Notes on Feynman Diagrams

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Abstract

We introduce in a pedagogical manner how to compute probability amplitudes from Feynman diagrams, starting with ϕ^4 model. We introduce the notion of renormalization in this model at the one-loop level. Then we review the Dirac equation, and introduce QED. We then perform several example calculations in QED. The appendices gives a survey of Gamma matrices and use of Feynman diagrams in computing decay rates.

1 Feynman Rules in a Nutshell with a Toy Model

We will work in a toy model[1] with massive spinless particles (so we won't have to worry about spin). This is the easiest nontrivial example of the use of Feynman diagrams. The basic ritual of Feynman diagrams is outlined thus:

1. (Notation) Label the incoming and outgoing four-momenta $p_1, p_2, \ldots, p_n$. Label the internal momenta $q_1, q_2, \ldots$. Put an arrow on each line, keeping track of the "positive" direction (antiparticles move "backward" in time).

2. (Coupling Constant) At each vertex, write a factor of
$$-ig$$
where g is called the "**coupling constant**"; it specifies the strength of the interaction. In our toy model, g will have dimensions of momentum, but in the real world it is dimensionless.

3. (Propagator) For each internal line, write a factor
$$\frac{i}{q_j^2 - m_j^2 c^2}$$
where q_j is the four-momentum of the line ($q_j^2 = q_j^\mu q_{j\mu}$; i.e. j is just a label keeping track of which internal line we are dealing with) and m_j is the mass of the particle the line describes. (Note that for virtual particles, we don't have the $E^2 - \vec{p}\cdot\vec{p} = m^2c^2$ relation that's for external legs only!)

4. (Conservation of Momentum) For each vertex, write a delta function of the form
$$(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3)$$
where the k's are the three four-momenta coming *into* the vertex (if the arrow leads outward, then k is *minus* the four-momentum of that line). This factor imposes conservation of energy and momentum at each vertex, since the delta function is zero unless the sum of the incoming momenta equals the sum of the outgoing momenta.

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*This is a page from https://pqnelson.github.io/notebk/
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[1]It is commonly referred to as the ϕ^4 model in the literature.

5. (Integration over Internal Momenta) For each internal line, write down a factor

$$\frac{1}{(2\pi)^4} d^4 q_j$$

and integrate over all internal momenta.

6. (Cancel the Delta Function) The result will include a delta function

$$(2\pi)^4 \delta^{(4)}(p_1 + p_2 + \cdots - p_n)$$

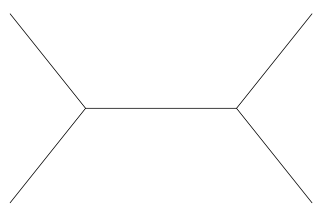
enforcing overall conservation of energy and momentum. Erase this factor, and what remains is $i\mathcal{M}$ that is $-i$ times the contribution to the amplitude from this process.

What we do with these rules is we form an integrand by multiplying everything together, so at the end it should look something like this:

$$i\mathcal{M} \text{ ``=''} \begin{pmatrix}\text{coupling}\\ \text{constants}\end{pmatrix} \int (\text{propagators}) \begin{pmatrix}\text{delta}\\ \text{functions}\end{pmatrix} d \begin{pmatrix}\text{internal}\\ \text{lines}\end{pmatrix} \tag{1.1}$$

1.1 Example

We will consider the process $A + A \to B + B$, which is represented by the following Feynman diagram (note that the x axis is the spatial dimension, the y axis is the time dimension):

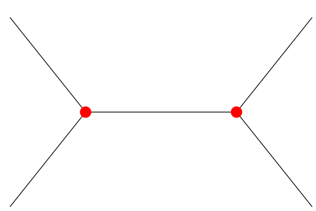


Step One: We drew it careful about notation (note the internal momentum line q and the external lines p_j).

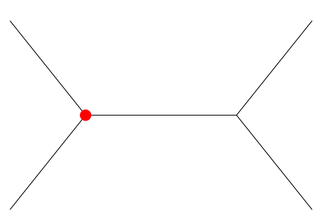
Step Two: We have to worry about the vertices, at each one we have to award a term of

$$-ig$$

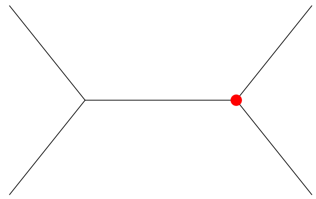
So here is the Feynman diagram with the vertices enlarged in red:



We see that there are two vertices, one where A emits C and becomes B:



and the other where A receives C and turns into B:



By our rules, this means we get two factors of

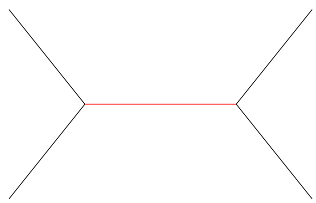
$$-ig.$$

That is to say, our integrand is thus

$$(-ig)^2 \tag{1.2}$$

and we will add even more to it!

Step Three: (Let m_C be the mass of a C particle.) We also need a propagator for the internal line; we have below the Feynman diagram with the internal line in red:



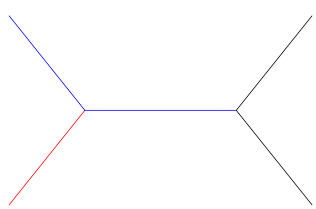
This means we have a factor of

$$\frac{i}{q^2 - m_C^2 c^2}.$$

We multiply this into our integrand which becomes

$$(-ig)^2 \frac{i}{q^2 - m_C^2 c^2}. \tag{1.3}$$

Step Four: Now conservation of momentum demands two delta functions; we see that the momentum has to be conserved at the vertices, so we have two diagrams in color this time. At one vertex, we have the input momentum (the red line be) equal to the sum of the output momentum (blue lines):



This means we have the conservation of momentum:

$$p_1 = p_3 + q. \tag{1.4}$$

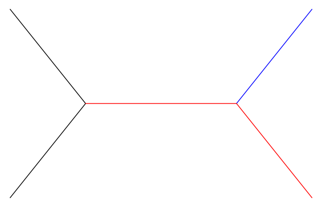
This corresponds to the delta function of

$$(2\pi)^4\delta^{(4)}(p_1 - p_3 - q)$$

(i.e. the left vertex has momentum conserved). We multiply this into the integrand, which becomes

$$(2\pi)^4(-ig)^2\frac{i}{q^2 - m_C^2c^2}\delta^{(4)}(p_1 - p_3 - q). \tag{1.5}$$

We have another vertex too, which we have the "input momenta lines" in red summed to have the same momentum as the "output momenta lines" in blue:



This corresponds to the conservation of momentum

$$p_2 + q = p_4 \Rightarrow p_2 + q - p_4 = 0 \tag{1.6}$$

and this corresponds to a delta function of

$$(2\pi)^4\delta^{(4)}(p_2 + q - p_4)$$

(i.e. the right vertex has momentum conserved). The integrand becomes

$$(2\pi)^8(-ig)^2\frac{i}{q^2 - m_C^2c^2}\delta^{(4)}(p_1 - p_3 - q)\delta^{(4)}(p_2 + q - p_4). \tag{1.7}$$

Step Five: We integrate over the internal lines, luckily we only have one! We have one integration thus one term

$$\frac{1}{(2\pi)^4}d^4q.$$

Combining rules 1 through 5 gives us the final expression

$$-i(2\pi)^4g^2\int\frac{1}{q^2 - m_C^2c^2}\delta^{(4)}(p_1 - p_3 - q)\delta^{(4)}(p_2 + q - p_4). \tag{1.8}$$

The second delta function serves to pick out the value of everything else at the point $q = p_4 - p_2$, so we have

$$-ig^2\frac{1}{(p_4 - p_2)^2 - m_C^2c^2}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4). \tag{1.9}$$

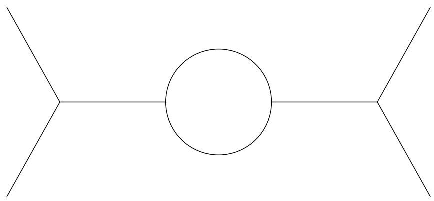
And we have one last delta function which tells us that we conserved the overall energy and momentum. We erase it by rule 6 and we get the amplitude for this particular process to be

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2c^2}. \tag{1.10}$$

This particular process is called a "**tree diagram**" because we do not have any internal loops. Lets consider such an example next.

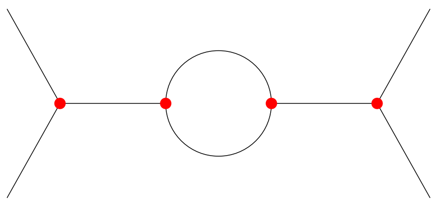
1.2 A Slightly More Complicated Example

This example will teach you that any idiot can complicate a simple scheme, consider the following diagram:



Step One: We drew it carefully and made special care of the notation used.

Step Two: We need to take note of how many vertices we have, so we have them enlarged in red to see how many:

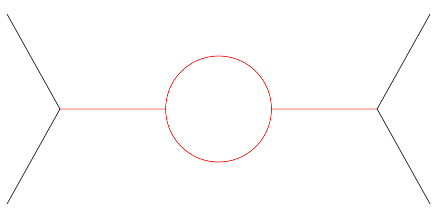


we have 4 vertex terms, that is we have

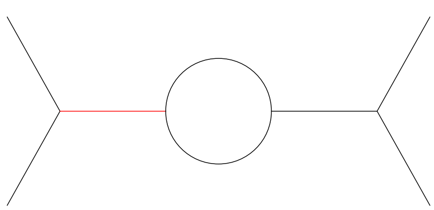
$$(-ig)^4$$

in the integrand so far.

Step Three: We need to take care of the internal lines now, so we will see which lines those are exactly:



Since we are being pedagogical, we will go through one by one and indicate which line we are dealing with and what we evaluate it to be. We will begin in any old arbitrary manner we please with this particular example, **it won't be that way in general!** We will first consider:



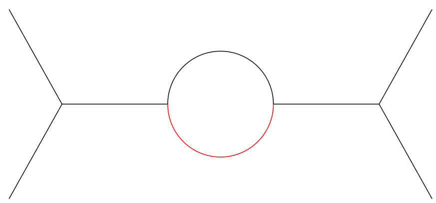
We evaluate this to be the propagator described by

$$\frac{i}{q_1^2 - m_C^2 c^2}$$

so we multiply it into the integrand. The integrand is then

$$(-ig)^4 \frac{i}{q_1^2 - m_C^2 c^2}. \tag{1.11}$$

We continue on and we see the term given by the internal line in red



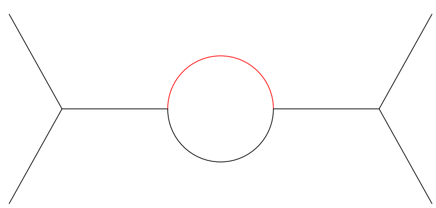
This corresponds to the propagator described by

$$\frac{i}{q_2^2 - m_A^2 c^2}.$$

We multiply this into our integrand now and we get

$$(-ig)^4 \frac{i}{q_1^2 - m_C^2 c^2} \frac{i}{q_2^2 - m_A^2 c^2} = -g^4 \frac{1}{q_1^2 - m_C^2 c^2} \frac{1}{q_2^2 - m_A^2 c^2}. \tag{1.12}$$

Similarly we can do likewise for the other part of the loop in red:



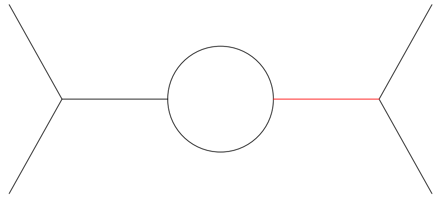
This corresponds to the propagator

$$\frac{i}{q_3^2 - m_B^2 c^2}$$

and we just multiply it into the integrand, which becomes

$$-g^4 \frac{1}{q_1^2 - m_C^2 c^2} \frac{1}{q_2^2 - m_A^2 c^2} \frac{i}{q_3^2 - m_B^2 c^2}. \tag{1.13}$$

We have one last internal line left! We highlight it in red:



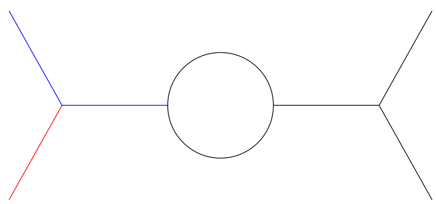
This corresponds to the propagator

$$\frac{i}{q_4^2 - m_C^2 c^2}$$

and we multiply it into the integrand, which becomes

$$-g^4 \frac{1}{q_1^2 - m_C^2 c^2} \frac{1}{q_2^2 - m_A^2 c^2} \frac{i}{q_3^2 - m_B^2 c^2} \frac{i}{q_4^2 - m_C^2 c^2} = g^4 \frac{1}{q_1^2 - m_C^2 c^2} \frac{1}{q_2^2 - m_A^2 c^2} \frac{1}{q_3^2 - m_B^2 c^2} \frac{1}{q_4^2 - m_C^2 c^2}. \tag{1.14}$$

Step Four: We need to enfore the conservation of momentum, so what do we do? We simply go back to our graph and go one vertex at a time and enforce conservation of momentum. At the first vertex, the input momentum is in red and the output momentum is in blue:



So we want to have momentum here conserved, i.e.

$$p_1 = p_3 + q_1 \tag{1.15}$$

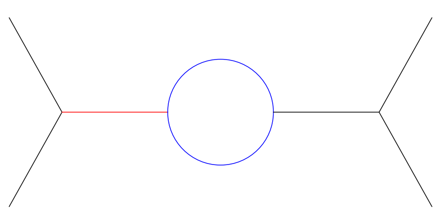
so we multiply the integrand by the term

$$(2\pi)^4 \delta^{(4)}(p_1 - p_3 - q_1).$$

Our integrand, which is ever expanding, is then

$$g^4 \frac{1}{q_1^2 - m_C^2 c^2} \frac{1}{q_2^2 - m_A^2 c^2} \frac{1}{q_3^2 - m_B^2 c^2} \frac{1}{q_4^2 - m_C^2 c^2} (2\pi)^4 \delta^{(4)}(p_1 - p_3 - q_1). \tag{1.16}$$

There are three other vertices which we must impose conservation laws on, so we will move right along to the next vertex; again the input momentum is in red, and the output momentum is in blue:



This corresponds to a conservation of momentum of

$$q_1 \approx q_2 + q_3 \tag{1.17}$$

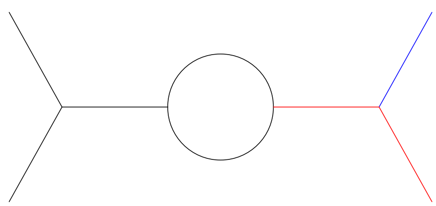
which corresponds to the dirac delta function term of

$$(2\pi)^4 \delta^{(4)}(q_1 - q_2 - q_3).$$

Multiplying this into our integrand, we get

$$(2\pi)^8 g^4 \frac{1}{q_1^2 - m_C^2 c^2} \frac{1}{q_2^2 - m_A^2 c^2} \frac{1}{q_3^2 - m_B^2 c^2} \frac{1}{q_4^2 - m_C^2 c^2} \delta^{(4)}(p_1 - p_3 - q_1)\delta^{(4)}(q_1 - q_2 - q_3). \tag{1.18}$$

Two vertices down, two to go! We simply move right along to find the next conservation of momentum to be at the next vertex. The input momentums are in red, and the output momentum is in blue:



This corresponds to the conservation

$$q_2 + q_3 \approx q_4 \tag{1.19}$$

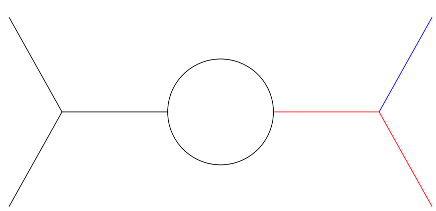
which means we have a delta function of the form

$$(2\pi)^4 \delta^{(4)}(q_2 + q_3 - q_4).$$

Our integrand becomes

$$(2\pi)^{12} g^4 \frac{1}{q_1^2 - m_C^2 c^2} \frac{1}{q_2^2 - m_A^2 c^2} \frac{1}{q_3^2 - m_B^2 c^2} \frac{1}{q_4^2 - m_C^2 c^2} \delta^{(4)}(p_1 - p_3 - q_1)\delta^{(4)}(q_1 - q_2 - q_3)\delta^{(4)}(q_2 + q_3 - q_4). \tag{1.20}$$

⟨*As we can see, this is getting really really messy! God help us when we try to feebly evaluate this beast!*⟩ Thank god only one vertex left! This is the last term to add prior to integration. The input momentums are in red and the output momentum is in blue:



This has the conservation of

$$q_4 + p_2 \approx p_4 \tag{1.21}$$

which takes the delta form of

$$(2\pi)^4 \delta^{(4)}(q_4 + p_2 - p_4) \tag{1.22}$$

and our integrand finally becomes

$$(2\pi)^{16} g^4 \frac{\delta^{(4)}(p_1 - p_3 - q_1)}{q_1^2 - m_C^2 c^2} \frac{\delta^{(4)}(q_1 - q_2 - q_3)}{q_2^2 - m_A^2 c^2} \frac{\delta^{(4)}(q_2 + q_3 - q_4)}{q_3^2 - m_B^2 c^2} \frac{\delta^{(4)}(q_4 + p_2 - p_4)}{q_4^2 - m_C^2 c^2} \tag{1.23}$$

At this point, if I were a professor, I would say "This is trivial...bye!" But I am no professor!

Step Five: We then integrate over the internal lines. This is the fun part, like when the dentist says he needs to give you a root canal and he's all outta Novocaine! We integrate over q_1, q_2, q_3, q_4. We notice that the factors of 2π completely cancel out which is nice, so all we have is

$$g^4 \int \frac{\delta^{(4)}(p_1 - p_3 - q_1)}{q_1^2 - m_C^2 c^2} \frac{\delta^{(4)}(q_1 - q_2 - q_3)}{q_2^2 - m_A^2 c^2} \frac{\delta^{(4)}(q_2 + q_3 - q_4)}{q_3^2 - m_B^2 c^2} \frac{\delta^{(4)}(q_4 + p_2 - p_4)}{q_4^2 - m_C^2 c^2} d^4q_1 d^4q_2 d^4q_3 d^4q_4. \tag{1.24}$$

We will take this slow and step by step, we see that in the first delta function we have the replacement of q_1 by $p_1 - p_3$ which is nice! We make this move:

$$\frac{g^4}{(p_1 - p_3)^2 - m_C^2 c^2} \int \frac{\delta^{(4)}(p_1 - p_3 - q_2 - q_3)}{q_2^2 - m_A^2 c^2} \frac{\delta^{(4)}(q_2 + q_3 - q_4)}{q_3^2 - m_B^2 c^2} \frac{\delta^{(4)}(q_4 + p_2 - p_4)}{q_4^2 - m_C^2 c^2} d^4q_2 d^4q_3 d^4q_4. \tag{1.25}$$

Similarly, we find the our last delta function allows us to make the switcheroo of q_4 for $p_2 - p_4$, so we make it so:

$$\frac{g^4}{(p_1 - p_3)^2 - m_C^2 c^2} \frac{1}{(p_2 - p_4)^2 - m_C^2 c^2} \int \frac{\delta^{(4)}(p_1 - p_3 - q_2 - q_3)}{q_2^2 - m_A^2 c^2} \frac{\delta^{(4)}(q_2 + q_3 - p_2 - p_4)}{q_3^2 - m_B^2 c^2} d^4q_2 d^4q_3. \tag{1.26}$$

The term $\delta^{(4)}(p_1 - p_3 - q_2 - q_3)$ tells us q_2 is replaced by $p_1 - p_3 - q_3$ so our last delta function (after monkeying around with integration) becomes

$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4).$$

We are left with

$$\begin{aligned} &\frac{g^4}{(p_1 - p_3)^2 - m_C^2 c^2} \frac{1}{(p_2 - p_4)^2 - m_C^2 c^2} \\ \times &\int \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2 c^2} \frac{\delta^{(4)}(p_1 - p_3 + p_2 - p_4)}{q_3^2 - m_B^2 c^2} d^4q_3. \end{aligned} \tag{1.27}$$

So we can skip ahead to rule 6 and assert: our contribution to the probability amplitude from this diagram is

$$\begin{aligned} \mathcal{M} &= i\left(\frac{g}{2\pi}\right)^4 \frac{1}{[(p_1 - p_3)^2 - m_C^2 c^2][(p_2 - p_4)^2 - m_C^2 c^2]} \\ &\quad \times \int \frac{1}{[(p_1 - p_3 - q_3)^2 - m_A^2 c^2][q_3^2 - m_B^2 c^2]} d^4q_3. \end{aligned} \tag{1.28}$$

2 Renormalization

You can go ahead and try to calculate out the integral in Eq (1.28) but I will tell you right now that it's not easy or finite. One could write the four-dimensional volume element as

$$d^4q = q^3 dq d\Omega \tag{2.1}$$

(where $d\Omega$ is the angular part; just like in two dimensions we have $rdrd\theta$ and in three dimensions $r^2 dr \sin\theta d\theta d\phi$). At large q the integrand is essentially $1/q^4$, so the q integral has the form

$$\int_0^\infty \frac{1}{q^4} q^3 dq = \ln(q)\Big|_{q=0}^{q=\infty} = \infty. \tag{2.2}$$

This is terrible! So what to do? The first step is to *regularize* the integral using some sort of "cut off" procedure that renders the integral finite without losing Lorentz invariance. So we introduce a factor into the Eq (1.28) of

The field of Renormalization consists of (1) regularization, and (2) radiative corrections.

$$\frac{-M^2c^2}{q^2 - M^2c^2} \tag{2.3}$$

under the integral sign. The *cutoff mass* M is assumed to be "very large" and will be taken to infinity when we are done (also observe that our fudge factor goes to 1 as $M \to \infty$). We can now calculate out the integral and seperate it into two terms: a finite term (one independent of M), and one (in this case) that is the logarithm of M (which goes to ∞ as $M \to \infty$).

At this point something rather magical happens: all the divergent, M-dependent terms appear in the final answer in the fomr of *additions to the masses and the coupling constant.* This means (if taken seriously) that the *physical* masses and couplings are not the m's and g's that appeared in the original Feynman rules but rather the "renormalized" ones, containing extra factors:

$$m_{\text{physical}} = m + \delta m, \quad g_{\text{physical}} = g + \delta g. \tag{2.4}$$

The fact that δm and δg are infinite (in the limit of $M \to \infty$) is disturbing but not lethal...we never measure them anyways! All we ever see in the lab are *physical* values, and these are trivially finite. As a practical matter, we take account of the infinities by using the *physical* values of m and g in Feynman rules, and then systematically ignoring the divergent contributions from higher-order diagrams.

Meanwhile, there were the finite contributions from the loop diagrams that we kinda were neglecting. They too lead to modifications in m and g (perfectly calculable in this case) – which are functions of the four-momentum of the line in which the loop is inserted (in our example, $p_1 - p_3$). This means that the *effective* masses and coupling constants actually depend on the *energies* of the particles involved; we call them "running" masses and "running" coupling constants. The dependence is typically rather slight, at least at low energies. They can be ignored but they are observable effects, such as the Lamb shift (in QED) and asymptotic freedom (in QCD).

Before Going to QED...

Before we can go ahead to Quantum Electrodynamics, we need to first introduce (or in some cases, review) the Dirac equation. We will proceed to do that now... For a more thorough treatment, see Dyson [Dys06]. Note for the most part, the inspiration of this entire article can be found in Griffiths [Gri87]. It is a good introductory text on general particle physics too.

3 Klein-Gordon Review

Recall that the Schrodinger equation for the free nonrelativistic particle is

$$\frac{-\hbar^2}{2m}\nabla^2 |\psi\rangle = -i\hbar\partial_t |\psi\rangle \tag{3.1}$$

which corresponds to a sort of quantized Newton's second law

$$\frac{p^2}{2m} \approx E. \tag{3.2}$$

However, in special relativity we have the mass shell constraint

$$p^\mu p_\mu = E^2 - \mathbf{p}\cdot\mathbf{p} = m^2 \tag{3.3}$$

(when $c = 1$) using Einstein summation convention. If we naively quantize this, we end up with

$$\partial_\mu \partial^\mu - m^2 |\psi\rangle = 0 \tag{3.4}$$

by moving the mass term onto the left hand side. This is the Klein-Gordon equation, it is plagued by problems such as negative probabilities, etc.

4 Dirac takes it up a notch...bam!

Naively, we want something simpler than this. We can rewrite Eq (3.3) to be

$$E^2 = \mathbf{p} \cdot \mathbf{p} + m^2 \Rightarrow E = \sqrt{\mathbf{p} \cdot \mathbf{p} + m^2} \tag{4.1}$$

then quantize it to be

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \sqrt{-\hbar^2 \nabla^2 + m^2} \, |\psi\rangle \, . \tag{4.2}$$

We end up being forced to use pseudo-differential operators, unfortunately, and it turns out that this results in nonlocality[2]. For further details see Laemmerzahl [Lae93].

The approach Dirac takes is basically taking the squareroot of the operator, but he does it with class. He uses a clifford algebra with generators γ^μ[3] such that the squareroot of the Klein Gordon equation breaks into two equations:

$$(i\hbar\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{4.3}$$

where $\psi(x)$ is a spinor wave function with 4 components. The adjoint field $\bar{\psi}(x)$ is defined by

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0 \tag{4.4}$$

and satisfies the *adjoint* Dirac equation

$$\bar{\psi}(x)(i\hbar\gamma^\mu \partial_\mu + m) = 0. \tag{4.5}$$

It is to be understood here the differential operator ∂^μ acts on the left. Observe that when we multiply the two operators together we get

$$(i\hbar\gamma^\mu \partial_\mu + m)(i\hbar\gamma^\mu \partial_\mu - m) = -\hbar^2 \left(\gamma^\mu\right)^2 \partial_\mu \partial^\mu - m^2 = \hat{p}^\mu \hat{p}_\mu - m^2 \tag{4.6}$$

which is *precisely* the Klein-Gordon operator (3.4)! We should be content now with the connection back to what we already know. ⟨*Note to self: And an added advantage is that the Dirac equation is a first order partial differential equation, whereas the Klein-Gordon equation is a second order one!*⟩

4.1 A Somewhat Rigorous Derivation of the Dirac Equation

We want the squareroot of the wave operator thus

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = (A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t) \tag{4.7}$$

We see multiplying out the right hand side all the cross-terms must vanish. To have this we want

$$AB + BA = 0, \tag{4.8}$$

and so on for all cross-term coefficients, with the property that

$$A^2 = B^2 = C^2 = D^2 = 1. \tag{4.9}$$

[2] In general, whenever there is a squareroot quantity, there is nonlocality.

[3] If the reader is unfamiliar with the Gamma Matrices, see the appendix A and/or CORE [BRS95].

Dirac had previously worked out rigorous results with Heisenberg's matrix mechanics, and concluded that these conditions could be met if $A, B, \ldots$ were *matrices* which has the implication that the wave function has *multiple components*.

In the mean time, Pauli had been working on quantum mechanics as well. Pauli had a model with two-component wave functions that was involved in a phenomenological theory of spin. At this point in time, spin was not well understood.

Given the factorization of these matrices, one can now write down immediately an equation

$$(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)\psi = \kappa\psi \tag{4.10}$$

with κ to be determined. Applying the same operation on either side yields

$$(\nabla^2 - \frac{1}{c^2}\partial_t^2)\psi = \kappa^2\psi. \tag{4.11}$$

If one take $\kappa = mc/\hbar$ we find that all the components of the wave function *individually* satisfy the mass-shell relation (3.3). Thus we have a first order differential equation in both space and time described by

$$(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t - \frac{mc}{\hbar})\psi = 0 \tag{4.12}$$

where $(A, B, C) = i\beta\alpha_k$ and $D = \beta$, which is precisely the Dirac equation for a spin-1/2 particle of rest mass m.

4.2 A Comparison to the Pauli Theory

The necessity of introducing half-integer spin goes back experimentally to the results of the Stern-Gerlach experiment. ⟨*Note to self: A beam of atoms is run through a strong inhomogeneous magnetic field, which then splits into N parts depending on the intrinsic angular momentum of the atoms. It was found that for silver atoms, the beam was split in two - the ground state therefore could not be integral, because even if the intrinsic angular momentum of the atoms were as small as possible, 1, the beam would be split into 3 parts, corresponding to atoms with* $L_z = -1, 0, +1$. *The conclusion is that silver atoms have net intrinsic angular momentum of 1/2.*⟩ Pauli set up a model which explained the splitting by introducing a two-component wave-function and a corresponding correction term in the Hamiltonian(representing a semiclassical coupling of this wave function to an applied magnetic field) as

$$H = \frac{1}{2m}(\sigma_i^I(p^i - \frac{e}{c}A^i)\sigma_{Ij}(p^j - \frac{e}{c}A^j)) + eA^0 \qquad (i, j, I = 1, 2, 3). \tag{4.13}$$

We have here A^μ is the magnetic potential, and the van Warden symbols σ_j^J which translates a vector into the Pauli matrix basis (if one is unfamiliar with Pauli matrices, see §A.1), e is the electric charge of the particle (here $e = -e_0$ for the electron), and m is the mass of the particle. Now we have just described the Hamiltonian of our system by a 2 by 2 matrix. The Schrodinger equation based on it

$$H\phi = i\hbar\frac{\partial\phi}{\partial t} \tag{4.14}$$

must use a two-component wave function. (If you're like me you're too lazy to flip to the appendix, so I'll reproduce some of it here) Pauli used the SU(2) matrices

SU(2) is the set of all 2 by 2 matrices that is self-adjoint and has a determinant of 1

$$\sigma_k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.15}$$

due to *phenomenological reasons* (explaining the Gerlach experiment). Dirac now has a *theoretical argument* that implies spin is a *consequence* of introducing special relativity into quantum theory.

The Pauli matrices share the same properties as the Dirac matrices – they are all self-adjoint, when squared are equal to the identity, and they anticommute. We can now use the Pauli matrices Eq (4.15) to describe a representation of the Dirac matrices:

$$\alpha_k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix}. \tag{4.16}$$

We now may write the Dirac equation as an equation coupling two-component spinors:

$$\begin{bmatrix} mc^2 & c\sigma \cdot p \\ c\sigma \cdot p & -mc^2 \end{bmatrix} \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}. \tag{4.17}$$

Observe that we have on the diagonal the rest mass. If we bring the particle to rest, we have

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \tag{4.18}$$

The equations for the individual two-spinors are now decoupled, and we see that the "spin-up" and "spin-down" (or "right-handed" and "left-handed", "positive frequency" and "negative frequency" respectively) are individual eigenfunctions ⟨*Eigenspinors?*⟩ of the energy with eigenvalues equal to $\pm$ the rest energy. The appearence of *negative* energy should not be alarming, it is completely consistent with relativity.

Note that this seperation is in the rest frame and **is not an invariant statement** – the bottom component does not generally represent antimatter. The *entire* four-component spinor represents an *irreducible whole* – in general states will have an admixture of positive *and* negative energy components.

4.3 Covariant Form and Relativistic Invariance

The explicity covariant form of the Dirac Equation is (using Einstein summation convention)

$$i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0, \tag{4.19}$$

where γ^μ are the Dirac gamma matrices. We have

$$\gamma^0 = \beta \qquad \gamma^k = \gamma^0 \alpha_k. \tag{4.20}$$

See the appendix for more details on this representation.

The Dirac equation may be interpreted as an eigenvalue expression, where the rest mass is proportional to an eigenvalue of the 4-momentum operator, the proportion being c:

$$\hat{P}\psi = mc\psi. \tag{4.21}$$

In practice we often work in units where we set $\hbar$ and c to be 1. The equation is multiplied by $-i$ and takes the form

$$(\gamma^\mu \partial_\mu + im)\,\psi = 0. \tag{4.22}$$

We may employ the Feynman slash notation to simplify this to

$$(\partial\!\!\!/ + im)\psi = 0. \tag{4.23}$$

For any two representations of the Dirac Gamma matrices, they are related by a unitary transformation. Likewise, the solutions in the two representations are related by the same way.

4.4 Conservation Laws and Canonical Structure

Recall the Dirac equation and its adjoint version, Eqns (4.3) and (4.5). We notice from the definition of the adjoint

$$\bar{\psi} = \psi^\dagger \gamma^0$$

that

$$(\gamma^\mu)^\dagger \gamma^0 = \gamma^0 \gamma^\mu \tag{4.24}$$

we can obtain the Hermitian conjugate of the Dirac equation and multiplying from the right by γ^0 we get its adjoint version:

$$\bar{\psi}(\gamma^\mu \overleftarrow{\partial}_\mu - im) = 0$$

where $\overleftarrow{\partial}_\mu$ acts on the left. When we multiply the Dirac equation by $\bar{\psi}$ from the left

$$\bar{\psi}(\gamma^\mu \overrightarrow{\partial}_\mu + im)\psi = 0 \tag{4.25}$$

(where $\overrightarrow{\partial}_\mu$ acts on the right) and multiply the adjoint equation by ψ on the right

$$\bar{\psi}(\gamma^\mu \overleftarrow{\partial}_\mu - im)\psi = 0 \tag{4.26}$$

then add the two together we get

Conservation Law of Dirac Current

$$\bar{\psi}(\gamma^\mu \overrightarrow{\partial}_\mu + im)\psi + \bar{\psi}(\gamma^\mu \overleftarrow{\partial}_\mu - im)\psi = \partial(\bar{\psi}\gamma^\mu\psi) = \partial_\mu J^\mu 0 \tag{4.27}$$

(where J^μ is the Dirac Current) which is the law of conservation of the Dirac current in covariant form. We see the huge advantage this has over the Klein-Gordon equation: this has conserved probability current desnity as required by relativistic invariance...only now its temporal component is *positive definite*:

$$J^0 = \bar{\psi}\gamma^0\psi = \bar{\psi}\psi. \tag{4.28}$$

From this we can find a conserved charge

$$Q = q\int \psi^\dagger(\mathbf{x})\psi(\mathbf{x})d^3x \tag{4.29}$$

where q is to be thought of as "charge".

We can now see that the Dirac equation (and its adjoint) are the Euler-Lagrange equations of motion of the four dimensional invariant action

$$S = \int \mathcal{L} d^4x \tag{4.30}$$

where the Dirac Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = c\bar{\psi}(x)\Big[i\hbar\gamma^\mu\partial_\mu - mc\Big]\psi(x) \tag{4.31}$$

and for purposes of variation, ψ and $\bar{\psi}$ are considered to be independent fields. Relativistic invariance follows from the variational principle.

Canonical Structure, Hamiltonian and Momentum operators

We can find the canonically conjugate momenta to the fields ψ and $\bar{\psi}$:

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^\dagger \qquad \bar{\pi}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}} = 0. \tag{4.32}$$

We can find the Hamiltonian of the Dirac field

$$H = \int d^3x(\pi_\alpha(x)\dot{\psi}^\alpha(x) - \mathcal{L}) = \int d^3x\bar{\psi}(x)[-i\hbar c\gamma^j\partial_j + mc]\psi(x). \tag{4.33}$$

Similarly, the momentum of some field ϕ to be given by

$$cP^{\alpha} \equiv \int d^3x \mathcal{T}^{0\alpha} = \int d^3x \left[c\pi_r(x) \frac{\partial \phi_r(x)}{\partial x_\alpha} - \mathcal{L}\eta^{0\alpha} \right]$$

where $\mathcal{T}^{\alpha\beta}$ is the stress-energy density tensor of the field ϕ. Recall that we define the stress-energy density tensor by the equation

$$\mathcal{T}^{\alpha\beta} \equiv \frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \frac{\partial \phi_r}{\partial x_\beta} - \mathcal{L}\eta^{\alpha\beta}. \tag{4.34}$$

Using this, we can find the momentum of the Dirac Field to be

$$\mathbf{P} = -i\hbar \int d^3x \psi^\dagger(x) \nabla \psi(x). \tag{4.35}$$

Of course, the Hamiltonian given by (4.33) could have been discovered by finding the Hamiltonian density applied to the current case.

Angular Momentum

We can similarly find the angular momentum of the Dirac Field by simply following the scheme of finding the angular momentum for a general field. That is, an infinitesmal transformation of the coordinates

$$x_\alpha \to x'_\alpha \equiv x_\alpha + \delta x_\alpha = x_\alpha + \varepsilon_{\alpha\beta} x^\beta + \delta_\alpha \tag{4.36}$$

(where δ_α is an infinitesmal displacement and $\varepsilon_{\alpha\beta}$ is an infinitesmal antisymmetric tensor to ensure invariance of $x_\alpha x^\alpha$ under homogeneous Lorentz transformations, i.e. ones with $\delta_\alpha = 0$) induces an infinitesmal transformation of the field ϕ:

$$\phi_r(x) \to \phi'_r(x') = \phi_r(x) + \frac{1}{2}\varepsilon_{\alpha\beta} S^{\alpha\beta}_{rs} \phi_s(x). \tag{4.37}$$

Here the coefficients $S^{\alpha\beta}_{rs}$ are antisymmetric in α and β, like $\varepsilon_{\alpha\beta}$, and are determined by the transformation properties of the fields.

For a rotation (i.e. $\delta_\alpha = 0$) we have the continuity equation

$$\frac{\partial \mathcal{M}^{\alpha\beta\gamma}}{\partial x^\alpha} = 0 \tag{4.38}$$

where

$$\mathcal{M}^{\alpha\beta\gamma} \equiv \frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} S^{\beta\gamma}_{rs} \phi_s(x) + [x^\beta \mathcal{T}^{\alpha\gamma} - x^\gamma \mathcal{T}^{\alpha\beta}], \tag{4.39}$$

We interpret $M^{\alpha\beta}$ as angular momentum

(note that $\mathcal{M}^{\alpha\beta\gamma} = -\mathcal{M}^{\alpha\gamma\beta}$) and the six conserved quantities are

$$\begin{aligned} cM^{\alpha\beta} &= \int d^3x \mathcal{M}^{0\alpha\beta} \\ &= \int d^3x \Big([x^\beta \mathcal{T}^{0\alpha} - x^\beta \mathcal{T}^{0\alpha}] + c\pi_r(x) S^{\alpha\beta}_{rs} \phi_s(x) \Big). \end{aligned} \tag{4.40}$$

We have stated that $\mathcal{T}^{0i}/c$ is the momentum density of the field, so we interpret the square brackets of Eq (4.40) as the orbital momentum, and the last term as the intrinsic spin angular momentum.

We can apply similar technqiues to the Dirac field. The transformation of the Dirac field under an infinitesmal Lorentz transformation is given by

$$\psi_\alpha \to \psi'_\alpha(x') = \psi_\alpha(x) - \frac{i}{4}\epsilon_{\mu\nu} \sigma^{\mu\nu}_{\alpha\beta} \psi_\beta(x), \tag{4.41}$$

where summation over $\mu, \nu = 0, \ldots, 3$ and $\beta = 1, \ldots, 4$ is implied, and where $\sigma^{\mu\nu}_{\alpha\beta}$ is the (α, β) matrix element of the 4×4 matrix

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{4.42}$$

We can now "plug and chug" to find the angular momentum of the Dirac field

$$\mathbf{M} = \int d^3x \psi^\dagger(x)[\mathbf{x} \wedge (-i\hbar\nabla)]\psi(x) + \int d^3x \psi^\dagger \left(\frac{\hbar}{2}\sigma\right)\psi(x) \tag{4.43}$$

where

$$\sigma = (\sigma^{23}, \sigma^{31}, \sigma^{12}) \tag{4.44}$$

are 4×4 matrices generalizing Pauli matrices. We also observe that Eq (4.43) represent the orbital and spin angular momentum of particles of spin 1/2.

4.5 Solutions to the Dirac Equation

The easiest approach to find solutions to the Dirac equation is to insist that the solution is independent of spatial position:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0. \tag{4.45}$$

This really describes a particle with zero momentum, since the momentum operator is $i\hbar\partial_\mu$ and all the spatial eigenvalues vanish. The Dirac equation simplifies to

$$\frac{i\hbar}{c}\gamma^0\frac{\partial\psi}{\partial t} - mc\psi = 0 \tag{4.46}$$

or equivalently

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \partial\psi_A/\partial t \\ \partial\psi_B/\partial t \end{bmatrix} = -i\frac{mc^2}{\hbar}\begin{bmatrix} \psi_A \\ \psi_B \end{bmatrix} \tag{4.47}$$

where

$$\psi_A = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{4.48}$$

carries the upper two components and

$$\psi_B = \begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix} \tag{4.49}$$

carries the lower two components. Thus

$$\frac{\partial\psi_A}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_A, \quad -\frac{\partial\psi_B}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_B \tag{4.50}$$

and the solutions are

$$\psi_A(t) = \exp[-i(mc^2/\hbar)t]\psi_A(0), \quad \psi_B(t) = \exp[i(mc^2/\hbar)t]\psi_B(0). \tag{4.51}$$

We should know that in Quatum mechanics, the term

$$\exp(-iEt/\hbar) \tag{4.52}$$

is the characteristic for time dependence of a quantum state with energy E. It follows that at rest with $\mathbf{p} = 0$, the energy of the particle is $E = mc^2$. So ψ_A is what we expect.

What about ψ_B? It has negative energy! What the heck?! This is a famous disaster, and Dirac's response was like the Hindenberg of physics. He suggested something called the Hole theory, we will not discuss it here.

We interpret these "negative" energy particles as *antiparticles* with *positive* energy. So for us in our Dirac equation, ψ_B describes positrons (or antielectrons if one prefers to be outlandish) and ψ_A describes electrons. Each of them is a 2 component spinor (a 2 column vector). This is ideal as such a mathematical object describes a spin 1/2 particle. So, to sum up, we have 2 particles that are each 2 solutions for a grand total of 4 independent solutions with momentum $\mathbf{p} = 0$:

$$\psi^{(1)} = \exp(i(mc^2/\hbar)t) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \psi^{(2)} = \exp(i(mc^2/\hbar)t) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{4.53}$$

$$\psi^{(3)} = \exp(-i(mc^2/\hbar)t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \psi^{(4)} = \exp(-i(mc^2/\hbar)t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.54}$$

describing (respectively) an electron with spin up, an electron with spin down, a positron with spin up and an electron with spin down.

So to look at this from the perspective of solving differential equations, we have a solution to the homogeneous equation and we will use the method of variation of parameters to get solutions to the Dirac equation. What does this mean? Well, it means we are looking for "plane wave solutions" that look like

$$\psi(\mathbf{r}, t) = ae^{-i(Et-\mathbf{p}\cdot\mathbf{r})/\hbar}u(E, \mathbf{p}) \tag{4.55}$$

where a is a normalization constant (so probabilities add up to 1). We want to solve for $u(E, \mathbf{p}) = u(p)$ (we will use $p = (E/c, \mathbf{p})$ which is a 4 vector, and similarly $x = (ct, \mathbf{x})$, which is a mathematical object called a "bispinor". We don't want any old bispinor, we want one that will solve Dirac's equation! We have x dependence only in the exponent, so we find

$$\partial_\mu \psi = \frac{-i}{\hbar} p_\mu a e^{-(i/hbar)x^\mu p_\mu} u \tag{4.56}$$

By plugging this into Dirac's equation, we get

$$\gamma^\mu p_\mu a e^{-(i/\hbar)x\cdot p} u - mcae^{-(i/\hbar)x\cdot p} u = 0 \tag{4.57}$$

or if we want a neater and cleaner way to present it

$$(\gamma^\mu p_\mu - mc)u = 0. \tag{4.58}$$

This is the "momentum space Dirac equation" (which we get by taking the Fourier Transform of the Dirac equation we all know and love). Notice this is purely algebraic, no derivatives! That's the beauty of Fourier transforms in solving differential equations! If u satisfies (4.58) then ψ satisfies the Dirac equation.

Now to *prove* this (because an assertion is always meaningless without a rigorous proof – take note of this social "scientists") we need to use a lot of gamma matrix manipulations. Remember all representations are "equivalent" in the sense that they are related by unitary transformations. First we have

$$\gamma^\mu p_\mu = \gamma^0 p^0 - \gamma \cdot \mathbf{p} = \frac{E}{c} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \mathbf{p} \cdot \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix} = \begin{bmatrix} E/c & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -E/c \end{bmatrix} \tag{4.59}$$

SO it follows that

$$(\gamma^\mu p_\mu - mc)u \quad = \quad \begin{bmatrix} \left(\frac{E}{c} - mc\right) & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & \left(\frac{-E}{c} - mc\right) \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{E}{c} - mc\right) u_A & -\mathbf{p}\cdot\sigma u_B \\ \mathbf{p}\cdot\sigma u_A & \left(\frac{-E}{c} - mc\right) u_B \end{bmatrix}$$

where the subscript A is for the upper two components and the B stands for the lower two. In order to satisfy the momentum space Dirac equation, we must have

$$u_A = \frac{c}{E - mc^2}(\mathbf{p}\cdot\sigma)u_B, \quad u_B = \frac{c}{E + mc^2}(\mathbf{p}\cdot\sigma)u_A \tag{4.60}$$

We substitute the second into the first to give us

$$u_A = \frac{c^2}{E^2 - m^2c^4}(\mathbf{p}\cdot\sigma)^2 u_A \tag{4.61}$$

Observe

$$\begin{aligned} \mathbf{p}\cdot\sigma &= p_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + p_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{bmatrix} \end{aligned}$$

We find then by matrix multiplication (we will not calculate this out with every detail, but we will show the result):

$$(\mathbf{p}\cdot\sigma)^2 = \begin{bmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & p_z(p_x - ip_y) - p_z(p_x - ip_y) \\ p_z(p_x + ip_y) - p_z(p_x + ip_y) & (p_x + ip_y)(p_x - ip_y) + p_z^2 \end{bmatrix} = \mathbf{p}^2 I \tag{4.62}$$

where I is the 2 by 2 identity matrix. We see then that by plugging this into our equation for u_A

$$u_A = \frac{\mathbf{p}^2c^2}{E^2 - m^2c^4}u_A \tag{4.63}$$

which can be rearranged to be

$$\begin{aligned} (E^2 - m^2c^4)u_A &= \mathbf{p}^2c^2u_A \\ \Rightarrow (E^2 - \mathbf{p}^2c^2)u_A &= m^2c^4u_A \end{aligned}$$

and thus

$$E^2 - \mathbf{p}^2c^2 = m^2c^4 \tag{4.64}$$

which is the famous Einstein equation we all know and love. This tells us that in order to satisfy the Dirac equation, we have to obey the mass shell constraint. This admits two solutions for E:

$$E = \pm\sqrt{m^2c^4 + \mathbf{p}^2c^2} \tag{4.65}$$

where the positive root is associated with particle states, and the negative root with antiparticle states.

Using Eq (4.60), it is straightforward to calculate out the solutions to the Dirac equation (ignoring normalization constants):

$$\text{Pick } u_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{then } u_B = \frac{c}{E + mc^2}(\mathbf{p}\cdot\sigma)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{c}{E + mc^2}\begin{bmatrix} p_z \\ p_x + ip_y \end{bmatrix}$$

$$\text{Pick } u_A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{then } u_B = \frac{c}{E + mc^2}(\mathbf{p}\cdot\sigma)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{c}{E + mc^2}\begin{bmatrix} p_x - ip_y \\ -p_z \end{bmatrix}$$

$$\text{Pick } u_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{then } u_A = \frac{c}{E - mc^2}(\mathbf{p}\cdot\sigma)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{c}{E - mc^2}\begin{bmatrix} p_z \\ p_x + ip_y \end{bmatrix}$$

$$\text{Pick } u_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{then } u_A = \frac{c}{E - mc^2}(\mathbf{p} \cdot \sigma) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{c}{E - mc^2} \begin{bmatrix} p_x - ip_y \\ -p_z \end{bmatrix}$$

For the first two of these, we must use the positive energy otherwise we have division by zero, and if you divide by zero you go to hell. For the same reason, the energy in the latter two are negative. It is convenient to "normalize" these spinors in such a way that

$$u^\dagger u = 2|E|/c \tag{4.66}$$

where the dagger indicates the transpose conjugate ("Hermitian conjugate") is used:

$$u = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Rightarrow \quad u^\dagger = (a^*, b^*, c^*, d^*)$$

so that

$$u^\dagger u = |a|^2 + |b|^2 + |c|^2 + |d|^2. \tag{4.67}$$

So we find that the four solutions are:

$$u^{(1)} = N \begin{bmatrix} 1 \\ 0 \\ \frac{cp_z}{E + mc^2} \\ \frac{c(p_x + ip_y)}{E + mc^2} \end{bmatrix} \tag{4.68}$$

$$u^{(2)} = N \begin{bmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{E + mc^2} \\ \frac{-cp_z}{E + mc^2} \end{bmatrix} \tag{4.69}$$

with $E = +\sqrt{m^2c^4 + \mathbf{p}^2c^2}$

$$u^{(3)} = N \begin{bmatrix} \frac{cp_z}{E - mc^2} \\ \frac{c(p_x + ip_y)}{E - mc^2} \\ 1 \\ 0 \end{bmatrix} \tag{4.70}$$

$$u^{(4)} = N \begin{bmatrix} \frac{c(p_x - ip_y)}{E - mc^2} \\ \frac{c(-p_z)}{E - mc^2} \\ 0 \\ 1 \end{bmatrix} \tag{4.71}$$

with $E = -\sqrt{m^2c^4 + \mathbf{p}^2c^2}$, and the normalization constant is

$$N = \sqrt{(|E| + mc^2)/c}. \tag{4.72}$$

Now we are really tempted to say that $u^{(1)}$ is an electron with spin up, and $u^{(2)}$ is an electron with spin down, and so on, but this is not quite so. For Dirac Particles, the spin matrices are

$$S = \frac{\hbar}{2}\Sigma \quad \text{with } \Sigma \equiv \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \tag{4.73}$$

and it's easy to check that $u^{(1)}$ is *not* an eigenstate of Σ. However, if we orient the z axis so it points along the direction of motion (in which case $p_x = p_y = 0$) then $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, and $u^{(4)}$ are eigenspinors of S_z; $u^{(1)}$ and $u^{(3)}$ are spin up, and $u^{(2)}$ and $u^{(4)}$ are spin down[4]

Now we have to discuss the importance of E and $\mathbf{p}$, which are mathematical parameters which correspond physically to energy and momentum. At least, this is true for the electron states $u^{(1)}$ and $u^{(2)}$; but in $u^{(3)}$ and $u^{(4)}$ the $E < 0$...so it *cannot* represent positron energy. All free particles – electrons and positrons alike – carry *positive* energy. The "negative-energy" solutions must be reinterpreted as *positive* energy *antiparticle* states. To express these solutions in terms of the *physical* energy and momentum of the positron, we flip the signs of E and $\mathbf{p}$:

$$\psi(\mathbf{r}, t) = ae^{i/\hbar(Et-\mathbf{p}\cdot\mathbf{r})}u(-E, -\mathbf{p}) \tag{4.74}$$

for solutions (3) and (4) of course. These are the same solutions, we just have changed the signs of two parameters so it is physically appealing. It is customary to use v for positron states, expressed in terms of the physical energy and momentum:

$$v^{(1)}(E, \mathbf{p}) = u^{(4)}(-E, -\mathbf{p}) = N \begin{bmatrix} \frac{c(p_x - ip_y)}{E + mc^2} \\ \frac{c(-p_z)}{E + mc^2} \\ 0 \\ 1 \end{bmatrix} \tag{4.75}$$

$$v^{(2)}(E, \mathbf{p}) = u^{(4)}(-E, -\mathbf{p}) = N \begin{bmatrix} \frac{c(p_z)}{E + mc^2} \\ \frac{c(p_x + ip_y)}{E + mc^2} \\ 1 \\ 0 \end{bmatrix} \tag{4.76}$$

(with $E = \sqrt{m^2c^4 + \mathbf{p}^2c^2}$).

So we will no longer be working with $u^{(3)}$ and $u^{(4)}$; instead, the set of solutions we will be working with are $u^{(1)}$, $u^{(2)}$ (representing the two spin states of an electron with energy E and momentum $\mathbf{p}$), and $v^{(1)}$, $v^{(2)}$ (representing the two spin states of a positron with energy E and momentum $\mathbf{p}$). Notice that whereas the u's satisfy the momentum space Dirac equation in the form

$$(\gamma^\mu p_\mu - mc)u = 0 \tag{4.77}$$

the v's obey the equation with the sign of p_μ reversed:

$$(\gamma^\mu p_\mu + mc)v = 0. \tag{4.78}$$

Sure this is interesting, but it's only the special case of plane waves. Why bother? Well, they are of interest because they describe particles with specified energies and momenta, and in a typical experiment that's what we control and measure.

5 Some Notes on Spinor Technology

It was mentioned that the Dirac spinor does not transform as a four-vector when one changes from one inertial reference frame to another. So how exactly do they transform?

[4]It is actually mathematically impossible to construct spinors that satisfies the momentum Dirac equation and are simultaneously eigenspinors of S_z (except for the special case $\mathbf{p} = p_z\hat{z}$). The reason is that S by itself is *not a conserved quantity.* Only the *total* angular momentum $L + S$ is conserved. It is possible to construct eigenspinors of *helicity*, $\Sigma \cdot \hat{p}$ (there's no *orbital* angular momentum about the direction of motion), but these are cumbersome and in practice we like to work with the spinors we have constructed, even though it is difficult to have a physical intuition to what they mean. In the end, all that really matters is that we have a complete set of solutions.

Well, it's quite a bit of work to do, but we will simply quote the result. If we go to a system moving with speed v in the x direction, the transformation rule is

$$\psi \to \psi' = S\psi \tag{5.1}$$

where S is the 4×4 matrix

$$S = a_+ + a_-\gamma^0\gamma^1 = \begin{bmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{bmatrix} \tag{5.2}$$

with

$$a_\pm = \pm\sqrt{(\gamma \pm 1)/2} \tag{5.3}$$

and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor as usual.

Suppose we want to construct a scalar quantity out of a spinor ψ (we can do this with vectors, it's just the dot product). It would be reasonable to follow suite with the dot product and try the following:

$$\psi^\dagger\psi = \begin{bmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2. \tag{5.4}$$

Unfortunately this doesn't quite work as well as we would like. We can illustrate this by transforming coordinates:

$$(\psi^\dagger\psi)' = (\psi')^\dagger\psi' = \psi^\dagger S^\dagger S\psi \neq (\psi^\dagger\psi) \tag{5.5}$$

In fact

$$S^\dagger S = S^2 = \gamma \begin{bmatrix} 1 & -v\sigma_1/c \\ -v\sigma_1/c & 1 \end{bmatrix} \neq 1. \tag{5.6}$$

Of course we shouldn't expect this to be invariant, with 4-vectors we have (if we are particle physicists) the time component squared minus the sum of the space components squared. We see now that we can introduce a notion of *adjointness*, that is an adjoint spinor:

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 = \begin{bmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{bmatrix} \tag{5.7}$$

We can see that

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 \tag{5.8}$$

is a relativistic invariant. Why? Well, $S^\dagger\gamma^0 S = \gamma^0$ so we avoid the problems from our first attempt.

Please Take Note!

We will be covering everything relevant here, and when the time comes we will be performing in excrutiating detail every Feynman diagram of significance in QED.

6 Maxwell's Equations in a Nutshell

Recall in classical electromagnetism we have it summed in Maxwell's equations [Jac98]. In the presence of a charge density $\rho(\vec{x}, t)$ and a current density $\vec{j}(\vec{x}, t)$, the electric and magnetic fields $\vec{E}$ and $\vec{B}$ satisfy the equations

$$\nabla \cdot \vec{E} = \rho \tag{6.1a}$$

$$\nabla \times \vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{6.1b}$$

$$\nabla \cdot \vec{B} = 0 \tag{6.1c}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{6.1d}$$

where cgs units are used.

In the second pair of equations (Eqs 6.1c and 6.1d) follows the existence of scalar and vector potentials $\phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ defined by

$$\vec{B} = \nabla \times \vec{A}, \quad \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}. \tag{6.2}$$

However, this does not determine the system uniquely, since for an *arbitrary* function $f(\vec{x}, t)$ the transformation

$$\phi \to \phi' = \phi + \frac{1}{c}\frac{\partial f}{\partial t}, \quad \vec{A} \to \vec{A}' = \vec{A} - \nabla f \tag{6.3}$$

leaves the fields $\vec{E}$ and $\vec{B}$ unaltered. The transformation (6.3) is known as a gauge transformation of the second kind[5]. Since all observable quantities can be expressed in terms of $\vec{E}$ and $\vec{B}$, it is a fundamental requirement of any theory formulated in terms of potentials that is gauge; i.e. the predictions for the observable quantities are invariant under such gauge transformations.

When we express Maxwell's equations in terms of potentials, the second pair are automatically satisfied. The first pair (6.1a and 6.1b) become

$$-\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \Box\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A}\right) = \rho \tag{6.4a}$$

$$\Box\vec{A} + \nabla\left(\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A}\right) = \frac{1}{c}\vec{j} \tag{6.4b}$$

where

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{6.5}$$

is called the "D'Alembertian".

We can now consider the so-called "free field" case. That is, we have no charge or current so $\rho = 0$ and $\vec{j} = 0$. We can choose a gauge for the system such that

$$\nabla \cdot \vec{A} = 0. \tag{6.6}$$

The condition (6.6) defines the **Coulomb or radiation gauge**. A vector field with vanishing divergence (ie satisfying Eq (6.6)) is called a "transverse field" since for a wave

$$\vec{A}(\vec{x}, t) = \vec{A}_0 \exp(i(\vec{k} \cdot \vec{x} - \omega t)) \tag{6.7}$$

gives

$$\vec{k} \cdot \vec{A} = 0, \tag{6.8}$$

or in other words $\vec{A}$ is perpendicular to the direction of propagation $\vec{k}$ of the wave. In the Coulomb gauge, the vector potential is a transverse vector.

7 A Comically Brief Review

The following is a table summarizing the properties of the solutions of the Dirac equation.

[5] I.e. it is described mathematically in differential geometry as a connection form.

Property	Electrons	Positrons
Spinor components	$\psi(x) = au^{(s)}(p)\exp[-(i/\hbar)p\cdot x]$	$\psi(x) = av^{(s)}(p)\exp[-(i/\hbar)p\cdot x]$
Momentum Space Dirac Equation	$(\gamma^\mu p_\mu - mc)u = 0$	$(\gamma^\mu p_\mu + mc)v = 0$
Adjoint Dirac Equation	$\bar{u}(\gamma^\mu p_\mu - mc) = 0$	$\bar{v}(\gamma^\mu p_\mu + mc) = 0$
Orthogonality	$\bar{u}^{(1)}u^{(2)} = 0$	$\bar{v}^{(1)}v^{(2)} = 0$
Normalization	$\bar{u}u = 2mc$	$\bar{v}v = -2mc$
Complete	$\sum_s u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc)$	$\sum_s v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - mc)$

A free photon, on the other hand, of momentum $p = (E/c, \mathbf{p})$ with $E = |\mathbf{p}|c$ is represented by the wave function

$$A^\mu(x) = ae^{-(i/\hbar)p\cdot x}\epsilon^\mu_{(s)} \tag{7.1}$$

where ϵ^μ is a spin dependent vector, $s = 1, 2$ for the two polarizations ("spin states") of the photon. The polarization vectors $\epsilon^\mu_{(s)}$ satisfy the *momentum space Lorentz condition:*

$$\epsilon^\mu p_\mu = 0. \tag{7.2}$$

They are orthogonal in the sense that

$$\epsilon^*_{\mu(1)}\epsilon^\mu_{(2)} = 0. \tag{7.3}$$

They are further normalized

$$\epsilon^*_\mu \epsilon^\mu = 1. \tag{7.4}$$

In the Coulomb gauge

$$\epsilon^0 = 0, \quad \epsilon\cdot p = 0 \tag{7.5}$$

and the polarization three-vectors obey the completeness relation

$$\sum_{s=1,2}(\epsilon_{(s)})_i(\epsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i\hat{p}_j. \tag{7.6}$$

8 Quantum Electrodynamics

8.1 The Rules to the Game

So we want to calculate out the probability amplitude $\mathcal{M}$ associated with a particular Feynman diagram, we proceed as follows:

1. (Notation) We must be more careful here! We label the incoming and outgoing four-momenta p_1, p_2, …, p_n and the corresponding spins s_1, s_2, …, s_n. We label the inernal four-momenta q_1, q_2, …. Assign arrows to the lines as follows: the arrows on *external* fermion lines indicates whether it is an electron or positron (if the arrow points forward in time, it is an electron; backwards in time it is a positron); arrows on *internal* fermion lines are assigned so that the "direction of the flow" through the diagram is preserved (i.e. every vertex must have at least one arrow entering and one arrow leaving). The arrows on photon lines (which is optional, since arrows are used to indicate whether the particle is an antiparticle or not; bosons are their own antipartners) point "forward" in time.

2. (External Lines) External lines contribute factors as follows:

$$\text{Electrons: } \begin{cases}\text{Incoming: } u\\ \text{Outgoing: } \bar{u}\end{cases}$$

$$\text{Positrons: } \begin{cases}\text{Incoming: } \bar{v}\\ \text{Outgoing: } v\end{cases}$$

$$\text{Photons: } \begin{cases} \text{Incoming: } \epsilon^\mu \\ \text{Outgoing: } (\epsilon^\mu)^* \end{cases}$$

3. (Vertex Factors) Each vertex contributes a factor

$$ig_e\gamma^\mu \tag{8.1}$$

The dimensionless coupling constant g_e is related to the charge of the positron $g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha_E}$[6]

4. (Propagators) Each internal line contributes a factor as follows

$$\text{Electrons and Positrons: } \frac{i\gamma^\mu q_\mu + mc}{q^2 - m^2c^2} \tag{8.2}$$

$$\text{Photons: } \frac{-ig_{\mu\nu}}{q^2} \tag{8.3}$$

5. (Conservation of Energy and Momentum) For each vertex, write a delta function of the form

$$(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3) \tag{8.4}$$

This enforces the conservation of momentum at the vertex.

6. (Integrate Over Internal Momenta) For each internal momentum q, write a factor

$$\frac{d^4q}{(2\pi)^4} \tag{8.5}$$

and integrate.

7. (Cancel the Delta Function) The result will include a factor

$$(2\pi)^4\delta^{(4)}(p_1 + p_2 + \cdots - p_n) \tag{8.6}$$

which corresponds to the overall energy-momentum conservation. Cancel this factor, and we get $-i\mathcal{M}$.

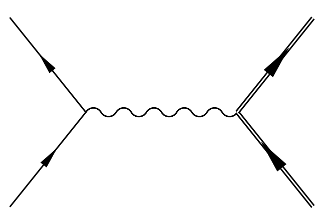
8. (Antisymmetrization) Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) electrons (or positrons), or of an incoming electron with an outgoing positron (or vice versa).

9 Elastic Processes

An elastic (relativistic) process is one where kinetic energy, rest energy, and mass are all conserved. We will explore such examples in QED.

9.1 Electron-Muon Scattering

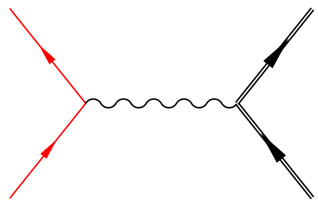
We draw the diagram (note the use of μ and ν at the vertices, which are used to sum over in the integral):



[6]Here α_E is the coupling constant of the electromagnetic force. In *general*, the QED coupling is $-q\sqrt{4\pi/\hbar c}$ where q is the charge of the *particle* (as opposed to antiparticle). For electrons $q = -e$, for an up quark $q = (2/3)e$.

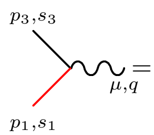
We will now evaluate it in a haphazard manner. Observe how it is done when spinors are in the game.

Step One: We will evaluate the part emboldened in Red first.

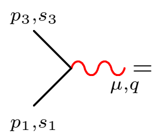


We will now analyze it in careful detail so we will "pull it out" and "dissect" it carefully.

We evaluate it in the following manner: since we write quantum mechanics like we write chinese (from right to left), we begin with

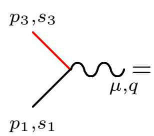


$$= u(s_1, p_1),$$



$$= (ig_e\gamma^\mu)u(s_1, p_1) \tag{9.1}$$

We have one last step to do



$$= \bar{u}(s_3, p_3)(ig_e\gamma^\mu)u(s_1, p_1) \tag{9.2}$$

So this contributes

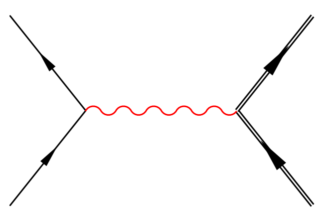
$$(\bar{u}(s_3, p_3))(ig_e\gamma^\mu)(u(s_1, p_1)) \tag{9.3}$$

to the integrand. Our integrand is going to take the form

$$[(\bar{u}(s_3, p_3))(ig_e\gamma^\mu)(u(s_1, p_1))] \begin{pmatrix}\text{photon}\\ \text{propagator}\end{pmatrix} \begin{pmatrix}\text{muon}\\ \text{terms}\end{pmatrix} \begin{pmatrix}\text{conservation of}\\ \text{momentum delta}\\ \text{functions}\end{pmatrix} \tag{9.4}$$

We will now move on to step two.

Step Two: We will consider the photon propagator, which corresponds to the red line in the following diagram



This corresponds to the term

$$\frac{-ig_{\mu\nu}}{q^2} \tag{9.5}$$

giving our integrand to be

$$[(\bar{u}(s_3, p_3))(ig_e\gamma^\mu)(u(s_1, p_1))] \frac{-ig_{\mu\nu}}{q^2} \begin{pmatrix}\text{muon}\\ \text{terms}\end{pmatrix} \begin{pmatrix}\text{conservation of}\\ \text{momentum delta}\\ \text{functions}\end{pmatrix}. \tag{9.6}$$

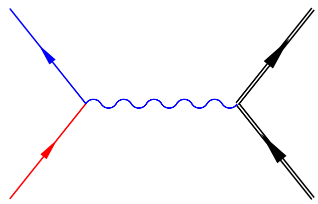
Step Three and Four: Moving right along to the Muon terms, we have exactly a term analagous to Eq (9.3). Muons are fermions with spin 1/2, with the same electric charge as an electron. So translating this into Feynman diagram terms, it is translated in the exact same fashion we translated the electron terms. So we have a contribution of

$$(\bar{u}(s_4,p_4))(ig_e\gamma^\nu)(u(s_2,p_2)). \tag{9.7}$$

Our integrand now becomes

$$[(\bar{u}(s_3,p_3))(ig_e\gamma^\mu)(u(s_1,p_1))]\frac{-ig_{\mu\nu}}{q^2}[(\bar{u}(s_4,p_4))(ig_e\gamma^\nu)(u(s_2,p_2))]\begin{pmatrix}\text{conservation of}\\ \text{momentum delta}\\ \text{functions}\end{pmatrix}. \tag{9.8}$$

Step Five: We kind of "fudged up" steps 1 through 4 because they are so interconnected it is hard to seperate them out from each other. We are now safely onto step 5 of the Feynman rules of QED: conservation of momentum! We have two places to do this (at the μ and ν vertices). We have for μ (chosen randomly) the input momentum in red and output momentum in blue:



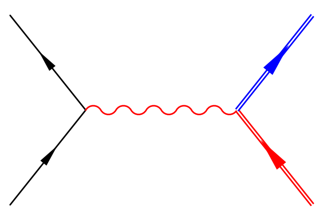
This corresponds to the conservation of momentum

$$p_1 = p_3 + q \quad \Rightarrow \quad p_1 - p_3 - q = 0 \tag{9.9}$$

which gives us the delta function

$$(2\pi)^4\delta^{(4)}(p_1 - p_3 - q). \tag{9.10}$$

We have another conservation of momentum point, which is at the vertex ν:



Which corresponds to a conservation of momentum

$$p_2 + q = p_4 \quad \Rightarrow \quad p_2 + q - p_4 = 0 \tag{9.11}$$

and thus contributes the delta function term

$$(2\pi)^4\delta^{(4)}(p_2 + q - p_4) \tag{9.12}$$

rendering our integrand to be

$$[(\bar{u}(s_3,p_3))(ig_e\gamma^\mu)(u(s_1,p_1))]\frac{-ig_{\mu\nu}}{q^2}[(\bar{u}(s_4,p_4))(ig_e\gamma^\nu)(u(s_2,p_2))](2\pi)^8$$

$$\times\delta^{(4)}(p_1 - p_3 - q)\delta^{(4)}(p_2 + q - p_4)d^4q.$$

Step Six: We integrate over the internal momenta (in our case the photon's momentum), so we have the integral expression:

$$\begin{aligned} i\mathcal{M} \quad \text{``=''} \quad & (2\pi)^4 \int [(\bar{u}(s_3,p_3))(ig_e\gamma^\mu)(u(s_1,p_1))]\frac{-ig_{\mu\nu}}{q^2}[(\bar{u}(s_4,p_4))(ig_e\gamma^\nu)(u(s_2,p_2))] \\ & \times\delta^{(4)}(p_1 - p_3 - q)\delta^{(4)}(p_2 + q - p_4)d^4q. \end{aligned}$$

Observe this is harder than it looks because we are taking the trace of gamma matrices! That is the whole point of having the metric tensor $g^{\mu\nu}$ here. So it is a bit tricky to compute...

We will integrate over q and take advantage of the delta function term (9.10) to make the switch

$$q \to p_1 - p_3$$

giving us the result from the integral

$$(2\pi)^4\frac{ig_e^2}{(p_1-p_3)^2}[(\bar{u}(s_3,p_3))(ig_e\gamma^\mu)(u(s_1,p_1))][(\bar{u}(s_4,p_4))(ig_e\gamma_\mu)(u(s_2,p_2))]\delta^{(4)}(p_2+p_1-p_3-p_4). \tag{9.13}$$

Step Seven: We simply set Eq (9.13) to be equal to $-i\mathcal{M}\delta^{(4)}(p_2 + p_1 - p_3 - p_4)$, and we solve to find

$$\mathcal{M} = \frac{-g_e^2}{(p_1-p_3)^2}[(\bar{u}(s_3,p_3))(ig_e\gamma^\mu)(u(s_1,p_1))][(\bar{u}(s_4,p_4))(ig_e\gamma_\mu)(u(s_2,p_2))] \tag{9.14}$$

is the probability amplitude. In spite of this nightmarish appearence, with four spinors and eight γ matrices, this is still just a number. We can figure it out when the spins are specified.

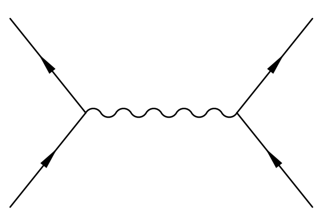
9.2 Moller Scattering

Moller scattering is the scattering of electrons

$$e^- + e^- \to e^- + e^-. \tag{9.15}$$

We have two diagrams to consider this time! In fact, from here on out, we will always have two diagrams to consider (the exception being one third order example, which is the most important third order example because it explains the anamolous magnetic moment of an electron – we'll burn that bridge when we get to it).

Step One: The first diagram to consider is the following:



This is precisely the electron-muon diagram with the exception that the muon has been replaced by an electron. Thus we will simply use the **exact same steps** we did in the first example; we will copy/paste the results here.

The integrand should take the form

$$[(\bar{u}(s_3,p_3))(ig_e\gamma^\mu)(u(s_1,p_1))]\frac{-ig_{\mu\nu}}{q^2}[(\bar{u}(s_4,p_4))(ig_e\gamma^\nu)(u(s_2,p_2))](2\pi)^8$$

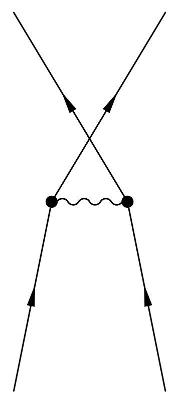
$$\times \delta^{(4)}(p_1 - p_3 - q)\delta^{(4)}(p_2 + q - p_4) d^4 q.$$

This has the contribution to the total probability amplitude that this process will happen of

$$\mathcal{M}_1 = \frac{-g_e^2}{(p_1 - p_3)^2}[(\bar{u}(s_3, p_3))(ig_e\gamma^\mu)(u(s_1, p_1))][(\bar{u}(s_4, p_4))(ig_e\gamma_\mu)(u(s_2, p_2))] \tag{9.16}$$

We will add it to the probability amplitude from the other graph to get the total probability amplitude of the process happening.

The second diagram is odd:



We make the switch of $(s_3, p_3) \iff (s_4, p_4)$ for this diagram, and low and behold we have a rule that takes care of this!

Step Eight: (Yes we are hopping right along!) We have by the eighth rule a change in signs. So the probability amplitude from this second diagram is (when we make the switches of $p_3 \mapsto p_4$, $p_4 \mapsto p_3$, $s_3 \mapsto s_4$, $s_4 \mapsto s_3$)

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1 - p_4)^2}[(\bar{u}(s_4, p_4))(ig_e\gamma^\mu)(u(s_1, p_1))][(\bar{u}(s_3, p_3))(ig_e\gamma_\mu)(u(s_2, p_2))] \tag{9.17}$$

So the total probability amplitude is then

$$\begin{aligned} \mathcal{M} &= \frac{-g_e^2}{(p_1 - p_3)^2}[(\bar{u}(s_3, p_3))(ig_e\gamma^\mu)(u(s_1, p_1))][(\bar{u}(s_4, p_4))(ig_e\gamma_\mu)(u(s_2, p_2))] \\ &\quad + \frac{g_e^2}{(p_1 - p_4)^2}[(\bar{u}(s_4, p_4))(ig_e\gamma^\mu)(u(s_1, p_1))][(\bar{u}(s_3, p_3))(ig_e\gamma_\mu)(u(s_2, p_2))]. \end{aligned}$$

Concluding Remarks

We have just covered quite a bit in excrutiating detail, but you should have some idea of how to compute Feynman diagrams now. This is actually more than enough to have you begin reading books like Peskin and Schroeder [PS95], and that was the secret hope and aim of this paper.

Perhaps in the future, we will include a section on being able to read the Feynman rules and begin computing directly from that; or perhaps a discussion of non-Abelian gauge theories in the Feynman diagrams. Or to derive Feynman rules from the Lagrangian alone! Who knows what the future will hold...

A Gamma Matrices

For an extensive reference, see [BRS95]. The defining property for the gamma matrices is that they form a Clifford algebra with the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I \tag{A.1}$$

where $\eta^{\mu\nu}$ is the Minkowski metric with signature (+—) and I is the unit (identity) matrix.

We can also define covariant gamma matrices by

$$\gamma_\mu = \eta_{\mu\nu}\gamma^\nu = \left(\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\right) \tag{A.2}$$

where Einstein summation is used.

Remark A.1. We may define a fifth element of our Clifford algebra,

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.3}$$

or equivalently

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \tag{A.4}$$

which is true due to the anticommutation relations (A.1). It has the following properties:

1. (Hermitian) $(\gamma^5)^\dagger = \gamma^5$
2. (Eigenvalues are ± 1) $\left\{\gamma^5, \gamma^\mu\right\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$
3. (Anticommutes with other 4 generators) $\left\{\gamma^5, \gamma^\mu\right\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$

Remark A.2. We can project a Dirac field onto its left-handed and right-handed components by

$$\psi_L = \frac{1-\gamma^5}{2}\psi, \qquad \psi_R = \frac{1+\gamma^5}{2}\psi \tag{A.5}$$

which is often useful when dealing with chirality in a quantum mechanical setting.

We should think of the tuple $\gamma^\mu = \left(\gamma^0, \gamma^1, \gamma^2, -\gamma^3\right) = \gamma^0 e^0 + \gamma^1 e^1 + \gamma^2 e^2 + \gamma^3 e^3$ sort of as a 4-vector (where e^μ is the basis vectors). But this is misleading! We should view the γ^μ as a mapping operator that "eats up" a 4-vector a^μ and "spits out" the corresponding vector in the Clifford representation.

Such a result would be represented by the **Feynman Slash**

$$\not{a} := \gamma^\mu a_\mu. \tag{A.6}$$

It should be noted that this beast, $\not{a}$, "lives" in the Clifford space so any changes to the basis vectors are irrelevant.

A quick review of some of the properties of the Dirac Gamma matrices!

Property 1. (Normalisation) Due to the anticommutation relations (A.1), we can show

$$\left(\gamma^0\right)^\dagger = \gamma^0 \qquad \text{and} \ \left(\gamma^0\right)^2 = I \tag{A.7}$$

and for the other gamma matrices (for $k = 1, 2, 3$) we have

$$\left(\gamma^k\right)^\dagger = -\gamma^k \qquad \text{and} \ \left(\gamma^k\right)^2 = -I \tag{A.8}$$

which results in a generalized relationship which encapsulates all this information:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{A.9}$$

Remark A.3. These relationships described below, and the property described above, are in the (+—) signature; if we used the (-+++) signature, things would be different.

We also have a list of identities the Gamma matrices obey:

1. $\gamma^\mu\gamma_\mu = 4I$,
2. $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$,

3. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}I$,

4. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$.

Similarly, there are 5 trace identities the Gamma matrices obey

1. The trace of the product of an odd number of γ is 0,
2. $\text{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$,
3. $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$,
4. $\text{tr}(\gamma^5) = \text{tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$,
5. $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$.

A.1 Representations of the Gamma Matrices

We can represent the gamma matrices in various different ways that satisfy the anti-commutation relations and all the above identities and properties. First recall the Pauli matrices, as they will prove useful in our discussion:

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{A.10}$$

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{A.11}$$

$$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.12}$$

We will let the 2 by 2 identity be denoted by I_2 in this section.

One representation is the **Dirac Basis**

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{A.13}$$

Another common one used is the **Weyl (chiral) basis** which basically changes the "temporal" gamma matrix while leaving the others the same. This causes the γ^5 quantity to change too. We can succinctly describe it as:

$$\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}. \tag{A.14}$$

This has the advantage that the chiral projections are merely

$$\psi_L = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\psi, \quad \psi_R = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\psi. \tag{A.15}$$

By slightly abusing notation, we can identify

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}, \tag{A.16}$$

where ψ_L and ψ_R are left-handed and right-handed two-component Weyl spinors.

The third, and for our investigations last, basis is the Majorana basis, in which all the Dirac matrices are imaginary. We can write them as

$$\gamma^0 = \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{bmatrix}$$

$$\gamma^2 = \begin{bmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{bmatrix}. \tag{A.17}$$

A.2 Euclidean Representation

Oftentimes in path integral approaches, we can Wick Rotate from Minkowski to Euclidean spacetime by making time imaginary[7]. We then are forced to work with Euclidean gamma matrices. There are two major representations in the Euclidean framework for them.

The first is the chiral representation, defined by

$$\gamma^{1,2,3} = \begin{bmatrix} 0 & -i\sigma^{1,2,3} \\ i\sigma^{1,2,3} & 0 \end{bmatrix}, \quad \gamma^4 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{A.18}$$

This is different from the Minkowski set by the relation

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \gamma^{5+}. \tag{A.19}$$

So in a Chiral basis we have

$$\gamma^5 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{A.20}$$

The other form is the nonrelativistic form, which is succinctly described by

$$\gamma^{1,2,3} = \begin{bmatrix} 0 & -i\sigma^{1,2,3} \\ i\sigma^{1,2,3} & 0 \end{bmatrix}, \quad \gamma^4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}. \tag{A.21}$$

B Decay Rates and Feynman Diagrams

Remember if we have some collection of particles (e.g. Muons) and they decay, the decay rate Γ (the probability per unit time that any given muon will disintegrate) satisfies a particular relation. If $N(t)$ is the number of particles at time t, the infinitesmal change in N from t to $t + dt$ is

$$dN = -\Gamma N(t) dt \tag{B.1}$$

which tells us the number is decreasing when we move forward in time. It follows that

$$\begin{aligned} \frac{1}{N} dN &= -\Gamma dt \\ \int \frac{1}{N} dN &= -\int \Gamma dt \\ \ln(N(t)) &= -\Gamma t + C \\ N(t) &= \exp(-\Gamma t)\exp(C) \\ &= N(0)\exp(-\Gamma t) \end{aligned}$$

where $N(0)$ is the initial number of particles, and C is the constant of integration.

The **mean lifetime** of the particle is simply the reciprocal of the decay rate

$$\tau = \frac{1}{\Gamma}. \tag{B.2}$$

If there are several different ways for the particle decay, each with different decay rates, the total decay rate is given by the sum of the individual rates:

$$\Gamma_{\text{tot}} = \sum_{j=1}^{n} \Gamma_j \tag{B.3}$$

and the mean lifetime is the reciprocal of this quantity

$$\tau = \frac{1}{\Gamma_{\text{tot}}}. \tag{B.4}$$

[7]Not as in "eleventeen is an imaginary number" but as in $\sqrt{-5}$ is an imaginary number.

B.1 Fermi's Golden Rule

Suppose we have one particle that decays into several others

$$1 \to 2 + 3 + \cdots + n \tag{B.5}$$

If $\mathcal{M}$ is the total probability amplitude from the various Feynman diagram representations of this process, then the infinitesmal decay rate is given by

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar m_1} \left[\left(\frac{c d^3 p_2}{(2\pi)^3 2E_2} \right) \left(\frac{c d^3 p_3}{(2\pi)^3 2E_3} \right) \cdots \left(\frac{c d^3 p_n}{(2\pi)^3 2E_n} \right) \right] \times (2\pi)^4 \delta^{(4)}(p_1 - (p_2 + p_3 + \cdots + p_n)) \tag{B.6}$$

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