Notes on An Algebraic Formulation of Tangent Spaces

Alex Nelson*
Email: pqnelson@gmail.com

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1 Differentiable Structures

We want to consider a so-called "differentiable structure" that we can equip a topological space with. The intuition already should be that this is a mapping of some sort, where for our topological space M we have a mapping

$$U \mapsto \text{Diff}(U) \tag{1.1}$$

where $U \subseteq M$ is open. As we are working with a topological space, we demand consistancy on overlaps, so if $U \cap V \neq \emptyset$ and $U, V \subseteq M$ are open, then we would want

$$\text{Diff}(U \cap V) = \text{Diff}(U|_{U\cap V}) = \text{Diff}(V|_{U\cap V}) \tag{1.2}$$

that is to say, if we restrict our attention of $\text{Diff}(U)$ on the overlap $\text{Diff}(U|_{U\cap V})$, it should be equal to the differentiable structure on the overlap.

But what exactly is this "differentiable structure" we are assigning to each open subset of M? Already it sounds like a presheaf or sheaf, so intuitively we can think of it as a generalization of a vector space. Instead of assigning a vector at each point of M, it assigns something "smooth" to each point in such a way that varies "nicely" as we vary the base-point. We will use something that we know very well: an algebra of smooth functions.

So basically, given some topological space M, we have

$$\text{Diff} : U \mapsto C^\infty(U). \tag{1.3}$$

That is, for each open subset U of M, we assign an algebra of C^∞ functions (i.e. infinitely differentiable functions) which includes the unit function $1(x) = 1$. **N.B:** we will *change notation* to use instead of $\text{Diff}(U)$ the mathematically clearer $C^\infty(U)$.

Notation Change! $\text{Diff}(U)$ *changed to* $C^\infty(U)$

Is it enough to assign "any old" algebra of smooth functions to open subsets $U \subseteq M$? Well, we should make some restrictions. Namely, we want it to be "consistent on overlaps".

*This is a page from https://pqnelson.github.io/notebk/
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So let $U, V \subseteq M$ are open, $U \cap V \neq \emptyset$, and $f \in C^{\infty}(U)$. We will denote the restriction mapping as

$$r_{U\cap V,U} : U \to U \cap V \tag{1.4}$$

we then *demand*

$$f \circ r_{U\cap V,U} \in C^{\infty}(U \cap V) \tag{1.5}$$

that f restricted to the overlap lie in the differentiable structure assigned to the overlap. This is the naive way to demand consistency on overlaps.

So we have a "differentiable structure" be an assignment of "sufficiently nice" unital algebras of smooth functions to open subsets of a topological space, but is that all? Well, for starters, how do we define a derivative? It requires a choice of some coordinates! This is a bit of a problem, so we need to equip each open subset $U \subseteq M$ some extra information which permits some notion of coordinates. Locally each open subset $U \subseteq M$ is "the same" as $\mathbb{R}^n$ (for some fixed $n \in \mathbb{N}$ called the "**Dimension of** M"). What does this rigorously translate to? How can we rigorously translate the "sameness" of two topological spaces? Well, we use a mapping. From the category theoretic perspective, it should be an isomorphism. But for topological spaces that is a *homeomorphism* (a continuous map with a continuous inverse — so we can translate open subsets of the domain into open subsets of the codomain and vice-versa). So we equip each open subset $U \subseteq M$ with a homeomorphism

Notion of dimension

$$\varphi \colon U \to \mathbb{R}^n \tag{1.6}$$

specified by components

$$\varphi \colon y \mapsto (x_1(y), x_2(y), \ldots, x_n(y)) \tag{1.7}$$

where $y \in U$, $x_1, \ldots, x_n \in C^{\infty}(U)$. These functions $x_i \in C^{\infty}(U)$ are called the "**Coordinate Functions**".

So we just introduced some extra structure, some special functions defined for each open subset $U \subseteq M$ called the coordinate functions on U, which allows us to solve the problem of specifying differentiability! Why? Well, observe that the coordinate function is invertible, so

$$\varphi^{-1} \colon \mathbb{R}^n \to U \tag{1.8}$$

can be composed with a function with domain U. Why is this important? Well, if

$$f \colon U \to \mathbb{R}, \tag{1.9}$$

then the composition

$$f \circ \varphi^{-1} \colon \mathbb{R}^n \to \mathbb{R} \tag{1.10}$$

can be differentiable. That is, *we can use the familiar notion of differentiability from ordinary calculus!* We just work "locally" in $\mathbb{R}^n$ when differentiating.

So to reiterate our specification of a differentiable structure, it is a mapping from a topological space M to a unital associative algebra of smooth C^{∞} functions on M. This mapping is "sufficiently nice" on overlaps of open subsets of M. Further it has preferred functions which are the "coordinate functions," whose *raison d'être* is to permit differentiation in the obvious way. This is precisely sufficient information to have some notion of "smoothness" in M.

Remark 1.1. The pair (M, Diff) is often referred to as a "**Smooth Manifold**". However, we will abuse language and simply refer to M as the smooth manifold with the understanding that it really is equipped with a smooth structure.

2 Tangent Vectors and Differential Expressions

We will present a purely (or mostly) algebraic formulation of tangent vectors and differential expressions. It is completely foreign to the unsuspecting observer, something phenomenally alien. We will begin very slowly by analyzing the components involved very slowly.

2.1 Germs

Let M be a C^∞ manifold of dimension m. We want to consider smooth functions on the open subsets of M. What can be said of the local properties of these functions? That is, how can we tell if two functions $f, g \in C^\infty(U)$ (for some $U \subseteq M$) have the same "local properties"? This is the problem we wish to consider as motivation for studying germs.

Namely, there is a problem we haven't really considered: what if two distinct functions agree everywhere on some open subset $U \subseteq M$? That is, if $f, g \colon U \to \mathbb{R}$ and

$$f(x) = g(x) \qquad \forall x \in U, \tag{2.1}$$

are the functions "different" in $C^\infty(U)$? Working within some coordinate system, we could Taylor expand and find they have *identical* Taylor series within U. So they have *locally* the same properties.

Well, in the obvious way they are "equivalent." That is, we have an equivalence relation for all $f, g \in C^\infty(U)$ specified by

$$f \sim g \quad \iff \quad f(x) = g(x) \quad \forall x \in U. \tag{2.2}$$

The *distinct functions* determined by their distinct values are specified by equivalence classes of $C^\infty(U)$. That is, if $f \sim g$ on U, then we can form a collection of functions that are equivalent to f. They have the same value as f on U, so without loss of generality we may intuitively think of them as "the same." We specify such equivalence classes by

$$[f]_x = \{g \in C^\infty(U) |\ \ g \sim f\} \tag{2.3}$$

where we are working in a neighborhood U of x (an open subset which contains $x \in U \subseteq M$). The equivalence class of a given function in a neighborhood of $x \in M$ *share all local properties* (e.g. continuity, differentiability, etc.).

So to summarize, we were concerned about local properties of functions. We want to study the local properties, but the first maneuver to do so is to consider the equivalence classes of functions. That is to say, we consider "equivalent functions" on the domain $U \subseteq M$ by finding equivalence classes of functions. And now for something quite dramatic: *we can study all local properties of a function by studying its equivalence class!* This should be a complete surprise, there has yet been motivation as to *why* this could even be considered!

Definition 2.1. Let M be a smooth manifold, let $x \in M$ be a point, then "**Germs of C^∞ Functions at** x" consist of equivalence classes given by the relation described by Eq (2.2).

So a germ is just an *alias* for an equivalence class $[f]_x$. Each germ corresponds to a different equivalence class. We will use the notation that D_x is the germs at x. *Notation for Germs at x: D_x*

Now, it should be noted that we can *induce* a differentiable structure on the germs in "the obvious way." That is, germs are working with functions in $C^\infty(U)$, so if we simply work with the differentiable structure $C^\infty(U)$ for each "germ representative" (the f for the germ $[f]_x$) we get for free a differentiable structure. This is the induced differentiable structure. This induced structure is quite dramatic, *it converts germs at x into an associative unital algebra!* Moreover that makes it a vector space.

2.2 Tangent Space From Germs

Recall that we have, for any vector space V over a field $\mathbb{F}$, a so-called "dual" vector space V^* which consists of "covectors". That is, mappings from V to $\mathbb{F}$. So if $\omega \in V^*$, then we know that

$$\omega \colon V \to \mathbb{F} \tag{2.4}$$

is a "linear mapping." More or less, for finite dimensional vector spaces, V^* is the "row-vector" space, and acts on V by matrix multiplication.

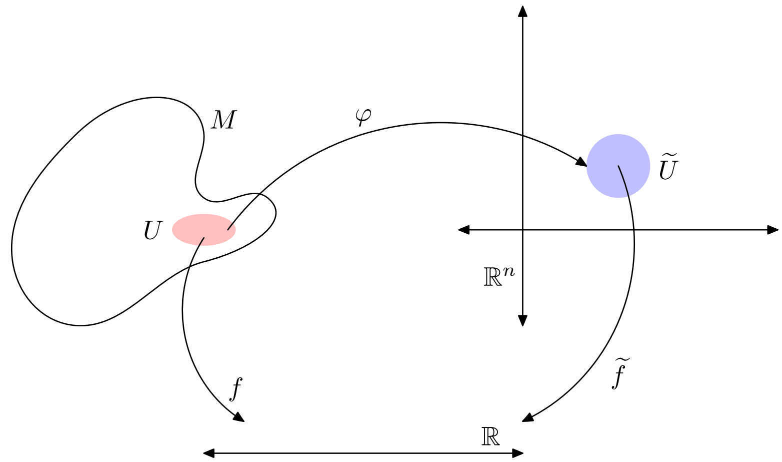

Figure 1: Functions in different local coordinates.

We just introduced the notion of germs at a point $x \in M$ for some smooth manifold M. It turned out to be a vector space (strong still, an associative unital algebra). The natural question is: what is the structure of its dual space?

Lets try to deduce its structure from what little we know about dual spaces and D_x. We know that D_x is a vector space over $\mathbb{R}$, so it would logically follow that its dual space D_x^* would "eat in" elements of D_x and "spit out" real numbers. So these gadgets map, at least ethically, functions to numbers. It is straightforward and possible to generalize the codomain of these gadgets from $\mathbb{R}$ to $\mathbb{C}$ complex numbers. If $v \in D_x^*$, then we want to consider all such v which satisfy

Notation: dual space of D_x is denoted D_x^*

$$v(fg) = v(f) \cdot g(x) + f(x) \cdot v(g) \tag{2.5}$$

which should make the reader think of the product rule for derivatives!

However, note that derivatives are mappings of the form

$$\partial \colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \tag{2.6}$$

which map functions to functions. Compare this with the covectors D_x^* of the form

$$v \colon C^\infty(U) \to \mathbb{R} \tag{2.7}$$

which map differentiable functions to numbers "somehow."

▸ **Exercise 1.** [*M10*] Prove that the subset of elements of D_x^* which satisfy the property (2.5) form a linear subspace of D_x^*.

The reader, upon proving exercise 1, should realize that we are working with a vector space. We will denote this vector space by T_xM. We will dub this space the "**Tangent Space of M at x**" This is another alias, we have qualitatively analysed its elements which we will call (appropriately enough) "**Tangent Vectors at x**".

Tangent Vector Space T_xM

This surprise is probably more than unwelcome, as there is no logical reason why anyone in their rightmind would ever consider this. However, it does have advantages. For one thing, we have a *purely algebraic* formulation of tangent vectors. It allows for generalization to other settings.

If we consider an example to solidify our understanding of this notion of tangent vectors, we should consult figure 1. We have a smooth manifold M, and some open subset $U \subseteq M$. We have some chart $\varphi \colon U \to \widetilde{U} \subseteq \mathbb{R}^n$. This is local coordinates describing our open subset

$U \subseteq M$. We have a function $f\colon U \to \mathbb{R}$ be described "locally" within "local coordinates" as $\widetilde{f}\colon \widetilde{U} \to \mathbb{R}$. That is

$$f = \widetilde{f} \circ \varphi \tag{2.8}$$

or equivalently, the diagram described by figure 1 is a commutative diagram. We want to consider a maps

$$f = \widetilde{f} \circ \varphi \mapsto \left(\frac{\partial \widetilde{f}}{\partial \widetilde{x}_j}\right)\Bigg|_{\widetilde{x}_j = x_j(x)} \tag{2.9}$$

induces a map from $D_x \to \mathbb{C}$ as we are evaluating the derivative at the point x. It's a map $C^{\infty}(U) \to \mathbb{C}$. We see that this induces a map on the germs D_x at x to $\mathbb{C}$. These mappings obey the product rule, which is precisely the property of particular interest with tangent vectors. It follows that

$$\frac{\partial}{\partial \widetilde{x}_j}\Bigg|_{\widetilde{x}_j = x_j(x)} = \frac{\partial}{\partial x_j} \tag{2.10}$$

form a basis of T_xM.

2.3 Generalizations, Differential Expressions

We want to generalize the notion of a tangent vector. This is a long and involved process that may seem roundabout and needlessly complicated, but it provides a nice generalization of the concept. We will first study the set of functions with roots at x_0.

We first want to consider the element $1_x \in D_x^*$ by

$$1_x(f) = f(x) \qquad \forall f \in D_x. \tag{2.11}$$

We see 1_x is real and linearly independent of T_xM.

Exercise 2. [*HM15*] Prove that 1_x is real and linearly independent of T_xM.

However, observe that if we have an element $v \in D_x^*$ to belong to the complex linear span of 1_x and T_xM, it is necessary and sufficient that $v(f_1 f_2) = 0$ for all $f_1,\ f_2 \in D_x$ which vanish at x.

▸ **Exercise 3.** [*HM20*] Why?

N.B.: this is what leads us to a natural generalization of the notion of a tangent vector.

We will introduce notation for the collection of functions with a zero at x, that is to say the germs $f \in D_x$ such that $f(x) = 0$.

Notation. Let

$$J_x = \{f \mid f \in D_x,\ f(x) = 0\} \tag{2.12}$$

be the collection of germs which vanish at the point x.

Note: J_x is the set of all germs which vanish at the point x

▸ **Exercise 4.** [*M10*] Prove that J_x is an ideal in D_x.

Exercise 5. [*10*] Show that any $f \in J_{x_0}$ is of the form $(x - x_0)^r g(x)$ for some function $g \in D_{x_0}$ and positive integer $r \in \mathbb{N}$.

Now, let us review what we have just done. We have considered the germs which vanish at a given point x_0. Collectively these germs form a set $J_{x_0} \subseteq D_{x_0}$. These germs form an ideal (exercise 4). Moreover, they are of the form (in some local coordinates)

$$f(x) = (x - x_0)^r g(x) \tag{2.13}$$

where r is a positive integer called the "multiplicity" of x_0, $g \in D_{x_0}$, and $f \in J_{x_0}$. We can introduce notation to specify the multiplicity of the root, namely

$$J_{x_0}^p = \{f_1 \cdots f_p \mid f_1, \dots, f_p \in J_{x_0}^p\} \tag{2.14}$$

Notation: J_x^p are germs with zeros at x with multiplicity p

for an integer $p \geq 1$. This is another way of saying that the multiplicity of the root x_0 is *at least* p.

Exercise 6. [*03*] Prove that $J_{x_0}^p$ is an ideal in D_{x_0} for any $p \in \mathbb{N}$.

Now, so far, the keen observer will note we have just been discussing functions which vanish at x_0. So far no attempt to generalize tangent vectors has been made. The critical property of tangent vectors that deserves generalization is, if $v \in T_xM$ and $f, g \in D_x$,

$$v(fg) = v(f)g(x) + f(x)v(g) \tag{2.15}$$

but observe now if we restrict our attention to $f, g \in J_x$ we get

$$v(fg) = v(f)g(x) + f(x)v(g) = 0 + 0 = 0. \tag{2.16}$$

This is precisely the property we use as grounds for generalization.

Definition 2.2. Let $r \geq 0$ be an integer. A "**Differential Expression of Order** $\leq r$" consists of

1. an element $v \in D_x^*$

such that

1. for each $f \in J_x^{r+1}$, $v(f) = 0$.

Now why is this a good generalization? Well, we see by the notion of a differential expression of order r satisfies this very property by the product rule. In fact, we have

$$\frac{d^r}{dx^r}\left[(x - x_0)^{r+1}g(x)\right] = (r+1)!(x - x_0)g(x) + \mathcal{O}([x - x_0]^2) \tag{2.17}$$

which vanishes at x_0. So in other words,

$$\left.\frac{d^r}{dx^r}f(x)\right|_{x=x_0} = 0 \qquad \forall f \in J_{x_0}^{r+1} \tag{2.18}$$

is precisely the property that we generalize.

▸ **Exercise 7.** [*M12*] Show that the collection of differential expressions of order $\leq r$ form a linear subspace of D_x^*.

So to review, we generalize the notion of derivatives by simply generalizing the product rule. This notion that $v \in D_x^*$ is a differential expression if

$$v(f) = 0 \tag{2.19}$$

for all $f \in J_x^{r+1}$ is *more general than* (i.e. extends beyond) the basic notion of a derivative in usual calculus.

3 Solution to Exercises

Exercise 1. [*M10*] We see that we can simply write any two v, w which satisfy the product rule into a linear combination

$$(c_1 v + c_2 w)f = c_1 v(f) + c_2 w(f). \tag{3.1}$$

We can then deduce that this linear combination also satisfies the product rule

$$\begin{aligned}(c_1 v + c_2 w)(fg) &= c_1 v(fg) + c_2 w(fg) \\ &= c_1 [v(f)g(x) + f(x)v(g)] + c_2 [w(f)g(x) + f(x)w(g)] \\ &= [(c_1 v + c_2 w)(f)]g(x) + f(x)[(c_1 v + c_2 w)(g)]\end{aligned} \tag{3.3}$$

since each term on the right hand side obeys the product rule. This is sufficient to show that a linear combination of elements of D_x^* (which obey the product rule) also obeys the product rule. It follows that this subset of elements obeying the product rule is a subspace of D_x^* as a vector space.

Exercise 2. [*HM15*] We see that $f(x) \in \mathbb{R}$, so it follows that 1_x is real. We need to show that it is linearly independent of $T_x M$. Well, if $1_x \in T_x M$, i.e. if it is linearly *dependent* of $T_x M$, then it would obey the property that

$$1_x(fg) = 1_x(f)g(x) + f(x)1_x(g)$$

but we see by definition that this is

$$1_x(fg) = f(x)g(x).$$

If we set these two equal we see that

$$1_x(f)g(x) + f(x)1_x(g) = f(x)g(x)$$

if and only if

$$2f(x)g(x) = f(x)g(x)$$

or equivalently

$$f(x)g(x) = 0$$

for all $f, g \in D_x$. As this is not true, we have a contradiction, and thus $1_x \notin T_x M$.

Exercise 3. [*HM20*] Consider any f_1, $f_2 \in D_x$. Let $u \in T_x M$, and write

$$v = c_1 u + c_2 1_x \tag{3.5}$$

where $c_1, c_2 \in \mathbb{C}$. We see by direct computation

$$v(f_1 f_2) = c_1 u(f_1) f_2(x) + c_1 f_1(x) u(f_2) + c_2 f_1(x) f_2(x) \tag{3.7}$$

which we can rewrite as

$$v(f_1 f_2) = [c_1 u(f_2) + c_2 f_2(x)]\, f_1(x) + c_1 u(f_1) f_2(x) \tag{3.9}$$

which vanishes iff $f_1(x) = f_2(x) = 0$ or $c_1 = c_2 = 0$.

Exercise 4. [*M10*] We see that for any $g \in D_x$ and $f \in J_x$ that

$$(fg)(x) = f(x)g(x) = 0 \cdot g(x) = 0 \tag{3.11}$$

so $fg \in J_x$ for any $f \in J_x$ and $g \in D_x$. Thus by definition J_x is an ideal.

Exercise 5. [*10*] It is obvious. By using the induced differentiable structure, we can Taylor expand

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots \tag{3.13}$$

and find when $f^{(r)}(x_0) \neq 0$. We can factorize then set

$$f(x) = (x - x_0)^r \left(f^{(r)}(x_0) + \cdots\right) = (x - x_0)^r g(x) \tag{3.15}$$

where we just define $g(x)$ as the parenthetic term.

Exercise 5. (Alternate answer) It is obvious immediately from the fundamental theorem of algebra that it must be of this form, if we work with $\mathbb{C}$ or if we embed $\mathbb{R}$ into $\mathbb{C}$.

Exercise 6. [*03*] We see that by the exact same reasoning as for exercise 4 that $J^p_{x_0}$ is an ideal in D_{x_0}.

Exercise 7. [*M12*] We see that if u, v are both differential expressions of order $\leq r$, and if c_1, $c_2 \in \mathbb{R}$ are constants, then

$$(c_1 u + c_2 v)(f) = c_1 u(f) + c_2 v(f) = 0 \tag{3.17}$$

for all $f \in J^{r+1}_x$. Thus the linear combination is also a differential expression of order $\leq r$, and thus the collection of all such expressions form a linear subspace of D^*_x.

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