficient for us. Given such an action, every orbit O_x is in one-to-one correspondence with G. We have

$$g \mapsto xg \tag{26.2}$$

using the right action for notational convenience. So

$$G \to O_x \tag{26.3}$$

is bijective and continuous. This is a topological equivalence! Moreover, if G is compact, then every continuous one-to-one mapping has a continuous inverse. Then every orbit is topologically equivalent to G. We have a map $E \to E/G$ which is a fibration.

Theorem 26.1. *A principal fibration has a section if and only if it is a trivial fibration.*

We should additionally assume that G is compact, and $B = E/G$ is at least locally compact. Lets suppose we have a section

$$q\colon B \to E, \tag{26.4}$$

consider the mapping

$$f\colon B \times G \to E \tag{26.5a}$$

defined by

$$f(b,g) = q(b)g \tag{26.5b}$$

which is continuous. So F is one-to-one and continuous. If everything is compact, it's a homeomorphism. It's a condition of triviality for a principal bundle. Also note

$$\begin{pmatrix}\text{Local Triviality}\\ \text{of Bundle}\end{pmatrix} \implies \begin{pmatrix}\text{Existence of}\\ \text{Local Sections}\end{pmatrix}. \tag{26.6}$$

We will do the following trick: include existence of lcoal sections into the definition of a free action.

There is an important case, namely a subgroup $H \subset G$ of a topological group. Itacts on the left or on the right, so lets consider the action

$$(g,h) \mapsto gh \tag{26.7}$$

for some $g \in G$ and $h \in H$. What is the space of orbits? It is G/H, the space of cosets. We have

$$G \xrightarrow{H} G/H \tag{26.8}$$

and it is a principal fibre bundle. We assume compactness everywhere (otherwise, we need to worry about the existence of sections, etc.).

Suppose G acts transitively on B. Lets take a point $b \in B$, then we may map $G \to B$ by considering

$$g \mapsto \varphi_g(b) \tag{26.9}$$

where

$$\varphi\colon G \to \mathrm{Aut}(B). \tag{26.10}$$

But this is precisely the picture we had before. If we denote $\mathrm{Stab}(b) = H$, then B corresponds to G/H.

26.2 Returning to Stiefel Manifolds

Now we would like to consider two different cases: the real case, and the complex case. But we don't really want to talk about quaternionic case, but everything we say may be repeated for quaternions. So what are the Stiefel Manifold $V_{n,k}$? It is precisely k column vectors in $\mathbb{R}^n$ which are orthonormal. But

$$V_{n,n}(\mathbb{R}) = \mathrm{O}(n) \tag{26.11}$$

may be considered as an identity. For the complex case, we see we get

$$V_{n,n}(\mathbb{C}) = \mathrm{U}(n) \tag{26.12}$$

the unitary group! The quaternionic case, we also have something of this kind

$$V_{n,n}(\mathbb{H}) = \mathrm{Sp}(n) \tag{26.13}$$

the Symplectic group.

We had explained

$$V_{n,n-1}(\mathbb{R}) = \mathrm{SO}(n), \tag{26.14}$$

and some consideration gives the fact

$$V_{n,n-1}(\mathbb{C}) = \mathrm{SU}(n). \tag{26.15}$$

In particular, what we would like to say is that

$$V_{2,1}(\mathbb{C}) = \mathrm{SU}(2). \tag{26.16}$$

So this is a pair of complex numbers $x, y \in \mathbb{C}$ such that

$$|x|^2 + |y|^2 = 1, \tag{26.17}$$

which is a sphere in $\mathbb{C}^2 \cong \mathbb{R}^4$. So $\mathrm{SU}(2) \cong S^3$.

Now it is easy to check that

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2. \tag{26.18}$$

This may be done in many different ways. One is to consider a 3-dimensional representation of SU(2). Or we may consider an action of SU(2) on Hermitian matrices. We won't go into detail here. By the way, this fact implies that

$$\mathrm{SO}(3) \cong \mathbb{RP}^3 \tag{26.19}$$

as topological spaces, and we used this before. Homotopy groups care about the topological structure of spaces, so we have

$$\pi_k\big(\mathrm{SO}(3)\big) \cong \pi_k\big(\mathrm{SU}(2)\big) \tag{26.20}$$

for any $k \in \mathbb{N}$.

After these remarks, we want to construct some fibrations. We do this in two ways. One way take $V_{n,k}$ and maps

$$(e_1, \dots, e_k) \mapsto e_1. \tag{26.21}$$

Note we could have mapped it to e_k, it doesn't really matter. This gives us a mapping

$$V_{n,k} \to V_{n,1}. \tag{26.22}$$

What's the fibre? Well, we fix one vector e_1 and we have $(k-1)$ vectors orthogonal to it. Thus we have $V_{n-1,k-1}$ be the fibre. So we have for each k a fibration of this kind. Observe

$$V_{n,n} \to V_{n,1} \tag{26.23}$$

has fibre $V_{n-1,n-1}$. But we know these guys! It's the fibration

$$\mathrm{O}(n) \to S^{n-1} \tag{26.24}$$

with fibre $\mathrm{O}(n-1)$, in the $\mathbb{R}$ case. (For the $\mathbb{C}$ case we have $\mathrm{U}(n) \to S^{2n-1}$ with fibre $\mathrm{U}(n-1)$.) We see that $\mathrm{O}(n)$ acts on $\mathbb{R}^n$ which preserves the scalar product and length (likewise describes the action of $\mathrm{U}(n)$ on $\mathbb{C}^n$). So it follows every sphere is a quotient

$$\mathrm{O}(n)/\mathrm{O}(n-1) = S^{n-1} \tag{26.25}$$

in the real case, and

$$\mathrm{U}(n)/\mathrm{U}(n-1) = S^{2n-1} \tag{26.26}$$

for the complex case.

We can get information about the connection of $\mathrm{U}(n-1)$, $\mathrm{U}(n)$ if we know the homotopy groups for S^{2n-1}. We know for small k that

$$\pi_k(S^{2n-1}) = 0. \tag{26.27}$$

Thus

$$\pi_k\big(\mathrm{U}(n-1)\big) \cong \pi_k\big(\mathrm{U}(n)\big). \tag{26.28}$$

We have a similar situation for the orthogonal group, for $k < n$.

EXERCISES

▸ **Exercise 23.** Prove that every graph (one-dimensional cell complex) has trivial homotopy groups in dimensions > 1.

Hint. Every simply connected graph is contractible. (This is true also for infinite graphs, but to solve the problem it is sufficient to check this for finite graphs.)

▸ **Exercise 24.** Calculate relative homotopy groups $\pi_k(S^n, S^1,)$ where $k \leq n$, $n \geq 3$. Here S^1 stand for a circle embedded into n-dimensional sphere S^n.

▸ **Exercise 25.** Let us consider a letter Φ as three-dimensional object (in other words we consider Φ as a small neighborhood of a graph in $\mathbb{R}^3$). One can say also that we consider three-dimensional body Φ bounded by a sphere with handles $\partial\Phi$. Calculate relative homotopy groups $\pi_n(\Phi, \partial\Phi,)$ where $\in \partial\Phi$.

Lecture 27.

Consider a topological group G and a closed subgroup $H \subset G$, we are assuming that G is a compact Lie group for simplicity. Then we may say that H acts on G by means of multiplication — specifically, by means of multiplication on the right. That is

$$\varphi_h(g) = gh \tag{27.1}$$

for $g \in G$ and $h \in H$. We have cosets gH by considering the orbits of such maps. We also have a fibration where the fibres are H and

$$p\colon G \to G/H \tag{27.2}$$

is the fibration. So G acts on G/H but on the left. If we take any $\gamma \in G$ and $\gamma(gH) = (\gamma g)H$. This action is transitive. Ifwe start with H and consider

$$\gamma(eH) = \gamma H \tag{27.3}$$

we get every orbit, therefore it is transitive. If G acts transitively on X, then X may be identified with the space of orbits $X = G/H$ where H is a stable subgroup. We may write down the exact homotopy sequnce for this fibration.

Example 27.1. We may consider $\mathrm{U}(n)/\mathrm{U}(n-1) = S^{2n-1}$ and this is simply because $\mathrm{U}(n)$ acts on $\mathbb{C}^n$ by definition. The orbits of this action are spheres. The stable subgroup is $\mathrm{SU}(n)$. We may also write

$$\mathrm{SU}(n)/\mathrm{SU}(n-1) = S^{2n-1} \tag{27.4}$$

which is the same stuff.

But we may repeat the same consideration on $\mathbb{R}^n$, where the orthogonal group replaces the unitary group. We have that

$$\mathrm{O}(n)/\mathrm{O}(n-1) = S^{n-1}. \tag{27.5}$$

We may also consider instead

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1}. \tag{27.6}$$

There is another consideration using $\mathbb{H}^n$ quaternionic space. The analog of unitary or orthogonal group, here, is the symplectic group $\mathrm{Sp}(n)$. We obtain

$$\mathrm{Sp}(n)/\mathrm{Sp}(n-1) = S^{4n-1} \tag{27.7}$$

by similar reasoning as the complex case.

Recall that for the exact sequence for fibrations we have

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \dots \tag{27.8}$$

but usually one of these spaces is contractible, so its homotopy group is trivial for all n. Thus we get an isomorphism between homotopy groups.

We thus deduce that, for "small k",

$$\pi_k\big(\mathrm{U}(n)\big) \cong \pi_k\big(\mathrm{U}(n-1)\big) \tag{27.9}$$

This means we may consider "**Stable Homotopy Groups**" which are denoted by $\pi_k(\mathrm{U})$, $\pi_k(\mathrm{O})$, and $\pi_k(\mathrm{Sp})$. This corresponds to $\pi_k\big(\mathrm{U}(n)\big)$, etc., for "large enough n". We can compute how large n has to be. There then exists a wonderful statement:

Bott Periodicity Theorem. *We have* $\pi_k(\mathrm{U}) \cong \pi_{k+2}(\mathrm{U})$, $\pi_k(\mathrm{O}) \cong \pi_{k+8}(\mathrm{O})$, *and* $\pi_k(\mathrm{Sp}) \cong \pi_{k+4}(\mathrm{O})$.

So we should know a small amount of these guys. First of all, all even guys

$$\pi_{2k}(\mathrm{U}) = 0. \tag{27.10a}$$

We see $\pi_0(\mathrm{U})$ is the set of connected components, and thus the unitary group is connected. We also can see

$$\pi_{2k}(\mathrm{U}) \cong \mathbb{Z} \tag{27.10b}$$

We see $\mathrm{SU}(n) \subset \mathrm{U}(n)$ is a subgroup. Moreover, if $u \in \mathrm{U}(n)$, $v \in \mathrm{SU}(n)$, we have

$$u = \lambda v \tag{27.11}$$

for some $\lambda \in \mathbb{C}$. So this quotient

$$\mathrm{U}(n)/\mathrm{SU}(n) = S^1 \tag{27.12}$$

as topological spaces, so we have

$$\pi_1\big(\mathrm{SU}(n)\big) = 0. \tag{27.13}$$

Thus we deduce that $\pi_1\big(\mathrm{U}(n)\big) \cong \mathbb{Z}$ and we may use Bott periodicity.

For the orthogonal group, we must compute $\pi_0, \ldots, \pi_7$. We see that

$$\pi_0(\mathrm{O}) \cong \mathbb{Z}_2 \tag{27.14}$$

since there are two connected components, by Euler's theorem[4]. We also observe

$$\pi_1\big(\mathrm{SO}(3)\big) \cong \pi_1\big(\mathbb{RP}^3\big) \cong \mathbb{Z}_2 \tag{27.15}$$

so

$$\pi_1(\mathrm{O}) \cong \mathbb{Z}_2. \tag{27.16}$$

Now we see that

$$\pi_2(\mathrm{O}) = 0 \tag{27.17}$$

is trivial. More interestingly, for any Lie group G we have

$$\pi_2(G) = 0. \tag{27.18}$$

But we have

$$\pi_3(\mathrm{O}) \cong \mathbb{Z}, \tag{27.19}$$

which is a general fact for every *simple* noncommutative Lie group G we have $\pi_3(G) \cong \mathbb{Z}$. We may now write down a table

$\pi_0(\mathrm{O})$	$\pi_1(\mathrm{O})$	$\pi_2(\mathrm{O})$	$\pi_3(\mathrm{O})$	$\pi_4(\mathrm{O})$	$\pi_5(\mathrm{O})$	$\pi_6(\mathrm{O})$	$\pi_7(\mathrm{O})$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Only π_0 depends on the components, all other homotopy groups are computed on the component connected to the identity element.

We may compute π_0, π_1, π_2, π_3 for all simple Lie groups U, O, Sp. Every Lie group is homotopy equivalent to a compact Lie group. So really, look, first of all both $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are connected but $\mathrm{O}(n)$ has two components. We know

$$\pi_3\big(\mathrm{SU}(2)\big) = \pi_3(S^3) = \mathbb{Z} \tag{27.20}$$

and this gives us

$$\pi_3\big(\mathrm{SO}(3)\big) = \mathbb{Z}. \tag{27.21}$$

[4]Euler's theorem states: if X is an orthogonal matrix, then $\det(X) = \pm 1$. This can be seen by $\det(X^{\mathrm{T}}X) = \det(I) = 1$ and $\det(X^{\mathrm{T}}) = \det(X)$. Thus $\det(X)^2 = 1$, and there are only two real numbers that do this.

We observe
$$\mathrm{SO}(4) \cong \big(\mathrm{SU}(2) \times \mathrm{SU}(2)\big)/\mathbb{Z}_2. \tag{27.22}$$
But it follows that
$$\pi_3\big(\mathrm{SO}(4)\big) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{27.23}$$
We have $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, so
$$\pi_3\big(\mathrm{Sp}(1)\big) \cong \mathbb{Z}. \tag{27.24}$$
We can now apply stability to deduce
$$\pi_2\big(\mathrm{SU}(n)\big) = 0 \tag{27.25}$$
and
$$\pi_3\big(\mathrm{SU}(n)\big) \cong \mathbb{Z} \tag{27.26}$$
for $n \geq 2$. We see we may compute
$$\mathrm{SU}(3)/\mathrm{SU}(2) \cong S^5 \tag{27.27}$$
We have
$$\mathrm{SO}(6) \cong \mathrm{SU}(4), \tag{27.28}$$
so we now know the homotopy group for SO(6). We see
$$\mathrm{SO}(6)/\mathrm{SO}(5) \cong S^5 \tag{27.29}$$
and that enables us to deduce the homotopy groups for SO(5).

EXERCISES

▸ **Exercise 26.** Let us consider a locally trivial fibration with total space E, base $B = S^2$ and fiber S^1. Express the homotopy groups of E in terms of homotopy groups of the sphere S^2.

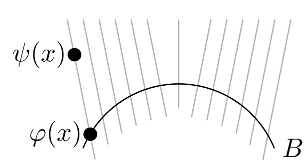
Hint. The data you have do not specify completely the homotopy groups of E. You should describe all possible answers.

▸ **Exercise 27.** A topological group G acts freely on contractible space E. Express the homotopy groups of the space of orbits $B_G = E/G$ in terms of homotopy groups of G. (The space B_G is called classifying space of G.)

▸ **Exercise 28.** The group $\mathrm{U}(k) \times \mathrm{U}(k)$ is embedded into group $\mathrm{U}(2k)$ as a subgroup consisting of block-diagonal matrices with two $k \times k$ blocks. Calculate the homotopy groups $\pi_n(\mathrm{U}(2k)/\mathrm{U}(k) \times \mathrm{U}(k))$ for $n < 2k$.

Hint. You can use the fact that the natural homomorphism $\pi_n(\mathrm{U}(k)) \to \pi_n(\mathrm{U}(m))$ where $k < m$ is an isomorphism for $n < 2k$. The groups $\pi_n\big(\mathrm{U}(k)\big)$ for $n < 2k$ are called stable homotopy groups; they are equal to $\mathbb{Z}$ for odd n and to 0 for even n

Lecture 28.



Now we will prove there exists a homotopy exact sequence of a fibration. Really we are proving the homotopy lifting property for locally trivial fibations. Lets suppose we have $p\colon E \to B$ for us the essential caase is a fibration where
$$p^{-1}\{b\} \sim F;$$
and if we have
$$\psi\colon X \to E$$
where X is some arbitrary (but fixed) topological space, then we can compose to get a map
$$p \circ \psi\colon X \to B.$$

Lets write $\varphi = p \circ \psi$. We may say that ψ lies above φ, or ψ is a "**Lifting**" of φ. Not every map can be lifted, in particular the map

$$\mathrm{id}_B \colon B \to B \tag{28.1}$$

this lifting map is prcisely what is called a "**Section**".

For a locally trivial fibration, every homotopy may be lifted provided the cell complex X is "good enough" (i.e., polyhedral). Precisely this means that if we have a map

$$\varphi \colon X \to B \tag{28.2}$$

and suppose we have lifted it to get a map

$$\psi \colon X \to E, \tag{28.3}$$

then

$$\varphi = p \circ \psi. \tag{28.4}$$

But now we know that ψ does not always exist. Assume we have a homotopy

$$\varphi_t \colon X \to B \tag{28.5}$$

with the property

$$\varphi_0 = \varphi, \tag{28.6}$$

then the statement is: if we start with a locally trivial fibration, there exists a homotopy ψ_t such that $\psi_0 = \psi$ and $p \circ \psi_t = \varphi_t$.

We will prove this by induction (in some sense). We will prove something stronger, namely this lifting property for a pair. We assume X is a cell complex, $A \subset X$ is a subcomplex. Or better, a polyhedron and subpolyhedron. This is the homotopy lifting property for pairs. We have the additional assumption that homotopy is lifted on A.

Theorem 28.1. *Let (A, X) be a cellular pair, $\psi \colon X \to A$ There exists a homotopy ψ_t such that $\psi_0 = \psi$, $p \circ \psi_t = \varphi_t$, and the lifted homotopy has the given value on A.*

In other words, we may begin on A, then extend it to X. We will now prove a particular case.

TODO: write up the proof.

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