COMMENTARY ON BOURBAKI'S "DESCRIPTION OF FORMAL MATHEMATICS"

ALEX NELSON

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0. Preface

0.1. **Urtext.** This is my commentary on the first chapter of Bourbaki's *Theory of Sets* [5], specifically the first chapter. I am mostly relying on the English translation — apparently the 1968 English translation of the French 1970 edition. Apparently the English translators had access to a time-machine. (I suspect the publisher erred in the dates, but it's the softcover English translation available right now by Springer.)

There are only one or two times I return to the French edition being translated. The English translation appears to be faithful to the point of being a literal translation. However, there is a critical passage which omits an important phrase, and fails to capture the idiomatic meaning of the passage in the original.

0.2. **Organization of these notes.** The "sections" and "subsections" are organized after Bourbaki's sections and subsections. I make a number of observations, which are organized into "subsubsections". So note "1.3.4" refers to section 1, subsection 3, and it's my fourth note on that section.

0.2.1. Puzzles. There are a number of puzzles which are numbered sequentially. These are questions for myself, which are really points of discussion for the reader as well.

0.3. **Pronouns.** Bourbaki is both a group and an individual. I'm not sure what pronoun to use to refer to Bourbaki, therefore I will switch between "he/him" and "they/them" according to what sounds good to my tone-deaf ear.

0.4. **Exegesis/Eisegesis.** I'm not trying to "read in what I want", but there are times when it is too ambiguous for clearly making sense of what Bourbaki "really meant". Therefore, I cannot claim to be clarifying Bourbaki's underlying intent.

But I knowingly deviate from Bourbaki's notation in the following ways: Bourbaki uses Polish notation ("lispy syntax") and then immediately discards it, I just avoid it; Bourbaki uses a 2-dimensional syntax for linkages, I just intuitively think of using de Bruijn indices.

0.4.1. Anachronisms. Bourbaki didn't actually adopt Hilbert's ε-calculus until July 1950, when Chevellay wrote the fifth draft of the chapter on formal mathematics [6]. The previous four drafts appear to be written *before* World War 2, so there's a span of more than a decade separating these drafts. I mention this because a lot of the concepts which Bourbaki introduces are rather clunky, and predate the conventional ideas of a formal grammar (introduced by Chomsky in 1956 [7]) and compiler technology (introduced around *c.*1954, but not popularized until well into the 1960s).

0.4.2. Methodology. Raymond Chandler once remarked something along the lines of, "People who write about writing know nothing about how to write well." I fear the same could be said about writing commentaries. Roughly, I am guided by a desire to write a proof assistant. Consequently, I am interested in forming a sort of "rational reconstruction" of Bourbaki's formal system.

Loosely, what I mean by a "rational reconstruction" is that I am trying to implement these ideas in a computer program, so how can I faithfully implement this? A lot of the ideas "sound" very close to ideas that emerged later (formal grammars, production rules, judgements, inference rules, inductive definitions, etc.). For me, a "rational reconstruction" is a *translation* of the original text into a different formal language which tries to preserve the critical aspects of the original text. Because "translation is interpretation", this forces us to interpret the original text and weigh what is important.

There is one or two instances where I have a "close reading" of certain key passages of Bourbaki, consulting the French original. I'd like to thank Stanley Oropesa for his help with certain French phrases.

0.5. **Other commentaries.** Wayne Aitken [2] has written a good commentary about the first chapter of Bourbaki [5]. Aitken has similar goals as me, but wishes to clarify Bourbaki's rushed chapter (which really is rather austere as a text).

Adrian Mathias has written a number of articles [9, 10, 11] about Bourbaki's formal system and its flaws. Mathias's articles are classics and worth reading in their own right.

1. Terms and Relations

1.1. Signs and Assemblies.

1.1.1. Definition. A **"sign"** refers to a member of an alphabet, specifically

1. logical signs: $\square$ [corresponding to bound variables], τ [the Hilbert choice operator], $\vee$, $\neg$;
2. "letters" (variables and parameters);
3. theory-specific signs, which correspond to primitive notions.

Of course, we call it an "alphabet", Bourbaki didn't have such a notion available to them. We think of "signs" as letter in an ambient alphabet.

Puzzle 1. *Although redundant, it is preferable to include a quantifier as a logical sign. This would reduce the size of their definition of the number* 1 *from* $\mathcal{O}(10^9)$ *signs to a couple dozen signs. But which quantifier should we introduce? Logicians appear to prefer* $\forall$*, but as we will see* $\exists$ *seems more natural for Bourbaki's system.*

1.1.2. Definition. An **"Assembly"** is a "string" over the ambient alphabet. Bourbaki avoids using bound variables by using "linkages", which are lines connecting $\square$ to τ. This is a horrible kludge. Nowadays, we would use de Bruijn indices.[1]

We will abuse language and refer to an assembly as an "expression".

1.1.3. Provisional definition. Bourbaki provisionally defines a **"Mathematical Theory"** (or just "theory") as consisting of

1. rules which tell us if an assembly is a term or relation of the theory, and
2. rules which assert certain assemblies are *theorems* of the theory.

This is fully defined in (§2.1).

1.1.4. Metavariables. To simplify the discussion, Bourbaki uses the convention that bold italicized variables (e.g., $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, …) are metavariables representing arbitrary assemblies. For reasons beyond me, Bourbaki does not call them "metavariables" [even though this term should be available to them at the time]. At any rate, we will follow their conventions writing bold italicized symbols for metavariables.

[1] Of course, de Bruijn introduced these indices in 1972, more than 20 years after Bourbaki drafted this chapter.

1.1.5. Example. We can concatenate assemblies, just as we can concatenate strings. If we let $\boldsymbol{A}$ and $\boldsymbol{B}$ be assemblies, then we can form a new assembly $\boldsymbol{AB}$ which is formed by just writing down the signs appearing in $\boldsymbol{A}$ followed by the signs appearing in $\boldsymbol{B}$.

1.1.6. Grammar for Hilbert operator. Bourbaki gives the rules for forming an assembly using the Hilbert choice operator τ. They denote by $\tau_{\boldsymbol{x}}(\boldsymbol{A})$ the assembly formed by the following rules:

(1) form $\tau\boldsymbol{A}$; then
(2) link all occurrences of $\boldsymbol{x}$ in $\tau\boldsymbol{A}$ to the leading τ prefix; then
(3) replace all instances of $\boldsymbol{x}$ by $\square$.

Observe that $\boldsymbol{x}$ no longer appears in the resulting assembly $\tau_{\boldsymbol{x}}(\boldsymbol{A})$. (If we use de Bruijn indices for bound variables, then we preserve this result: $\boldsymbol{x}$ no longer appears in $\tau_{\boldsymbol{x}}(\boldsymbol{A})$.)

1.1.7. Ambiguities. Bourbaki never defines what an "occurrence" of a variable in an assembly means, nor what it means for an assembly to "appear" in another assembly. Presumably, we can safely assume they mean the variable (considered as a string) is a substring of the assembly, and the assembly $\boldsymbol{A}$ appears in $\boldsymbol{B}$ if $\boldsymbol{A}$ is a substring of $\boldsymbol{B}$.

Furthermore, a second ambiguity, Bourbaki does not define what it means for two assemblies to be identical with each other. Presumably this is syntactic equality.

Puzzle 2. *Should we use de Bruijn indices or de Bruijn levels for bound variables in $\tau_{\boldsymbol{x}}(\boldsymbol{A})$?*

1.1.8. Substitution. Bourbaki introduces the metalinguistic notation $(\boldsymbol{B} \mid \boldsymbol{x})\boldsymbol{A}$ for replacing all instances of $\boldsymbol{x}$ by $\boldsymbol{B}$ in $\boldsymbol{A}$.

1.1.9. Notation: Parametrized assemblies. Bourbaki adopts the notation $\boldsymbol{A}[\boldsymbol{x}]$ for the assembly $\boldsymbol{A}$ explicitly parametrized by the variable $\boldsymbol{x}$. Substitution is denoted $\boldsymbol{A}[\boldsymbol{B}]$ which is a synonym for $(\boldsymbol{B} \mid \boldsymbol{x})\boldsymbol{A}$. This can be generalized to multiple variables parametrizing an assembly, e.g., $\boldsymbol{A}[\boldsymbol{x}, \boldsymbol{y}]$ and parallel substitution occurs when writing $\boldsymbol{A}[\boldsymbol{B}, \boldsymbol{C}]$. If $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are fresh variables (they do not occur in $\boldsymbol{A}$ or $\boldsymbol{B}$ or $\boldsymbol{C}$), then $\boldsymbol{A}[\boldsymbol{B}, \boldsymbol{C}]$ is identical to $(\boldsymbol{C} \mid \boldsymbol{y}')(\boldsymbol{B} \mid \boldsymbol{x}')(\boldsymbol{y}' \mid \boldsymbol{y})(\boldsymbol{x}' \mid \boldsymbol{x})\boldsymbol{A}$.

1.1.10. Definitions, Abbreviating Symbols. The notion of an abbreviating symbol is rather vague. Presumably Bourbaki is borrowing what was common knowledge at the time, I guess tracing back to Russell and Whitehead's *Principia Mathematica*. As I understand it, abbreviating symbols amount to something like macros in the C programming language. Bourbaki gives the example "$\boldsymbol{A} \implies \boldsymbol{B}$" as an abbreviating symbol for "$(\neg\boldsymbol{A}) \vee \boldsymbol{B}$".

Abbreviating symbols belong to the metalanguage, not the object language.

1.2. Criteria of Substitution.

1.2.1. On "Criteria". Since abbreviating symbols lead to Brobdingnagian expressions, this would force us to endure long chains of reasoning. This is clearly unmanageable. Bourbaki states, in my amended translation (my insertions in blue, deletions in red),

> For this reason we shall establish in this book[2] *criteria* relating to metavariables [*assemblages indeterminé*] ~~indeterminate assemblies~~; each of these criteria will describe ~~once for all~~ definitively [*une fois pour tout*] the final result of a ~~definite~~ determined [*déterminée*] sequence of manipulations on these metavariables [*assemblages*] ~~assemblies~~.

The modern way we would do this would be to use judgements for term rewriting and grammatical well-formedness, and inductively define them using inference rules.

Bourbaki explains, "These criteria are therefore not ~~indispensable to the theory~~ theoretically essential; their justification belongs to *metamathematics*." I personally have great difficulty not to read this blue text as saying the criteria is essential to the object language. This reinforces the suspicion that these are prototypes of inference rules.

1.2.2. Families of criteria. There are several families of criteria in Bourbaki's formal system. Again, this is because we have several operations in the metalanguage (anachronistically we'd call them "judgements") and each correspond to a family of criteria. The first family of criteria Bourbaki introduces concerns the "*Criteria of Substitution*".

These are enumerated, prefixed by "CS".

1.2.3. Criteria of Substitution. Essentially, the "criteria of substitution" are rules governing how substitution behaves.

Puzzle 3. *Could we use "explicit substitutions" [1] in Bourbaki's formal system? What are the costs and benefits?*

CS1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be assemblies, let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be variables. If $\boldsymbol{x}'$ does not appear in $\boldsymbol{A}$, then $(\boldsymbol{B} \mid \boldsymbol{x})\boldsymbol{A}$ is identical to $(\boldsymbol{B} \mid \boldsymbol{x}')(\boldsymbol{x}' \mid \boldsymbol{x})\boldsymbol{A}$.

CS2. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be assemblies. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be distinct variables. If $\boldsymbol{y}$ does not appear in $\boldsymbol{B}$, then $(\boldsymbol{B} \mid \boldsymbol{x})(\boldsymbol{C} \mid \boldsymbol{y})\boldsymbol{A}$ is identical with $(\boldsymbol{C}' \mid \boldsymbol{y})(\boldsymbol{B} \mid \boldsymbol{x})\boldsymbol{A}$ where $\boldsymbol{C}'$ is identical with $(\boldsymbol{B} \mid \boldsymbol{x})\boldsymbol{C}$. (This is basically Barendregt's substitution lemma.)

CS3. Let $\boldsymbol{A}$ be an assembly. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be variables. If $\boldsymbol{x}'$ does not appear in $\boldsymbol{A}$, then $\tau_{\boldsymbol{x}}(\boldsymbol{A})$ is identical with $\tau_{\boldsymbol{x}'}(\boldsymbol{A}')$ where $\boldsymbol{A}'$ is $(\boldsymbol{x}' \mid \boldsymbol{x})\boldsymbol{A}$.

CS4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be assemblies, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be distinct variables. If $\boldsymbol{x}$ does not appear in $\boldsymbol{B}$, then $(\boldsymbol{B} \mid \boldsymbol{y})\tau_{\boldsymbol{x}}(\boldsymbol{A})$ is identical with $\tau_{\boldsymbol{x}}(\boldsymbol{A}')$ where $\boldsymbol{A}'$ is $(\boldsymbol{B} \mid \boldsymbol{y})\boldsymbol{A}$.

CS5. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be assemblies. Let $\boldsymbol{x}$ be a letter. Let $\boldsymbol{A}'$ be identical with $(\boldsymbol{C} \mid \boldsymbol{x})\boldsymbol{A}$ and let $\boldsymbol{B}'$ be identical with $(\boldsymbol{C} \mid \boldsymbol{x})\boldsymbol{B}$. Then:

(1) $(\boldsymbol{C} \mid \boldsymbol{x})\neg\boldsymbol{A}$ is identical with $\neg\boldsymbol{A}'$;
(2) $(\boldsymbol{C} \mid \boldsymbol{x})\boldsymbol{A} \vee \boldsymbol{B}$ is identical with $\boldsymbol{A}' \vee \boldsymbol{B}'$;
(3) $(\boldsymbol{C} \mid \boldsymbol{x})\boldsymbol{A} \implies \boldsymbol{B}$ is identical with $\boldsymbol{A}' \implies \boldsymbol{B}'$;
(4) If $\boldsymbol{s}$ is a specific sign, then $(\boldsymbol{C} \mid \boldsymbol{x})\boldsymbol{s}\boldsymbol{A}\boldsymbol{B}$ is identical with $\boldsymbol{s}\boldsymbol{A}'\boldsymbol{B}'$.

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[2] This phrase "in this book" appears in the French edition, but is missing in the English translation.

1.2.4. Insufficient criteria. We need to specify how substitution works on letters for this to be well-defined. It's literally the base case for the inductive definition.

Also note that the remarks about ambiguity (§1.1.7).

1.3. **Formative Constructions.**

1.3.1. Definition. A specific sign for a theory is either **"relational"** or else it is **"substantific"**. Bourbaki calls its arity (number of arguments) its **"weight"**.

1.3.2. Definition. Bourbaki says an assembly is of the **"first species"** [i.e., it's a "term"] if it begins with a τ or a substantific sign, or if it consists of a single letter. Otherwise, the assembly is of the **"second species"** [i.e., it's a "formula" or "proposition"].

1.3.3. Production rules. Loosely, Bourbaki is giving us "production rules" for the grammar of their formal language which underlies their formal system. However, their notion of a "formative construction" is a "firehose of words" which generate the language.

1.3.4. Definition. A **"Formative Construction"** in a theory $\mathcal{T}$ is a sequence of assemblies with the following property — for each assembly $\boldsymbol{A}$ of the sequence, one of the following holds:

(1) $\boldsymbol{A}$ is a letter;
(2) There is in the sequence an assembly $\boldsymbol{B}$ of the second species preceding $\boldsymbol{A}$ such that $\boldsymbol{A}$ is $\neg\boldsymbol{B}$;
(3) There are two assemblies $\boldsymbol{B}$ and $\boldsymbol{C}$ of the second species (possibly not distinct) preceding $\boldsymbol{A}$ such that $\boldsymbol{A}$ is identical with $\boldsymbol{B}\vee\boldsymbol{C}$;
(4) There is an assembly of the second species preceding $\boldsymbol{A}$, and there is a variable $\boldsymbol{x}$ such that $\boldsymbol{A}$ is identical with $\tau_{\boldsymbol{x}}(\boldsymbol{B})$;
(5) There is a specific sign $\boldsymbol{s}$ of weight n in $\mathcal{T}$ and n assemblies $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$ of the first species preceding $\boldsymbol{A}$ such that $\boldsymbol{A}$ is identical with $\boldsymbol{s}\boldsymbol{A}_1\dots\boldsymbol{A}_n$.

1.3.5. Formal language. The formative constructions of a theory is precisely the formal language underlying that theory, including both the terms and all possible formulas generated by the primitive notions of that theory.

1.3.6. First-order logic. Bourbaki's formal system is first-order, since variables range over terms, and this is ensured by the previous definition.

Also note that Bourbaki takes for primitive connectives in the underlying logic $\{\vee,\neg\}$. This appears to be inspired by Russell and Whitehead's *Principia Mathematica* or Hilbert and Ackermann's book (which was inspired by *Principia Mathematica*).

1.3.7. Definition. Bourbaki now announces that assemblies of the first species which appear in the formative constructions of $\mathcal{T}$ are called **"Terms"** in $\mathcal{T}$, and assemblies of the second species which appear in the formative constructions of $\mathcal{T}$ are called **"Relations"** (or, in modern terminology, **"Formulas"**) in $\mathcal{T}$.

1.3.8. Hilbert choice operator. The intuition for $\tau_{\boldsymbol{x}}(\boldsymbol{B})$ is that it corresponds to:

(1) If there is at least one term $\boldsymbol{T}$ in $\mathcal{T}$ such that $(\boldsymbol{T}\mid\boldsymbol{x})\boldsymbol{B}$ is satisfied, then $\tau_{\boldsymbol{x}}(\boldsymbol{B})$ corresponds to a distinguished object which satisfies $\boldsymbol{B}[\boldsymbol{x}]$;
(2) If there are no such terms satisfying $\boldsymbol{B}[\boldsymbol{x}]$, then $\tau_{\boldsymbol{x}}(\boldsymbol{B})$ is a fixed object about which nothing can be said.

1.4. **Formative Criteria.**

1.4.1. Grammar. These appear to be production rules from the grammar underlying Bourbaki's system, they just lack the vocabulary to call it such. There are 13 rules in the book, but only 8 or so are introduced in the first chapter.

CF1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are formulas in $\mathcal{T}$, then $\boldsymbol{A}\vee\boldsymbol{B}$ is a formula in $\mathcal{T}$.

CF2. If $\boldsymbol{A}$ is a formula in $\mathcal{T}$, then $\neg\boldsymbol{A}$ is a formula in $\mathcal{T}$.

CF3. If $\boldsymbol{A}$ is a formula in $\mathcal{T}$, then $\tau_{\boldsymbol{x}}(\boldsymbol{A})$ is a term in $\mathcal{T}$.

CF4. If $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$ are terms in $\mathcal{T}$ and $\boldsymbol{s}$ is a specific relational (resp., substantific) sign in $\mathcal{T}$, then $\boldsymbol{s}\boldsymbol{A}_1\cdots\boldsymbol{A}_n$ is a formula (resp., term) in $\mathcal{T}$.

CF5. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are formulas in $\mathcal{T}$, then $\boldsymbol{A}\implies\boldsymbol{B}$ is a formula in $\mathcal{T}$.

CF6. Let $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$ be a formative construction in $\mathcal{T}$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be letters. If $\boldsymbol{y}$ does not appear in any $\boldsymbol{A}_j$, then $(\boldsymbol{y}\mid\boldsymbol{x})\boldsymbol{A}_1, \dots, (\boldsymbol{y}\mid\boldsymbol{x})\boldsymbol{A}_n$ is a formative construction in $\mathcal{T}$.

CF7. Let $\boldsymbol{A}$ be a formula (resp., term) in $\mathcal{T}$, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be letters. Then $(\boldsymbol{y}\mid\boldsymbol{x})\boldsymbol{A}$ is a formula (resp., term) in $\mathcal{T}$.

CF8. Let $\boldsymbol{A}$ be a formula (resp., term) in $\mathcal{T}$, let $\boldsymbol{T}$ be a term in $\mathcal{T}$, let $\boldsymbol{x}$ be a variable. Then $(\boldsymbol{T}\mid\boldsymbol{x})\boldsymbol{A}$ is a formula (resp., term) in $\mathcal{T}$.

1.4.2. Proofs. The proofs of these rules are not really enlightening, and anachronistic. I skipped over them.

2. Theorems

2.1. **Axioms.**

2.1.1. Constructing theories. We have seen that the specific signs (opaque predicates and functions) determines the formulas and terms in a theory $\mathcal{T}$. In general, to construct $\mathcal{T}$, we proceed as follows:

(1) We write down a certain number of formulas in $\mathcal{T}$ which Bourbaki calls the **"explicit axioms"** (I'm told the modern terminology for these are "simple axioms") which govern the behaviour of **"Constants"** of the theory — any letter which appears in them is considered a "constant", but a better term would be "parameter" (e.g., for a group, its "constants" are G [the underlying set] and μ [the binary operator]);
(2) We write down one or more rules called the **"Schemes"** of $\mathcal{T}$, which have the following properties:
    (a) the application of such a rule $\mathcal{R}$ yields a formula in $\mathcal{T}$;
    (b) if $\boldsymbol{T}$ is a term in $\mathcal{T}$, if $\boldsymbol{x}$ is a letter, and if $\boldsymbol{R}$ is a relation in $\mathcal{T}$ obtained by applying the scheme $\mathcal{R}$, then the relation $(\boldsymbol{T}\mid\boldsymbol{x})\boldsymbol{R}$ can also be cosntructed by applying $\mathcal{R}$.

    The relations obtained by applying a scheme of $\mathcal{T}$ is called an *implicit axiom* of $\mathcal{T}$.

2.1.2. Schemes. We stress that Bourbaki requires at least one scheme for a mathematical theory.

I'm not really satisfied with Bourbaki's definition of a scheme, because it's unclear how to think of it. Specifically, Bourbaki's description of schemes have their variables range over terms, rather than using metavaraibles to range over formulas and terms, or [as Mizar does] using second-order variables to parametrize formulas. See Corcoran's commentary [8] on schemas in logic.[3]

2.1.3. Definition. Later on, Bourbaki imagines a theory as a triple

$$\langle \textit{signs},\ \textit{explicit axioms},\ \textit{schemes}\rangle$$

where "signs" are a finite set of specific signs, "explicit axioms" are a finite set of formulas, and "schemes" are a finite set of axiom schemes.

2.2. Proofs.

2.2.1. Definition. Bourbaki defines a **"Demonstrative Text"** (literal translation from French) of a theory $\mathcal{T}$ to consist of:

(1) "auxiliary" formative constructions for terms and formulas of $\mathcal{T}$;
(2) "Proofs" in $\mathcal{T}$.

Presumably these capture "definitions" and "theorems + proofs" in articles (or other forms of mathematical exposition).

2.2.2. Proofs. A **"proof in $\mathcal{T}$"** is a sequence of relations (formulas) in $\mathcal{T}$ which appear in the auxiliary formative construction, such that for every relation $\boldsymbol{R}$ in the sequence at least one of the following is satisfied:

(a_1) $\boldsymbol{R}$ is an explicit axiom of $\mathcal{T}$;
(a_2) $\boldsymbol{R}$ results from the application of a scheme of $\mathcal{T}$ to terms or relations which appear in the auxiliary formative construction;
(b) There are two relations $\boldsymbol{S}$ and $\boldsymbol{S} \implies \boldsymbol{R}$ in the sequence which precede $\boldsymbol{R}$.

The way Bourbaki describes it, a mathematical theory sounds like a kind of "stock ticker" which just prints theorem after theorem in such a manner that every formula printed is either an axiom or follows from *modus ponens* applied to previous theorems. (The modern terminology we'd use now would be a "Turing machine".)

2.2.2.1. *Remark.* In Bourbaki's *Elements of Mathematics* series of books, the astute reader will recognize there are no "Proof: .... □" environments, unlike modern mathematics texts. I suspect the reason is that their underlying philosophical view is that *mathematics is a stream of theorems.* If so, then it is especially bewildering to modern readers, since Bourbaki's formal system (as we shall see) is a Hilbert system with one inference rule and 7 or 8 axiom schemas—i.e., their foundations are what we'd expect from Model theorists, not Proof theorists. This is all said with the benefit of hindsight and some 70 years of collective experience, which Bourbaki lacked. (Although it does appear that Bourbaki is just following Hilbert without thinking too much about it.)

2.2.3. Definition. A **"Theorem"** in $\mathcal{T}$ is a formula which appears in a proof in $\mathcal{T}$.

[3]See also John Corcoran's contribution to the Stanford Encyclopedia of Philosophy about axiom schemas: https://plato.stanford.edu/entries/schema/

2.2.4. "Theoremhood". Bourbaki has an intriguing aside where he observes the assertion that "a formula is not a theorem" cannot be adequately established, since later on the formula may be proven. In other words, the ontological status of "theoremhood" is dynamic. This is all very interesting, but equally irrelevant.

2.2.5. True vs Provable. Bourbaki blurs the distinction between "A theorem $\boldsymbol{A}$ proven in $\mathcal{T}$" with "A true formula $\boldsymbol{A}$". For Bourbaki, there is no difference: all proven theorems are true. But logicians (both then and now) insist that "truth" is a semantical notion (a formula is true *relative to a model or interpretation*) whereas "proven" is a syntactic notion (a theorem is proven relative to a syntactic proof calculus).

Most working mathematicians probably adhere to the "Bourbakian creed": there is no distinction between a proven theorem and a true formula, all proven theorems are true. (I know I did.)

2.2.6. Grammar of Hilbert-style proofs. I am going to intentionally deviate from Bourbaki's presentation, and I will work with an explicit Hilbert-style proof system. The syntax for this system is quite conventional: let Γ be a finite set of formulas, let $\boldsymbol{A}$ be a formula, all in theory $\mathcal{T}$. We write $\Gamma\vdash_{\mathcal{T}}\boldsymbol{A}$ for "Assuming hypothesis Γ, we can prove $\boldsymbol{A}$ in theory $\mathcal{T}$". When $\Gamma=\emptyset$, we just write $\vdash_{\mathcal{T}}\boldsymbol{A}$. We will also suppress the subscript $\mathcal{T}$ on the turnstile when it is clear from context what the theory is.

Now, a *theorem* is a formula $\boldsymbol{A}$ together with an associated proof. A *proof* of $\boldsymbol{A}$ is a finite [ordered] sequence of *proof lines* whose final line is $\vdash\boldsymbol{A}$. A *proof line* is a triple consisting of the numerical line number, the assertion, and the justification. Usually the justification is a reference to a theorem or derived inference rule, and there may be arguments supplied.

The explicit grammar:

⟨*proof line*⟩	::=	⟨*line number*⟩ ⟨*assertion*⟩ ⟨*justification*⟩
⟨*line number*⟩	::=	`(.` ⟨*positive integer*⟩ `)`
⟨*justification*⟩	::=	`by` ⟨*theorem or axiom*⟩ ⟨*optional arguments*⟩
	|	`by MP(`⟨*line number*⟩ `,` ⟨*line number*⟩`)`
⟨*assertion*⟩	::=	⟨*hypotheses*⟩ ⊢ ⟨*formula*⟩
⟨*hypotheses*⟩	::=	⟨*blank*⟩
	|	⟨*comma-separated formulas*⟩
⟨*comma-separated formulas*⟩	::=	⟨*formula*⟩
	|	⟨*formula*⟩ `,` ⟨*comma-separated formulas*⟩
⟨*arguments*⟩	::=	⟨*blank*⟩
	|	`(`⟨*comma-separated arguments*⟩`)`
⟨*argument*⟩	::=	⟨*line number*⟩
	|	⟨*formula*⟩
	|	⟨*term*⟩
⟨*comma-separated arguments*⟩	::=	⟨*argument*⟩
	|	⟨*argument*⟩ `,` ⟨*comma-separated arguments*⟩

2.2.7. Definition. Let $\boldsymbol{R}$ be a relation in $\mathcal{T}$, let $\boldsymbol{x}$ be a variable, let $\boldsymbol{T}$ be a term in $\mathcal{T}$. If $\vdash (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{R}$ is a theorem in $\mathcal{T}$, then $\boldsymbol{T}$ is said to **"satisfy the relation"** $\boldsymbol{R}$ in $\mathcal{T}$ (or, to be a **"solution"** of $\boldsymbol{R}$) when $\boldsymbol{R}$ is considered as a relation in $\boldsymbol{x}$.

2.2.8. Definition. Bourbaki calls a theory $\mathcal{T}$ **"Contradictory"** if there is some formula $\boldsymbol{A}$ such that we can prove both $\vdash_{\mathcal{T}} \boldsymbol{A}$ and $\vdash_{\mathcal{T}} \neg\boldsymbol{A}$.

2.2.9. Deductive criteria. Bourbaki introduces metatheorems to help expedite formal proofs. They are called "deductive criteria", which are usually parametrized by metavariables ranging over relations, terms, or theorems. Recall (§1.2.1) our discussion of criteria as inference rules.

C1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be relations in a theory $\mathcal{T}$. If $\vdash_{\mathcal{T}} \boldsymbol{A} \implies \boldsymbol{B}$ and $\vdash_{\mathcal{T}} \boldsymbol{A}$ are theorems in $\mathcal{T}$, then $\vdash_{\mathcal{T}} \boldsymbol{B}$ is a theorem in $\mathcal{T}$.*

> *Remark.* This is precisely what we called `MP`. Its first argument is a reference to the theorem $\vdash \boldsymbol{A} \implies \boldsymbol{B}$, its second argument is a reference to the theorem $\vdash \boldsymbol{A}$. Bourbaki calls C1 "Syllogism".

2.3. Substitutions in a Theory.

2.3.1. Notation. Let $\mathcal{T}$ be a theory, let $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$ be its explicit axioms (§2.1.1), let $\boldsymbol{x}$ be a letter, let $\boldsymbol{T}$ be a term of $\mathcal{T}$. We denote by $(\boldsymbol{T} \mid \boldsymbol{x})\mathcal{T}$ the theory whose signs and schemes are the same as those of $\mathcal{T}$, and whose explicit axioms are $(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}_1$, $\dots$, $(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}_n$.

C2. *Let $\vdash \boldsymbol{A}$ be a theorem of $\mathcal{T}$, let $\boldsymbol{T}$ be a term in $\mathcal{T}$, and let $\boldsymbol{x}$ be a letter. Then $\vdash (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}$ is a theorem in $(\boldsymbol{T} \mid \boldsymbol{x})\mathcal{T}$.*

C3. *Let $\vdash \boldsymbol{R}$ be a theorem in $\mathcal{T}$, let $\boldsymbol{x}$ be a letter which is not a parameter ["constant"] in $\mathcal{T}$, let $\boldsymbol{T}$ be a term in $\mathcal{T}$. Then $\vdash (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{R}$ is a theorem in $\mathcal{T}$.*

This follows immediately from C2.

2.4. Comparing Theories.

2.4.1. Definition. A theory $\mathcal{T}'$ is said to be **"Stronger"** than a theory $\mathcal{T}$ if

(1) all signs of $\mathcal{T}$ are signs of $\mathcal{T}'$, and
(2) all schemes of $\mathcal{T}$ are schemes of $\mathcal{T}'$, and
(3) all explicit axioms of $\mathcal{T}$ are theorems in $\mathcal{T}'$.

(This defines a binary predicate.)

C4. *If a theory $\mathcal{T}'$ is stronger than a theory $\mathcal{T}$, then all theorems of $\mathcal{T}$ are theorems of $\mathcal{T}'$.*

2.4.2. Definition. A theory $\mathcal{T}'$ is **"Equivalent"** to a theory $\mathcal{T}$ if

(1) all signs of $\mathcal{T}$ are signs of $\mathcal{T}'$ and vice-versa, and
(2) all schemes of $\mathcal{T}$ are schemes of $\mathcal{T}'$ and vice-versa, and
(3) all explicit axioms of $\mathcal{T}$ are theorems of $\mathcal{T}'$ and vice-versa.

In other words, $\mathcal{T}'$ is equivalent to $\mathcal{T}$ if and only if $\mathcal{T}'$ is stronger than $\mathcal{T}$ and $\mathcal{T}$ is stronger than $\mathcal{T}'$.

2.4.3. Corollary. *Equivalent theories have the same terms, formulas, and theorems.*

(This follows immediately from C4.)

C5. *Let $\mathcal{T}$ be a theory. Let $\boldsymbol{T}_1$, ..., $\boldsymbol{T}_m$ be terms of $\mathcal{T}$. Let $\boldsymbol{a}_1$, ..., $\boldsymbol{a}_m$ be parameters ["constants"] of $\mathcal{T}$. Suppose the theory $\mathcal{T}'$ is such that*

(1) all signs of $\mathcal{T}$ are signs of $\mathcal{T}'$, and
(2) every scheme of $\mathcal{T}$ are schemes of $\mathcal{T}'$, and
(3) if $\boldsymbol{A}$ is an explicit axiom of $\mathcal{T}$, then $\vdash_{\mathcal{T}'} (\boldsymbol{T}_1 \mid \boldsymbol{a}_1)(\cdots)(\boldsymbol{T}_m \mid \boldsymbol{a}_m)\boldsymbol{A}$ is a theorem in $\mathcal{T}'$.

If $\vdash_{\mathcal{T}} \boldsymbol{B}$ is a theorem of $\mathcal{T}$, then $\vdash_{\mathcal{T}'} (\boldsymbol{T}_1 \mid \boldsymbol{a}_1)(\cdots)(\boldsymbol{T}_m \mid \boldsymbol{a}_m)\boldsymbol{B}$ is a theorem in $\mathcal{T}'$.

(This does not appear to be used anywhere in Bourbaki's *Theory of Sets*; maybe it is used somewhere much later on, in another volume of the Elements of Mathematics.)

3. Logical Theories

3.1. The Axioms.

3.1.0. Propositional logic. This is the Hilbert system for a fragment of Bourbaki's formal system. It is what we would now call "propositional logic".

3.1.1. Definition. A **"Logical Theory"** is a theory $\mathcal{T}$ which includes the following four schemes:

(S1) $(\boldsymbol{A} \lor \boldsymbol{A}) \implies \boldsymbol{A}$
(S2) $\boldsymbol{A} \implies (\boldsymbol{A} \lor \boldsymbol{B})$
(S3) $(\boldsymbol{A} \lor \boldsymbol{B}) \implies (\boldsymbol{B} \lor \boldsymbol{A})$
(S4) $(\boldsymbol{A} \implies \boldsymbol{B}) \implies ((\boldsymbol{C} \lor \boldsymbol{A}) \implies (\boldsymbol{C} \lor \boldsymbol{B}))$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are formulas in the theory.

3.1.2. Russell–Bernays axioms. These axioms are precisely those used by Russell and Whitehead's *Principia Mathematica*, and simplified by Paul Bernays [4]. Originally Russell and Whitehead had 5 axioms, but Bernays proved one of them was redundant. Recall (§1.1.10) Bourbaki defined $\boldsymbol{A} \implies \boldsymbol{B}$ as an abbreviation for $(\neg\boldsymbol{A}) \lor \boldsymbol{B}$. This choice follows the decision made by Russell and Whitehead, as well as Hilbert and Ackermann.

We will be painfully explicit in our proofs, so we introduce axioms for this syntactic sugar:

Syn1. If $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ then $\vdash \neg\boldsymbol{A} \lor \boldsymbol{B}$.

Syn2. If $\vdash \neg\boldsymbol{A} \lor \boldsymbol{B}$, then $\vdash \boldsymbol{A} \implies \boldsymbol{B}$.

Syn3. $\vdash (\boldsymbol{A} \implies \boldsymbol{B}) \implies (\neg\boldsymbol{A} \lor \boldsymbol{B})$

Syn4. $\vdash (\neg\boldsymbol{A} \lor \boldsymbol{B}) \implies (\boldsymbol{A} \implies \boldsymbol{B})$

3.1.3. Theorem. $\vdash \neg\boldsymbol{A} \implies (\boldsymbol{A} \implies \boldsymbol{B})$

Pf: (.1) $\vdash \neg\boldsymbol{A} \implies ((\neg\boldsymbol{A}) \lor \boldsymbol{B})$ by S4($\neg\boldsymbol{A}$, $\boldsymbol{B}$)
(.2) $\vdash \neg\boldsymbol{A} \implies (\boldsymbol{A} \implies \boldsymbol{B})$ by definition of implies.

3.1.4. Theorem. *If $\mathcal{T}$ is contradictory theory (§2.2.8) [i.e., if $\vdash_{\mathcal{T}} \boldsymbol{A}$ and $\vdash_{\mathcal{T}} \neg\boldsymbol{A}$], then $\vdash_{\mathcal{T}} \boldsymbol{B}$ for any formula $\boldsymbol{B}$ of $\mathcal{T}$.*

Pf: (.1) $\vdash \boldsymbol{A}$ by hypothesis
(.2) $\vdash \neg\boldsymbol{A}$ by hypothesis

(.3) $\vdash \neg\boldsymbol{A} \implies (\boldsymbol{A} \implies \boldsymbol{B})$ by Th3.1.3($\boldsymbol{A}$, $\boldsymbol{B}$)
(.4) $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ by MP(.3, .2)
(.5) $\vdash \boldsymbol{B}$ by MP(.4, .1)

3.1.5. Assumptions. If $\Gamma = \boldsymbol{A}_1, \Gamma'$ is our set of assumptions, we can write $\Gamma \vdash \boldsymbol{A}_1$ and justify it "by assumption". However, theorems must have $\Gamma = \emptyset$.

3.1.6. Theorem (Weakening). *If* $\Gamma_1 \subset \Gamma_2$ *and if* $\Gamma_1 \vdash \boldsymbol{A}$*, then we can infer* $\Gamma_2 \vdash \boldsymbol{A}$.

This is going to be so common we won't bother citing this explicitly, we'll just saying "by weakening ⟨*line number*⟩".

3.2. **First consequences.**

C6. *If* $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ *and* $\vdash \boldsymbol{B} \implies \boldsymbol{C}$*, then* $\vdash \boldsymbol{A} \implies \boldsymbol{C}$.

Pf:
(.1) $\vdash (\boldsymbol{A} \implies \boldsymbol{B})$ by hypothesis
(.2) $\vdash (\boldsymbol{B} \implies \boldsymbol{C})$ by hypothesis
(.3) $\vdash (\boldsymbol{B} \implies \boldsymbol{C}) \implies ((\neg\boldsymbol{A} \vee \boldsymbol{B}) \implies (\neg\boldsymbol{A} \vee \boldsymbol{C}))$ by S4($\boldsymbol{B}, \boldsymbol{C}, \neg\boldsymbol{A}$).
(.4) $\vdash ((\neg\boldsymbol{A} \vee \boldsymbol{B}) \implies (\neg\boldsymbol{A} \vee \boldsymbol{C}))$ by MP(.3,.2)
(.5) $\vdash (\neg\boldsymbol{A} \vee \boldsymbol{B})$ by Syn1(.1)
(.6) $\vdash (\neg\boldsymbol{A} \vee \boldsymbol{C})$ by MP(.4,.5).
(.7) $\vdash (\boldsymbol{A} \implies \boldsymbol{C})$ by Syn2(.6)

C7. $\vdash \boldsymbol{B} \implies (\boldsymbol{A} \vee \boldsymbol{B})$.

Pf:
(.1) $\vdash \boldsymbol{B} \implies (\boldsymbol{B} \vee \boldsymbol{A})$ by *S*2($\boldsymbol{B}, \boldsymbol{A}$)
(.2) $\vdash (\boldsymbol{B} \vee \boldsymbol{A}) \implies (\boldsymbol{A} \vee \boldsymbol{B})$ by *S*3($\boldsymbol{B}, \boldsymbol{A}$)
(.3) $\vdash \boldsymbol{B} \implies (\boldsymbol{A} \vee \boldsymbol{B})$ by C6(.1, .2)

C8. $\vdash \boldsymbol{A} \implies \boldsymbol{A}$

Pf:
(.1) $\vdash \boldsymbol{A} \implies (\boldsymbol{A} \vee \boldsymbol{A})$ by S2($\boldsymbol{A}$, $\boldsymbol{A}$)
(.2) $\vdash (\boldsymbol{A} \vee \boldsymbol{A}) \implies \boldsymbol{A}$ by S1($\boldsymbol{A}$)
(.3) $\vdash \boldsymbol{A} \implies \boldsymbol{A}$ by C6(.1, .2)

C9. *If* $\vdash \boldsymbol{B}$*, then for any proposition* $\boldsymbol{A}$ *we have* $\vdash \boldsymbol{A} \implies \boldsymbol{B}$.

Pf:
(.1) $\vdash \boldsymbol{B}$ by hypothesis
(.2) $\vdash \boldsymbol{B} \implies (\neg\boldsymbol{A} \vee \boldsymbol{B})$ by C7($\neg\boldsymbol{A}$, $\boldsymbol{B}$)
(.3) $\vdash \neg\boldsymbol{A} \vee \boldsymbol{B}$ by MP(.1, .2)
(.4) $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ by Syn2(.3)

C10. $\vdash \boldsymbol{A} \vee \neg\boldsymbol{A}$

Pf:
(.1) $\vdash \boldsymbol{A} \implies \boldsymbol{A}$ by C8($\boldsymbol{A}$)
(.2) $\vdash \neg\boldsymbol{A} \vee \boldsymbol{A}$ by Syn1(.1)
(.3) $\vdash ((\neg\boldsymbol{A}) \vee \boldsymbol{A}) \implies (\boldsymbol{A} \vee \neg\boldsymbol{A})$ by S3($\neg\boldsymbol{A}$, $\boldsymbol{A}$)
(.4) $\vdash \boldsymbol{A} \vee \neg\boldsymbol{A}$ by MP(.3, .2)

C11. $\vdash \boldsymbol{A} \implies (\neg\neg\boldsymbol{A})$

Pf:
(.1) $\vdash (\neg\boldsymbol{A}) \vee (\neg\neg\boldsymbol{A})$ by C10($\boldsymbol{A}$)
(.2) $\vdash \boldsymbol{A} \implies \neg\neg\boldsymbol{A}$ by Syn2(.1)

C12. $\vdash (\boldsymbol{A} \implies \boldsymbol{B}) \implies ((\neg\boldsymbol{B}) \implies (\neg\boldsymbol{A}))$

Pf: (.1) $\vdash \boldsymbol{B} \Longrightarrow \neg\neg\boldsymbol{B}$ by C11($\boldsymbol{B}$)
(.2) $\vdash (\boldsymbol{B} \Longrightarrow \neg\neg\boldsymbol{B}) \Longrightarrow ((\neg\boldsymbol{A} \vee \boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A} \vee \neg\neg\boldsymbol{B}))$ by S4($\boldsymbol{B}$, $\neg\neg\boldsymbol{B}$, $\neg\boldsymbol{A}$)
(.3) $\vdash (\neg\boldsymbol{A} \vee \boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A} \vee \neg\neg\boldsymbol{B})$ by MP(.2, .1)
(.4) $\vdash (\neg\boldsymbol{A} \vee \neg\neg\boldsymbol{B}) \Longrightarrow (\neg\neg\boldsymbol{B} \vee \neg\boldsymbol{A})$ by S3($\neg\boldsymbol{A}$, $\neg\neg\boldsymbol{B}$)
(.5) $\vdash (\neg\boldsymbol{A} \vee \boldsymbol{B}) \Longrightarrow (\neg\neg\boldsymbol{B} \vee \neg\boldsymbol{A})$ by C6(.3, .4)
(.6) $\vdash (\boldsymbol{A} \Longrightarrow \boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A} \vee \boldsymbol{B})$ by Syn3($\boldsymbol{A}$, $\boldsymbol{B}$)
(.7) $\vdash (\boldsymbol{A} \Longrightarrow \boldsymbol{B}) \Longrightarrow (\neg\neg\boldsymbol{B} \vee \neg\boldsymbol{A})$ by C6(.6, .5)
(.8) $\vdash (\neg\neg\boldsymbol{B} \vee \neg\boldsymbol{A}) \Longrightarrow (\neg\boldsymbol{B} \Longrightarrow \neg\boldsymbol{A})$ by Syn3($\neg\boldsymbol{B}$, $\neg\boldsymbol{A}$)
(.9) $\vdash (\boldsymbol{A} \Longrightarrow \boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{B} \Longrightarrow \neg\boldsymbol{A})$ by C6(.7, .8)

C13. *If* $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{B}$, *then* $\vdash (\boldsymbol{B} \Longrightarrow \boldsymbol{C}) \Longrightarrow (\boldsymbol{A} \Longrightarrow \boldsymbol{C})$

Pf: (.1) $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{B}$ by hypothesis
(.2) $\vdash (\boldsymbol{A} \Longrightarrow \boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{B} \Longrightarrow \neg\boldsymbol{A})$ by C12($\boldsymbol{A}$, $\boldsymbol{B}$)
(.3) $\vdash \neg\boldsymbol{B} \Longrightarrow \neg\boldsymbol{A}$ by MP(.2, .1)
(.4) $\vdash (\neg\boldsymbol{B} \Longrightarrow \neg\boldsymbol{A}) \Longrightarrow ((\boldsymbol{C} \vee \neg\boldsymbol{B}) \Longrightarrow (\boldsymbol{C} \vee \neg\boldsymbol{A}))$ by S4($\neg\boldsymbol{B}$, $\neg\boldsymbol{A}$, $\boldsymbol{C}$)
(.5) $\vdash (\boldsymbol{C} \vee \neg\boldsymbol{B}) \Longrightarrow (\boldsymbol{C} \vee \neg\boldsymbol{A})$ by MP(.4, .3)
(.6) $\vdash (\boldsymbol{B} \Longrightarrow \boldsymbol{C}) \Longrightarrow (\boldsymbol{C} \vee \neg\boldsymbol{B})$ by C6(Syn3($\boldsymbol{B}$, $\boldsymbol{C}$), S3($\boldsymbol{C}$, $\neg\boldsymbol{B}$))
(.7) $\vdash (\boldsymbol{B} \Longrightarrow \boldsymbol{C}) \Longrightarrow (\boldsymbol{C} \vee \neg\boldsymbol{A})$ by C6(.6, .5)
(.8) $\vdash (\boldsymbol{C} \vee \neg\boldsymbol{A}) \Longrightarrow (\boldsymbol{A} \Longrightarrow \boldsymbol{C})$ by C6(S3($\boldsymbol{C}$, $\neg\boldsymbol{A}$), Syn4($\boldsymbol{A}$, $\boldsymbol{C}$))
(.9) $\vdash (\boldsymbol{B} \Longrightarrow \boldsymbol{C}) \Longrightarrow (\boldsymbol{A} \Longrightarrow \boldsymbol{C})$ by C6(.7, .8)

3.2.1. Theorem. *If* $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{B}$, *then* $\vdash \boldsymbol{A} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{B}$.

Pf: (.1) $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{B}$ by hypothesis
(.2) $\vdash (\boldsymbol{A} \Longrightarrow \boldsymbol{B}) \Longrightarrow ((\boldsymbol{B} \vee \boldsymbol{A}) \Longrightarrow (\boldsymbol{B} \vee \boldsymbol{B}))$ by S4($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{B}$)
(.3) $\vdash \boldsymbol{A} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{B} \vee \boldsymbol{A}$ by S3($\boldsymbol{A}$, $\boldsymbol{B}$)
(.4) $\vdash \boldsymbol{B} \vee \boldsymbol{A} \Longrightarrow \boldsymbol{B} \vee \boldsymbol{B}$ by MP(.2, .1)
(.5) $\vdash \boldsymbol{A} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{B} \vee \boldsymbol{B}$ by C6(.3, .4)
(.6) $\vdash \boldsymbol{B} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{B}$ by S1($\boldsymbol{B}$)
(.7) $\vdash \boldsymbol{A} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{B}$ by C6(.5, .6)

3.2.2. Theorem. *If* $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{C}$ *and* $\vdash \boldsymbol{B} \Longrightarrow \boldsymbol{C}$, *then* $\vdash \boldsymbol{A} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{C}$.

Pf: (.1) $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{C}$ by hypothesis
(.2) $\vdash \boldsymbol{B} \Longrightarrow \boldsymbol{C}$ by hypothesis
(.3) $\vdash (\boldsymbol{B} \Longrightarrow \boldsymbol{C}) \Longrightarrow ((\boldsymbol{A} \vee \boldsymbol{B}) \Longrightarrow (\boldsymbol{A} \vee \boldsymbol{C}))$ by S4($\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{A}$)
(.4) $\vdash \boldsymbol{A} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{A} \vee \boldsymbol{C}$ by MP(.3, .2)
(.5) $\vdash \boldsymbol{A} \vee \boldsymbol{C} \Longrightarrow \boldsymbol{C}$ by Thm3.2.1(.1)
(.6) $\vdash \boldsymbol{A} \vee \boldsymbol{B} \Longrightarrow \boldsymbol{C}$ by C6(.4, .5)

3.3. **Methods of Proof.** Bourbaki introduces a number of metatheorems which directly relate to common phrases used in "informal" [i.e., ordinary] mathematical proofs. I borrow the anachronistic phrase "proof steps" (used to describe declarative formal proof languages for proof assistants) to show how certain "proof steps" would "compile" [translate] to Bourbaki's foundations.

It is tempting, therefore, to think of Bourbaki's formal system as the "machine code" of mathematics. We could "compile" a controlled language for proofs down to Bourbaki's "machine code". Bourbaki seems to suspect that such an approach is "on the table" for consideration as a strategy for doing formal mathematics, but dismisses it out of hand (see page 10 of the paperback edition of their book). Apparently Bourbaki feared that the "compilation process" would be unsound, or be a source of unsoundness. Although such concerns seem ridiculous *now*, that's because we have the benefit of 60 years of experience with compilers, whereas Bourbaki had pre-dated compilers.

Puzzle 4. *Construct a compiler for a formal language for a declarative proof assistant which produces output in Bourbaki's formal system. Is it possible? What considerations do we need to bear in mind for the backend? Is it possible to use an LCF-style prover for the backend, or will this run into performance problems?*

For inspiration on declarative proof systems, look at Mizar[4] [3] *and Wiedijk's vernacular* [12].

I. Method of Auxiliary Hypothesis

3.3.1. Lemma (Hypothetical syllogism). *If* $\vdash \boldsymbol{A} \implies (\boldsymbol{B} \implies \boldsymbol{C})$ *and* $\vdash \boldsymbol{A} \implies \boldsymbol{B}$, *then* $\vdash \boldsymbol{A} \implies \boldsymbol{C}$.

Pf:
(.1) $\vdash \boldsymbol{A} \implies (\boldsymbol{B} \implies \boldsymbol{C})$ by hypothesis
(.2) $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ by hypothesis
(.3) $\vdash (\boldsymbol{B} \implies \boldsymbol{C}) \implies (\boldsymbol{A} \implies \boldsymbol{C})$ by C13(.2, $\boldsymbol{C}$)
(.4) $\vdash \boldsymbol{A} \implies (\boldsymbol{A} \implies \boldsymbol{C})$ by C6(.1, .3)
(.5) $\vdash \neg\boldsymbol{A} \implies (\boldsymbol{A} \implies \boldsymbol{C})$ by Th3.1.3($\boldsymbol{A}$, $\boldsymbol{C}$)
(.6) $\vdash (\boldsymbol{A} \vee \neg\boldsymbol{A}) \implies (\boldsymbol{A} \implies \boldsymbol{C})$ by Th3.2.2(.4, .5)
(.7) $\vdash \boldsymbol{A} \vee \neg\boldsymbol{A}$ by C10($\boldsymbol{A}$)
(.8) $\vdash \boldsymbol{A} \implies \boldsymbol{C}$ by MP(.6, .7)

C14. *If* $\boldsymbol{A} \vdash \boldsymbol{B}$, *then* $\vdash \boldsymbol{A} \implies \boldsymbol{B}$.

Proof. By induction on the derivation of $\boldsymbol{A} \vdash \boldsymbol{B}$. Recall (§2.2.2) there are three possible cases:

Case 1: $\boldsymbol{B}$ is an axiom or an instance of a scheme. Then $\vdash \boldsymbol{B}$ holds, and we can obtain $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ by C4.

Case 2: $\boldsymbol{B}$ is identical to $\boldsymbol{A}$, in which case $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ follows from C9.

Case 3: $\boldsymbol{A} \vdash \boldsymbol{B}$ is obtained by a syllogism C1, which means we have $\boldsymbol{A} \vdash \boldsymbol{C}$ and $\boldsymbol{A} \vdash \boldsymbol{C} \implies \boldsymbol{B}$. We have the inductive hypotheses:
IH_1: $\vdash \boldsymbol{A} \implies (\boldsymbol{C} \implies \boldsymbol{B})$
IH_2: $\vdash \boldsymbol{A} \implies \boldsymbol{C}$
Then $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ by Lm3.3.1(IH_1, IH_2). ☐

3.3.2. Proof step. Bourbaki tells us that, in practice, we see this occur in the proof step "Suppose $\boldsymbol{A}$ [is true]." Then we just need to prove $\boldsymbol{B}$, and this constitutes a proof of $\vdash \boldsymbol{A} \implies \boldsymbol{B}$.

However, Bourbaki mistakenly believes that "Let $\boldsymbol{x}$ be a real number" introduces the auxiliary hypothesis "$\boldsymbol{x}$ is a real number". This is $\forall$-introduction, not $\implies$-introduction, so I'm not sure Bourbaki is correct. But the peculiarities of Hilbert's ε-calculus might allow the two to coincide.

II. Method of Reductio ad Absurdum

[4] https://mizar.uwb.edu.pl/

C15. *If adjoining to the theory $\mathcal{T}$ the axiom $\neg\boldsymbol{A}$ results in a contradictory theory (§§2.2.8, 3.1.4), then $\vdash_{\mathcal{T}} \boldsymbol{A}$.*

Pf:
(.1) $\neg\boldsymbol{A} \vdash \boldsymbol{B}$ by hypothesis
(.2) $\neg\boldsymbol{A} \vdash \neg\boldsymbol{B}$ by hypothesis
(.3) $\neg\boldsymbol{A} \vdash \neg\boldsymbol{B} \Longrightarrow (\boldsymbol{B} \Longrightarrow \boldsymbol{A})$ by Th3.1.3($\boldsymbol{B}$, $\boldsymbol{A}$)
(.4) $\neg\boldsymbol{A} \vdash \boldsymbol{B} \Longrightarrow \boldsymbol{A}$ by MP(.3, .2)
(.5) $\neg\boldsymbol{A} \vdash \boldsymbol{A}$ by MP(.4, .1)
(.6) $\vdash \neg\boldsymbol{A} \Longrightarrow \boldsymbol{A}$ by C14(.5)
(.7) $\vdash (\neg\boldsymbol{A} \Longrightarrow \boldsymbol{A}) \Longrightarrow ((\boldsymbol{A} \vee \neg\boldsymbol{A}) \Longrightarrow (\boldsymbol{A} \vee \boldsymbol{A}))$ by S4($\neg\boldsymbol{A}$, $\boldsymbol{A}$, $\boldsymbol{A}$)
(.8) $\vdash (\boldsymbol{A} \vee \neg\boldsymbol{A}) \Longrightarrow (\boldsymbol{A} \vee \boldsymbol{A})$
(.9) $\vdash \boldsymbol{A} \vee \neg\boldsymbol{A}$ by C10($\boldsymbol{A}$)
(.10) $\vdash \boldsymbol{A} \vee \boldsymbol{A}$ by MP(.8, .9)
(.11) $\vdash \boldsymbol{A} \vee \boldsymbol{A} \Longrightarrow \boldsymbol{A}$ by S1($\boldsymbol{A}$)
(.12) $\vdash \boldsymbol{A}$ by MP(.11, .10)

3.3.3. Proof step. Bourbaki says that we usually state "Suppose for the sake of contradiction that $\neg\boldsymbol{A}$". Then deduce both $\boldsymbol{B}$ and $\neg\boldsymbol{B}$. This is the desired contradiction. Therefore $\boldsymbol{A}$.

C16. $\vdash (\neg\neg\boldsymbol{A}) \Longrightarrow \boldsymbol{A}$

Pf:
(.1) $\neg\boldsymbol{A}, \neg\neg\boldsymbol{A} \vdash \neg\boldsymbol{A}$ by assumption
(.2) $\neg\boldsymbol{A}, \neg\neg\boldsymbol{A} \vdash \neg(\neg\boldsymbol{A})$ by assumption
(.3) $\neg\neg\boldsymbol{A} \vdash \boldsymbol{A}$ by C15(.1, .2)
(.4) $\vdash (\neg\neg\boldsymbol{A}) \Longrightarrow \boldsymbol{A}$ by C14(.3).

C17. (Contrapositive)$\vdash ((\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A})) \Longrightarrow (\boldsymbol{A} \Longrightarrow \boldsymbol{B})$

Pf:
(.1) $\neg\boldsymbol{B}, (\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A}), \boldsymbol{A} \vdash \neg\boldsymbol{B}$ by assumption
(.2) $\neg\boldsymbol{B}, (\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A}), \boldsymbol{A} \vdash (\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A})$ by assumption
(.3) $\neg\boldsymbol{B}, (\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A}), \boldsymbol{A} \vdash \neg\boldsymbol{A}$ by MP(.2, .1)
(.4) $\neg\boldsymbol{B}, (\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A}), \boldsymbol{A} \vdash \boldsymbol{A}$ by assumption
(.5) $(\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A}), \boldsymbol{A} \vdash \boldsymbol{B}$ by C15(.4)
(.6) $(\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A}) \vdash \boldsymbol{A} \Longrightarrow \boldsymbol{B}$ by C14(.5)
(.7) $\vdash ((\neg\boldsymbol{B}) \Longrightarrow (\neg\boldsymbol{A})) \Longrightarrow (\boldsymbol{A} \Longrightarrow \boldsymbol{B})$ by C14(.6).

Method of Disjunction of Cases

C18. *If $\vdash \boldsymbol{A} \vee \boldsymbol{B}$ and $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{C}$ and $\vdash \boldsymbol{B} \Longrightarrow \boldsymbol{C}$, then $\vdash \boldsymbol{C}$.*

(Bourbaki's proof is thrice as long, here's an optimized proof)

Pf:
(.1) $\vdash \boldsymbol{A} \vee \boldsymbol{B}$ by hypothesis
(.2) $\vdash \boldsymbol{A} \Longrightarrow \boldsymbol{C}$ by hypothesis
(.3) $\vdash \boldsymbol{B} \Longrightarrow \boldsymbol{C}$ by hypothesis
(.4) $\vdash (\boldsymbol{A} \vee \boldsymbol{B}) \Longrightarrow \boldsymbol{C}$ by Th3.2.2(.2, .3)
(.5) $\vdash \boldsymbol{C}$ by MP(.4, .1).

3.3.4. Proof step. This corresponds to "Per cases [by $\vdash \boldsymbol{A} \vee \boldsymbol{B}$]; suppose $\boldsymbol{A}$, [then proof of $\boldsymbol{C}$]; suppose $\boldsymbol{B}$ [, then proof of $\boldsymbol{C}$]".

IV. Method of the Auxiliary Constant

C19. *Let $\boldsymbol{x}$ be a letter, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be relations in $\mathcal{T}$ such that*

(1) $\boldsymbol{x}$ is not a constant of $\mathcal{T}$ and does not appear in $\boldsymbol{B}$; and
(2) there is a term $\boldsymbol{T}$ in $\mathcal{T}$ such that $(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}$ is a theorem in $\mathcal{T}$.

Let $\mathcal{T}'$ be the theory obtained by adjoining $\boldsymbol{A}$ to the axioms of $\mathcal{T}$. If $\vdash_{\mathcal{T}'} \boldsymbol{B}$ is a theorem in $\mathcal{T}'$, then $\vdash_{\mathcal{T}} \boldsymbol{B}$ is a theorem in $\mathcal{T}$.

Pf: (.1) $\vdash_{\mathcal{T}'} \boldsymbol{B}$ by hypothesis
(.2) $\boldsymbol{A} \vdash_{\mathcal{T}} \boldsymbol{B}$
(.3) $\vdash \boldsymbol{A} \implies \boldsymbol{B}$ by C14(.2)
(.4) $\vdash (\boldsymbol{T} \mid \boldsymbol{x})(\boldsymbol{A} \implies \boldsymbol{B})$ by C3(.3)
(.5) $\vdash \big((\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}\big) \implies \boldsymbol{B}$ by CS5(.4), hypothesis 1
(.6) $\vdash (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}$ by hypothesis 2
(.7) $\vdash \boldsymbol{B}$ by MP(.5, .6).

3.3.5. Theorem of Legitimation. Bourbaki calls the theorem $\vdash (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}$ a "theorem of legitimation", which is necessary to prove for us to use the method of auxiliary constant [i.e., auxiliary parameter]. Really, we will see (§4.1.4) that it suffices to use an existence theorem.

3.4. Conjunction.

3.4.1. Definition. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be assemblies. Then $\neg((\neg\boldsymbol{A}) \vee (\neg\boldsymbol{B}))$ is denoted "$\boldsymbol{A} \wedge \boldsymbol{B}$" (or "$\boldsymbol{A}$ and $\boldsymbol{B}$").

CS6. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{T}$ be assemblies and let $\boldsymbol{x}$ be a letter. Then $(\boldsymbol{T} \mid \boldsymbol{x})(\boldsymbol{A} \wedge \boldsymbol{B})$ is identical to $\boldsymbol{A}' \wedge \boldsymbol{B}'$ where $\boldsymbol{A}'$ is identical to $(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A}$ and $\boldsymbol{B}'$ is identical to $(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{B}$.

Proof. This is a direct consequence of CS5

$$(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A} \wedge \boldsymbol{B} = (\boldsymbol{T} \mid \boldsymbol{x})\neg((\neg\boldsymbol{A}) \vee (\neg\boldsymbol{B})) \tag{1a}$$
$$= \neg[(\boldsymbol{T} \mid \boldsymbol{x})((\neg\boldsymbol{A}) \vee (\neg\boldsymbol{B}))] \tag{1b}$$
$$= \neg([(\boldsymbol{T} \mid \boldsymbol{x})(\neg\boldsymbol{A})] \vee [(\boldsymbol{T} \mid \boldsymbol{x})(\neg\boldsymbol{B})]) \tag{1c}$$
$$= \neg([(\neg(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{A})] \vee [(\neg(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{B})]) \tag{1d}$$
$$= \boldsymbol{A}' \wedge \boldsymbol{B}', \tag{1e}$$

as desired. □

CF9. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are relations in $\mathcal{T}$, then $\boldsymbol{A} \wedge \boldsymbol{B}$ is a relation in $\mathcal{T}$ (called the **"Conjunction"** of $\boldsymbol{A}$ and $\boldsymbol{B}$).

This follows from CF1 and CF2.

C20. *If $\vdash \boldsymbol{A}$ and $\vdash \boldsymbol{B}$, then $\vdash \boldsymbol{A} \wedge \boldsymbol{B}$.*

I formalize Bourbaki's proof, there is almost certainly a more optimal proof.

Pf: (.1) $\vdash \boldsymbol{A}$ by hypothesis
(.2) $\vdash \boldsymbol{B}$ by hypothesis
(.3) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash\neg\neg((\neg\boldsymbol{A})\vee(\neg\boldsymbol{B}))$ by assumption, definition of conjunction
(.4) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash(\neg\neg((\neg\boldsymbol{A})\vee(\neg\boldsymbol{B})))\implies((\neg\boldsymbol{A})\vee(\neg\boldsymbol{B}))$ by C16($(\neg\boldsymbol{A})\vee(\neg\boldsymbol{B})$)
(.5) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash(\neg\boldsymbol{A})\vee(\neg\boldsymbol{B})$ by MP(.4, .3)
(.6) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash((\neg\boldsymbol{A})\vee(\neg\boldsymbol{B}))\implies(\boldsymbol{A}\implies\neg\boldsymbol{B})$ by Syn4($\boldsymbol{A}$, $\neg\boldsymbol{B}$)
(.7) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash\boldsymbol{A}\implies\neg\boldsymbol{B}$ by MP(.6, .5)
(.8) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash\boldsymbol{A}$ by weakening(.1)
(.9) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash\neg\boldsymbol{B}$ by MP(.7, .8)
(.10) $\neg(\boldsymbol{A}\wedge\boldsymbol{B})\vdash\boldsymbol{B}$ by weakening(.2)
(.11) $\vdash\boldsymbol{A}\wedge\boldsymbol{B}$ by C15(.10, .9).

C21. *We have the following results for any formulas $\boldsymbol{A}$ and $\boldsymbol{B}$:*
(1) $\vdash(\boldsymbol{A}\wedge\boldsymbol{B})\implies\boldsymbol{A}$
(2) $\vdash(\boldsymbol{A}\wedge\boldsymbol{B})\implies\boldsymbol{B}$

Pf: (1) (.1) $\vdash\ (\neg\boldsymbol{A})\implies\neg(\boldsymbol{A}\wedge\boldsymbol{B})$ by S3($\neg\boldsymbol{A}$, $\neg\boldsymbol{B}$), definition of conjunction
(.2) $\vdash((\neg\boldsymbol{A})\implies\neg(\boldsymbol{A}\wedge\boldsymbol{B}))\implies(\boldsymbol{A}\wedge\boldsymbol{B}\implies\boldsymbol{A})$ by C17($\boldsymbol{A}\wedge\boldsymbol{B}$, $\boldsymbol{A}$)
(.3) $\vdash\boldsymbol{A}\wedge\boldsymbol{B}\implies\boldsymbol{A}$ by MP(.2, .1).
(2) (.1) $\vdash\ \neg\boldsymbol{B}\implies\neg\boldsymbol{A}\vee\neg\boldsymbol{B}$ by C7($\neg\boldsymbol{A}$, $\neg\boldsymbol{B}$)
(.2) $\vdash(\neg\boldsymbol{B}\implies\neg\boldsymbol{A}\vee\neg\boldsymbol{B})\implies(\boldsymbol{A}\wedge\boldsymbol{B}\implies\boldsymbol{B})$ by C17($\boldsymbol{A}\wedge\boldsymbol{B}$, $\boldsymbol{B}$), definition of conjunction
(.3) $\vdash\boldsymbol{A}\wedge\boldsymbol{B}\implies\boldsymbol{B}$ by MP(.2, .1).

3.5. **Equivalence.**

3.5.1. Definition. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be assemblies. The assembly $(\boldsymbol{A}\implies\boldsymbol{B})\wedge(\boldsymbol{B}\implies\boldsymbol{A})$ will be denoted by "$\boldsymbol{A}\iff\boldsymbol{B}$".

CS7. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{T}$ be assemblies, let $\boldsymbol{x}$ be a letter. Then $(\boldsymbol{T}\mid\boldsymbol{x})(\boldsymbol{A}\iff\boldsymbol{B})$ is identical to $\boldsymbol{A}'\iff\boldsymbol{B}'$ where $\boldsymbol{A}'$ is $(\boldsymbol{T}\mid\boldsymbol{x})\boldsymbol{A}$ and $\boldsymbol{B}'$ is $(\boldsymbol{T}\mid\boldsymbol{x})\boldsymbol{B}$.

Proof sketch. This follows from CS5 and CS6. □

CF10. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are relations in $\mathcal{T}$, then $\boldsymbol{A}\iff\boldsymbol{B}$ is a relation in $\mathcal{T}$.

Proof sketch. This follows from CF5 and CF9. □

C22. *The biconditional is an equivalence relation. Bourbaki skips reflexivity, which I included as item (0).*
(0) $\vdash\boldsymbol{A}\iff\boldsymbol{A}$
(1) If $\vdash\boldsymbol{A}\iff\boldsymbol{B}$*, then* $\vdash\boldsymbol{B}\iff\boldsymbol{A}$
(2) If $\vdash\boldsymbol{A}\iff\boldsymbol{B}$ *and* $\vdash\boldsymbol{B}\iff\boldsymbol{C}$*, then* $\vdash\boldsymbol{A}\iff\boldsymbol{C}$*.*

Proof sketch. (0) follows from C20(C8($\boldsymbol{A}$), C8($\boldsymbol{A}$)).
(1) and (2) follows from C20 and C21. □

At this point, it's painfully clear that Bourbaki is "running out of steam", because the proofs are increasingly sketch for C22, and completely omitted afterwards. The remaining deductive criteria given are enumerated faithfully (i.e., Bourbaki just gives a list of results).

C23. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be equivalent relations (i.e.,* $\vdash \boldsymbol{A} \iff \boldsymbol{B}$*), let* $\boldsymbol{C}$ *be an arbitrary relation. Then:*

(1) $\vdash (\neg\boldsymbol{A}) \iff (\neg\boldsymbol{B})$
(2) $\vdash (\boldsymbol{A} \implies \boldsymbol{C}) \iff (\boldsymbol{B} \implies \boldsymbol{C})$
(3) $\vdash (\boldsymbol{A} \land \boldsymbol{C}) \iff (\boldsymbol{B} \land \boldsymbol{C})$
(4) $\vdash (\boldsymbol{C} \implies \boldsymbol{A}) \iff (\boldsymbol{C} \implies \boldsymbol{B})$
(5) $\vdash (\boldsymbol{A} \lor \boldsymbol{C}) \iff (\boldsymbol{B} \lor \boldsymbol{C})$

C24. *Let* $\boldsymbol{A}$*,* $\boldsymbol{B}$*,* $\boldsymbol{C}$ *be arbitrary relations. Then:*

(1) $\vdash (\neg\neg\boldsymbol{A}) \iff \boldsymbol{A}$ *[hint: C11(*$\boldsymbol{A}$*), C16(*$\boldsymbol{A}$*)]*
(2) $\vdash (\boldsymbol{A} \implies \boldsymbol{B}) \iff (\neg\boldsymbol{B} \implies \neg\boldsymbol{A})$ *[hint: C12(*$\boldsymbol{A}$*,* $\boldsymbol{B}$*), C17(*$\boldsymbol{A}$*,* $\boldsymbol{B}$*)]*
(3) $\vdash (\boldsymbol{A} \land \boldsymbol{A}) \iff \boldsymbol{A}$
(4) $\vdash (\boldsymbol{A} \land \boldsymbol{B}) \iff (\boldsymbol{B} \land \boldsymbol{A})$
(5) $\vdash \boldsymbol{A} \land (\boldsymbol{B} \land \boldsymbol{C}) \iff (\boldsymbol{A} \land \boldsymbol{B}) \land \boldsymbol{C}$
(6) $\vdash (\boldsymbol{A} \lor \boldsymbol{B}) \iff \neg((\neg\boldsymbol{A}) \lor (\neg\boldsymbol{B}))$
(7) $\vdash \boldsymbol{A} \lor \boldsymbol{A} \iff \boldsymbol{A}$
(8) $\vdash \boldsymbol{A} \lor \boldsymbol{B} \iff \boldsymbol{B} \lor \boldsymbol{A}$ *by S3(*$\boldsymbol{A}$*,* $\boldsymbol{B}$*) and S3(*$\boldsymbol{B}$*,* $\boldsymbol{A}$*)*
(9) $\vdash \boldsymbol{A} \lor (\boldsymbol{B} \lor \boldsymbol{C}) \iff (\boldsymbol{A} \lor \boldsymbol{B}) \lor \boldsymbol{C}$
(10) $\vdash \boldsymbol{A} \land (\boldsymbol{B} \lor \boldsymbol{C}) \iff (\boldsymbol{A} \land \boldsymbol{B}) \lor (\boldsymbol{A} \land \boldsymbol{C})$
(11) $\vdash \boldsymbol{A} \lor (\boldsymbol{B} \land \boldsymbol{C}) \iff (\boldsymbol{A} \lor \boldsymbol{B}) \land (\boldsymbol{A} \land \boldsymbol{C})$
(12) $\vdash \boldsymbol{A} \land (\neg\boldsymbol{B}) \iff \neg(\boldsymbol{A} \implies \boldsymbol{B})$
(13) $\vdash \boldsymbol{A} \lor \boldsymbol{B} \iff ((\neg\boldsymbol{A}) \implies \boldsymbol{B})$

C25. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be any formulas. Then:*

(1) If $\vdash \boldsymbol{A}$*, then* $\vdash (\boldsymbol{A} \land \boldsymbol{B}) \iff \boldsymbol{B}$*.*
(2) If $\vdash \neg\boldsymbol{A}$*, then* $\vdash (\boldsymbol{A} \lor \boldsymbol{B}) \iff \boldsymbol{B}$*.*

4. Quantified Theories

4.1. Definition of Quantifiers.

4.1.1. Definition. Bourbaki defines the **"Existential Quantifier"** $(\exists\boldsymbol{x})\boldsymbol{R}$ as identical to $(\tau_{\boldsymbol{x}}(\boldsymbol{R}) \mid \boldsymbol{x})\boldsymbol{R}$, which is common in Hilbert's ε-calculus.

The **"Universal Quantifier"** $(\forall\boldsymbol{x})\boldsymbol{R}$ is defined to be identical with $\neg(\exists\boldsymbol{x})(\neg\boldsymbol{R})$. We observe by plugging this back into the definition of the existential quantifier that this is identical with $\neg(\tau_{\boldsymbol{x}}(\neg\boldsymbol{R}) \mid \boldsymbol{x})\neg\boldsymbol{R}$ and by the law of double negation $(\tau_{\boldsymbol{x}}(\neg\boldsymbol{R}) \mid \boldsymbol{x})\boldsymbol{R}$.

4.1.2. Proposition. *The letter* $\boldsymbol{x}$ *does not appear in* $\tau_{\boldsymbol{x}}(\boldsymbol{R})$*. Therefore the letter* $\boldsymbol{x}$ *does not appear in either* $(\exists\boldsymbol{x})\boldsymbol{R}$ *or* $(\forall\boldsymbol{x})\boldsymbol{R}$*.*

For this reason, it seems appealing to use de Bruijn indices (or levels) for bound variables.

CS8. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be letters, let $\boldsymbol{R}$ be an assembly. Assume $\boldsymbol{x}'$ does not appear in $\boldsymbol{R}$. Then $(\exists\boldsymbol{x})\boldsymbol{R}$ is identical with $(\exists\boldsymbol{x}')\boldsymbol{R}'$, and $(\forall\boldsymbol{x})\boldsymbol{R}$ is identical with $(\exists\boldsymbol{x}')\boldsymbol{R}'$ where $\boldsymbol{R}'$ is identical with $(\boldsymbol{x}' \mid \boldsymbol{x})\boldsymbol{R}$.

Proof sketch. This follows from CS1 and CS3. □

CS9. Let $\boldsymbol{R}$ and $\boldsymbol{U}$ be assemblies, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be distinct letters. If $\boldsymbol{x}$ does not appear in $\boldsymbol{U}$, then $(\boldsymbol{U} \mid \boldsymbol{y})(\exists\boldsymbol{x})\boldsymbol{R}$ is identical with $(\exists\boldsymbol{x})\boldsymbol{R}'$, and $(\boldsymbol{U} \mid \boldsymbol{y})(\forall\boldsymbol{x})\boldsymbol{R}$ is identical with $(\forall\boldsymbol{x})\boldsymbol{R}'$, where $\boldsymbol{R}'$ is identical with $(\boldsymbol{U} \mid \boldsymbol{y})\boldsymbol{R}$.

Proof sketch. This follows from CS2 and CS4. □

CF11. If $\boldsymbol{R}$ is a formula in $\mathcal{T}$ and $\boldsymbol{x}$ is a letter, then $(\exists\boldsymbol{x})\boldsymbol{R}$ and $(\forall\boldsymbol{x})\boldsymbol{R}$ are formulas in $\mathcal{T}$.

4.1.3. Intuition of quantifiers. From the intuition of τ (§1.3.8), we see that $(\exists\boldsymbol{x})\boldsymbol{R}$ corresponds to our intuitive understanding that "There exists some $\boldsymbol{x}$ such that $\boldsymbol{R}$". Similarly, $\neg(\exists\boldsymbol{x})\neg\boldsymbol{R}$ corresponds to "There is no object $\boldsymbol{x}$ which has the property 'not $\boldsymbol{R}$'" (and therefore all objects have property $\boldsymbol{R}$).

4.1.4. Existence theorems. If we have a theorem $\vdash(\exists\boldsymbol{x})\boldsymbol{R}$, then it can act as a theorem of legitimation (§3.3.5). I suspect the reason we don't speak of "theorems of legitimation" is because "existence theorems" suffice.

C26. *Let $\boldsymbol{R}$ be a relation in theory $\mathcal{T}$, let $\boldsymbol{x}$ be a letter. Then* $\vdash\big((\forall\boldsymbol{x})\boldsymbol{R}\big)\iff\big((\tau_{\boldsymbol{x}}(\neg\boldsymbol{R})\mid\boldsymbol{x})\boldsymbol{R}\big)$.

Proof sketch.

$$\begin{aligned}(\forall\boldsymbol{x})\boldsymbol{R}&\iff\neg(\exists\boldsymbol{x})\neg\boldsymbol{R}\quad\text{by definition of }\forall\\&\iff\neg(\tau_{\boldsymbol{x}}(\neg\boldsymbol{R})\mid\boldsymbol{x})\neg\boldsymbol{R}\quad\text{by definition of }\exists\\&\iff\neg\neg(\tau_{\boldsymbol{x}}(\neg\boldsymbol{R})\mid\boldsymbol{x})\boldsymbol{R}\quad\text{by CS5}\\&\iff(\tau_{\boldsymbol{x}}(\neg\boldsymbol{R})\mid\boldsymbol{x})\boldsymbol{R}\quad\text{by C24.1}\end{aligned}$$
□

C27. (Generalization) *If* $\vdash\boldsymbol{R}$ *and* $\boldsymbol{x}$ *is any letter, then* $\vdash(\forall\boldsymbol{x})\boldsymbol{R}$.

Proof sketch. We have $\vdash(\tau_{\boldsymbol{x}}(\neg\boldsymbol{R})\mid\boldsymbol{x})\boldsymbol{R}$ by C3. □

> *Remark.* If $\boldsymbol{x}$ is a "constant" [i.e., parameter] of the theory $\mathcal{T}$, then $\vdash\boldsymbol{R}$ proves a property concerning the parameter. Generalization becomes vacuous, and we just obtain $\vdash\boldsymbol{R}$ in Bourbaki's system.

Puzzle 5. *We should expect generalization C27 to fail in the case of* $\Gamma\vdash\boldsymbol{R}$ *when* $\boldsymbol{x}$ *appears freely in* Γ *somewhere. But the proof given for C27 does not indicate that. What's the gimmick? What's wrong with this line of reasoning? [Hint: look at C3; what does it mean for* $\boldsymbol{x}$ *to be a parameter ["constant"] in a theory? What does that mean in terms of* Γ*?]*

C28. $\vdash\neg(\forall\boldsymbol{x})\boldsymbol{R}\iff(\exists\boldsymbol{x})\neg\boldsymbol{R}$.

Proof sketch.

$$\begin{aligned}\neg(\forall\boldsymbol{x})\boldsymbol{R}&\iff\neg\big(\neg(\exists\boldsymbol{x})\neg\boldsymbol{R}\big)\quad\text{by definition of }\forall\\&\iff(\exists\boldsymbol{x})\neg\boldsymbol{R}\quad\text{by C24.1.}\end{aligned}$$
□

4.2. Axioms of Quantified Theories.

4.2.1. Definition. A **"Quantified Theory"** is any theory with axiom schemes S1 through S4, plus the axiom scheme S5:

(S5) If $\boldsymbol{R}$ is a relation in theory $\mathcal{T}$, if $\boldsymbol{T}$ is a term in $\mathcal{T}$, if $\boldsymbol{x}$ is a letter, then $\vdash(\boldsymbol{T}\mid\boldsymbol{x})\boldsymbol{R}\implies(\exists\boldsymbol{x})\boldsymbol{R}$.

> *Remark.* Bourbaki spends an extraordinary amount of time proving that S5 is, in fact, a scheme. I found it rather unenlightening.

4.3. Properties of Quantified Theories.

4.3.1. Reservation. Bourbaki, for the rest of this section, considers $\mathcal{T}$ to be a quantified theory. Further, the theory $\mathcal{T}_0$ consists of the same signs as $\mathcal{T}$ but $\mathcal{T}_0$ has no explicit axioms and only schemes S1 through S5. Observe that $\mathcal{T}$ is stronger (§2.4.1) than $\mathcal{T}_0$.

C29. *Let* $\boldsymbol{R}$ *be a relation in* $\mathcal{T}$*, let* $\boldsymbol{x}$ *be a letter. Then* $\vdash \big(\neg(\exists\boldsymbol{x})\boldsymbol{R}\big) \iff \big((\forall\boldsymbol{x})\neg\boldsymbol{R}\big)$

C30. *Let* $\boldsymbol{R}$ *be a relation in* $\mathcal{T}$*, let* $\boldsymbol{T}$ *be a term in* $\mathcal{T}$*, let* $\boldsymbol{x}$ *be a letter. Then* $\vdash (\forall\boldsymbol{x})\boldsymbol{R} \implies (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{R}$.

Pf: (.1) $\vdash (\boldsymbol{T} \mid \boldsymbol{x})\neg\boldsymbol{R} \implies (\tau_{\boldsymbol{x}}(\neg\boldsymbol{R}) \mid \boldsymbol{x})\neg\boldsymbol{R}$ by S5($\neg\boldsymbol{R}$, $\boldsymbol{T}$, $\boldsymbol{x}$)
(.2) $\vdash \neg(\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{R} \implies \neg(\tau_{\boldsymbol{x}}(\neg\boldsymbol{R}) \mid \boldsymbol{x})\boldsymbol{R}$ by CS5
(.3) $\vdash (\tau_{\boldsymbol{x}}(\neg\boldsymbol{R}) \mid \boldsymbol{x})\boldsymbol{R} \implies (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{R}$ by contrapositive of .2
(.4) $\vdash (\forall\boldsymbol{x})\boldsymbol{R} \implies (\tau_{\boldsymbol{x}}(\neg\boldsymbol{R}) \mid \boldsymbol{x})\boldsymbol{R}$ by C26($\boldsymbol{R}$, $\boldsymbol{x}$)
(.5) $\vdash (\forall\boldsymbol{x})\boldsymbol{R} \implies (\boldsymbol{T} \mid \boldsymbol{x})\boldsymbol{R}$ by C6(.4, .3)

C31. *Let* $\boldsymbol{R}$ *and* $\boldsymbol{S}$ *be relations in* $\mathcal{T}$*. Let* $\boldsymbol{x}$ *be a variable (i.e., a letter which is not a constant in* $\mathcal{T}$*). Then:*

(1) If $\vdash \boldsymbol{R} \implies \boldsymbol{S}$*, then* $\vdash (\forall\boldsymbol{x})\boldsymbol{R} \implies (\forall\boldsymbol{x})\boldsymbol{S}$
(2) If $\vdash \boldsymbol{R} \implies \boldsymbol{S}$*, then* $\vdash (\exists\boldsymbol{x})\boldsymbol{R} \implies (\exists\boldsymbol{x})\boldsymbol{S}$
(3) If $\vdash \boldsymbol{R} \iff \boldsymbol{S}$*, then* $\vdash (\forall\boldsymbol{x})\boldsymbol{R} \iff (\forall\boldsymbol{x})\boldsymbol{S}$
(4) If $\vdash \boldsymbol{R} \iff \boldsymbol{S}$*, then* $\vdash (\exists\boldsymbol{x})\boldsymbol{R} \iff (\exists\boldsymbol{x})\boldsymbol{S}$

Observe that (3) follows from (1), and (4) follows from (2). Therefore it suffices to prove only (1) and (2).

Pf: (1) (.1) $\vdash \boldsymbol{R} \implies \boldsymbol{S}$ by hypothesis
(.2) $(\forall\boldsymbol{x})\boldsymbol{R} \vdash (\forall\boldsymbol{x})\boldsymbol{R} \implies \boldsymbol{R}$ by C30
(.3) $(\forall\boldsymbol{x})\boldsymbol{R} \vdash (\forall\boldsymbol{x})\boldsymbol{R}$ by assumption
(.4) $(\forall\boldsymbol{x})\boldsymbol{R} \vdash \boldsymbol{R}$ by MP(.2, .3)
(.5) $(\forall\boldsymbol{x})\boldsymbol{R} \vdash \boldsymbol{R} \implies \boldsymbol{S}$ by weakening .1
(.6) $(\forall\boldsymbol{x})\boldsymbol{R} \vdash \boldsymbol{S}$ by MP(.5, .4)
(.7) $(\forall\boldsymbol{x})\boldsymbol{R} \vdash (\forall\boldsymbol{x})\boldsymbol{S}$ by C27(.6, $\boldsymbol{x}$)
(.8) $\vdash (\forall\boldsymbol{x})\boldsymbol{R} \implies (\forall\boldsymbol{x})\boldsymbol{S}$ by C14(.7)

(2) (.1) $\vdash \boldsymbol{R} \implies \boldsymbol{S}$ by hypothesis
(.2) $\vdash \neg\boldsymbol{S} \implies \neg\boldsymbol{R}$ by contrapositive of .1
(.3) $\vdash (\forall\boldsymbol{x})\neg\boldsymbol{S} \implies (\forall\boldsymbol{x})\neg\boldsymbol{R}$ by applying (1) of this theorem to .2
(.4) $\vdash \neg(\forall\boldsymbol{x})\neg\boldsymbol{R} \implies \neg(\forall\boldsymbol{x})\neg\boldsymbol{S}$ by contrapositive of .3, and using Bourbaki's definition of $\forall$.

4.3.2. Addendum. Observe that $\vdash (\forall\boldsymbol{x})\boldsymbol{R} \implies \boldsymbol{R}$ by C30($\boldsymbol{R}$, $\boldsymbol{x}$, $\boldsymbol{x}$).

C32. *Let* $\boldsymbol{R}$ *and* $\boldsymbol{S}$ *be relations in* $\mathcal{T}$*.*

(1) $\vdash (\forall\boldsymbol{x})(\boldsymbol{R} \wedge \boldsymbol{S}) \iff \Big(\big((\forall\boldsymbol{x})\boldsymbol{R}\big) \wedge \big((\forall\boldsymbol{x})\boldsymbol{S}\big)\Big)$
(2) $\vdash (\exists\boldsymbol{x})(\boldsymbol{R} \vee \boldsymbol{S}) \iff \Big(\big((\exists\boldsymbol{x})\boldsymbol{R}\big) \vee \big((\exists\boldsymbol{x})\boldsymbol{S}\big)\Big)$

It suffices to prove (1), since (2) follows from (1) and C29.

Pf: (.1) $(\forall\boldsymbol{x})(\boldsymbol{R} \wedge \boldsymbol{S}) \vdash \boldsymbol{R} \wedge \boldsymbol{S}$ by C30($\boldsymbol{R}$, $\boldsymbol{x}$, $\boldsymbol{x}$)
(.2) $(\forall\boldsymbol{x})(\boldsymbol{R} \wedge \boldsymbol{S}) \vdash \boldsymbol{R}$
(.3) $(\forall\boldsymbol{x})(\boldsymbol{R} \wedge \boldsymbol{S}) \vdash (\forall\boldsymbol{x})\boldsymbol{R}$
(.4) $(\forall\boldsymbol{x})(\boldsymbol{R} \wedge \boldsymbol{S}) \vdash \boldsymbol{S}$

(.5) $(\forall\boldsymbol{x})(\boldsymbol{R}\wedge\boldsymbol{S})\vdash(\forall\boldsymbol{x})\boldsymbol{S}$
(.6) $(\forall\boldsymbol{x})(\boldsymbol{R}\wedge\boldsymbol{S})\vdash((\forall\boldsymbol{x})\boldsymbol{R})\wedge((\forall\boldsymbol{x})\boldsymbol{S})$
(.7) $\vdash(\forall\boldsymbol{x})(\boldsymbol{R}\wedge\boldsymbol{S})\implies((\forall\boldsymbol{x})\boldsymbol{R})\wedge((\forall\boldsymbol{x})\boldsymbol{S})$ by C14(.6)
(.8) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash(\forall\boldsymbol{x})\boldsymbol{R}$ by assumption
(.9) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash(\forall\boldsymbol{x})\boldsymbol{R}\implies\boldsymbol{R}$ by C30($\boldsymbol{R}$, $\boldsymbol{x}$, $\boldsymbol{x}$)
(.10) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash\boldsymbol{R}$ by MP(.9, .8)
(.11) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash(\forall\boldsymbol{x})\boldsymbol{S}$ by assumption
(.12) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash(\forall\boldsymbol{x})\boldsymbol{S}\implies\boldsymbol{S}$ by C30($\boldsymbol{S}$, $\boldsymbol{x}$, $\boldsymbol{x}$)
(.13) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash\boldsymbol{S}$ by MP(.12, .11)
(.14) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash\boldsymbol{R}\wedge\boldsymbol{S}$ by C20(.10, .13)
(.15) $(\forall\boldsymbol{x})\boldsymbol{R},(\forall\boldsymbol{x})\boldsymbol{S}\vdash(\forall\boldsymbol{x})(\boldsymbol{R}\wedge\boldsymbol{S})$ by C27(.14, $\boldsymbol{x}$)
(.16) $\vdash(\forall\boldsymbol{x})\boldsymbol{R}\wedge(\forall\boldsymbol{x})\boldsymbol{S}\implies(\forall\boldsymbol{x})(\boldsymbol{R}\wedge\boldsymbol{S})$

C33. *Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be relations in $\mathcal{T}$. Let $\boldsymbol{x}$ be a letter. Assume $\boldsymbol{x}$ does not appear in $\boldsymbol{R}$.*

(1) $\vdash(\forall\boldsymbol{x})(\boldsymbol{R}\vee\boldsymbol{S})\iff\Big(\boldsymbol{R}\vee\big((\forall\boldsymbol{x})\boldsymbol{S}\big)\Big)$
(2) $\vdash(\exists\boldsymbol{x})(\boldsymbol{R}\wedge\boldsymbol{S})\iff\Big(\boldsymbol{R}\wedge\big((\exists\boldsymbol{x})\boldsymbol{S}\big)\Big)$

Note the use of disjunction and conjunction is paired with opposite quantifiers as found in C32.

Observe (2) follows from (1) and C29.

C34. *Let $\boldsymbol{R}$ be a relation in $\mathcal{T}$.*

(1) $\vdash(\forall\boldsymbol{x})(\forall\boldsymbol{y})\boldsymbol{R}\iff(\forall\boldsymbol{y})(\forall\boldsymbol{x})\boldsymbol{R}$
(2) $\vdash(\exists\boldsymbol{x})(\exists\boldsymbol{y})\boldsymbol{R}\iff(\exists\boldsymbol{y})(\exists\boldsymbol{x})\boldsymbol{R}$
(3) $\vdash(\exists\boldsymbol{x})(\forall\boldsymbol{y})\boldsymbol{R}\implies(\forall\boldsymbol{y})(\exists\boldsymbol{x})\boldsymbol{R}$

4.4. **Typical Quantifiers.** Bourbaki generalizes the quantifiers to something like bounded quantifiers, but they're never really used anywhere. I'm going to skip my notes on this subsection for now, and perhaps I'll type them up at a later date. This subsection is parallel to the previous one, introducing analogous deductive criteria with "typical quantifiers" instead of "vanilla quantifiers".

5. Equalitarian Theories

5.1. The Axioms.

5.1.1. Definition. An **"Equalitarian Theory"** is any theory with axiom schemes S1 through S4, plus the axiom scheme S5 and the following two schemes:

(S6) Let $\boldsymbol{x}$ be a letter, let $\boldsymbol{T}$ and $\boldsymbol{U}$ be terms in $\mathcal{T}$, let $\boldsymbol{R}[\boldsymbol{x}]$ be a relation in $\mathcal{T}$. Then $\vdash(\boldsymbol{T}=\boldsymbol{U})\implies(\boldsymbol{R}[\boldsymbol{T}]\iff\boldsymbol{R}[\boldsymbol{U}])$.
(S7) Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be relations in $\mathcal{T}$, let $\boldsymbol{x}$ be a letter. Then $\vdash(\boldsymbol{R}\iff\boldsymbol{S})\implies(\tau_{\boldsymbol{x}}(\boldsymbol{R})=\tau_{\boldsymbol{x}}(\boldsymbol{S}))$.

Remark. We have not proven that equality is an equivalence relation yet. Bourbaki does this in the next subsection. Care must be taken in proofs until then, because our intuition will lead us awry.

C43. $\vdash(\boldsymbol{T}=\boldsymbol{U}\wedge\boldsymbol{R}[\boldsymbol{T}])\iff(\boldsymbol{T}=\boldsymbol{U}\wedge\boldsymbol{R}[\boldsymbol{U}])$

5.1.2. Abuse of language. Bourbaki (correctly) notes we abuse language saying "$\boldsymbol{T}$ is identical with $\boldsymbol{U}$" to mean "$\boldsymbol{T} = \boldsymbol{U}$", and "$\boldsymbol{T}$ is distinct from $\boldsymbol{U}$" to mean "$\boldsymbol{T} \neq \boldsymbol{U}$". Also note we use the conventional shorthand $\boldsymbol{T} \neq \boldsymbol{U}$ to mean $\neg(\boldsymbol{T} = \boldsymbol{U})$.

References

[1] M. Abadi, L. Cardelli, P-L. Curien and J-J. Levy, "Explicit Substitutions". *Journal of Functional Programming* **1** no.4 (1991) 375–416.

[2] Wayne Aitken, "Bourbaki, Theory of Sets, Chapter I, *Description of Formal Mathematics*: Summary and Commentary". Manuscript dated June 2022, https://public.csusm.edu/aitken_html/Essays/Bourbaki/BourbakiSetTheory1.pdf

[3] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pak, and Josef Urban. "Mizar: State-of-the-art and beyond." In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of Lecture Notes in Computer Science, pages 261–279. Springer International Publishing, 2015. doi:10.1007/978-3-319-20615-8 17.

[4] Paul Bernays, "Axiomatische Untersuchungen des Aussagen-Kalkuls der *Principia Mathematica*." *Mathematische Zeitschrift* **25** (1926) 305–320; translated into English in Richard Zach's *Universal Logic: An Anthology* (2012) pp.43–58.

[5] Nicolas Bourbaki, *Theory of Sets*. Springer, softcover reprint, 1968 English translation.

[6] Claude Chevalley, "Editorial No. 139. Sets. Chapter I (state 5)." Manuscript with handwritten note dating it as July 1950. Identifier R139_nbr042, accessed January 17, 2024 http://archives-bourbaki.ahp-numerique.fr/items/show/548 — first introduced the Hilbert ε operator into Bourbaki's system.

[7] Noam Chomsky, "Three models for the description of language". *IRE Transactions on Information Theory* **2** no.3 (1956) 113–124.

[8] John Corcoran, "Schemata: The Concept of Schema in the History of Logic". *Bulletin of Symbolic Logic* **12** no.2 (2006) 219–240; doi:10.2178/bsl/1146620060

[9] Adrian RD Mathias, "The ignorance of Bourbaki". *Mathematical Intelligencer* **14** no.3 (1992) 4–13

[10] Adrian RD Mathias, "A term of length 4,523,659,424,929". *Synthese* **133** (2002) 75–86

[11] Adrian RD Mathias, "Hilbert, Bourbaki and the scorning of logic". In *Infinity and truth* (World Scientific, 2014) pp.47–156. https://www.dpmms.cam.ac.uk/~ardm/lbmkemily5.pdf

[12] Freek Wiedijk, "The Mathematical Vernacular". Unpublished manuscript, 2000 http://www.cs.ru.nl/F.Wiedijk/notes/mv.pdf

Email address: pqnelson@gmail.com