Complex Analysis with Applications Notes

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September 29, 2011

Abstract

These are notes from Dr Dmitry B. Fuchs' course (Math 185B) from Spring 2009. This covers advanced complex analysis. The "applications" are to various fields of mathematics.

NOTE: any errors are due to me, and not Dr Fuchs; and corrections are welcome.

Contents

*This is a page from https://pqnelson.github.io/notebk/
Compiled: January 31, 2016 at 4:43pm (PST)

Introduction

These are a collection of notes on complex analysis. It is mostly from the course math 185B taught spring quarter 2009, by Dmitry Fuchs. Of course, there are several peculiarities with my notes worth mentioning.

First, I do use diagrams. This tends to make most people cringe (the puritans do not like pictures). But my diagrams include more than pictures: it includes commutative diagrams! For example

$$\begin{array}{ccc} a & = & b \\ \| & & \| \\ A & = & B \end{array} \tag{0.1}$$

would be used in place of four equations

$$a = b \tag{0.2a}$$
$$b = B \tag{0.2b}$$
$$a = A \tag{0.2c}$$
$$A = B \tag{0.2d}$$

which, in my modest opinion, takes up far too much room. My diagrams litter the notes, hopefully they are useful.

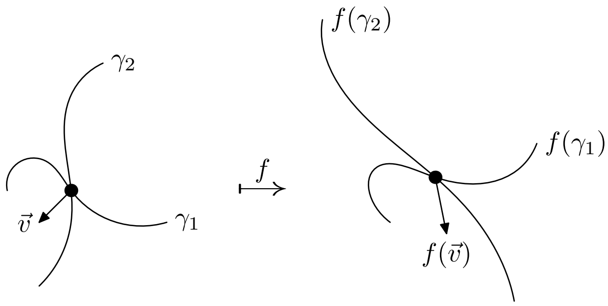
Second, this is currently written in the more “Russian” style. That is, it glosses over a lot of material giving the intuition underlying the theorems and few proofs. The idea is that the reader would be mathematically intelligent enough to supply the proofs instantly.

Lecture 1

A "**Conformal Map**", for us, is a smooth map from a plane into itself that preserves angles. Let $A \subseteq \mathbb{R}^2$, $B \subseteq \mathbb{R}^2$, and

$$f\colon A \to B \tag{1.1}$$

be smooth and bijective. Consider the following:



Additionally we have

$$\frac{\|f(\vec{v})\|}{\|\vec{v}\|} = \text{constant}. \tag{1.2}$$

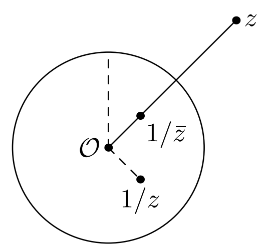
It is understood that it is an orientation preserving map, the correct term is an "orientation preserving conformal map". They precisely correspond to analytic functions of complex variables with nonzero first derivative.

In complex analysis this is equivalent to $A, B \subseteq \mathbb{C}$, we have $z \in A$ and $\omega \in B$, f be holomorphic. If

$$f'(z) \neq 0 \tag{1.3}$$

for all $z \in A$, then it is conformal.

NON-Example (Inversion)**.** We have some disc of radius R, and it has its center be $\mathcal{O}$. Inversion would be $z \mapsto 1/\bar{z}$ in the complex plane (if we make the origin $\mathcal{O}$ and dilate by $1/R$). It does not preserve orientation. In the complex plane, $f(z) = 1/z$ is a bit more interesting. This preserves angles but it *reverses* orientation. This situation is doodled on the right.



Inversion changes lines into circles if we have lines that do not pass through the origin. How can we see this? Consider

$$\gamma(t) = z_0 + z_1 t \tag{1.4}$$

where we have nonzero constants $z_0, z_1 \in \mathbb{C} - \{0\}$. Then γ is a line that does not pass through the origin. What happens when we invert it? It becomes

$$\widetilde{\gamma}(t) = \frac{1}{z_0 + z_1 t} = \frac{\bar{z}_0 + \bar{z}_1 t}{\|z_0 + z_1 t\|^2} \tag{1.5}$$

We see that the denominator is

$$\|z_0 + z_1 t\|^2 = z_0\bar{z}_0 + (\bar{z}_1 z_0 + \overline{\bar{z}_1 z_0})t + \bar{z}_1 z_1 t^2 \tag{1.6a}$$

$$= r_0 + 2\,\mathrm{Re}(\bar{z}_1 z_0)t + r_1^2 t^2 \tag{1.6b}$$

where $r_0, r_1 \in \mathbb{R}$ are positive real numbers. Observe, then, that

$$\lim_{t\to+\infty} \widetilde{\gamma}(t) = 0 \tag{1.7}$$

since it's of the form

$$\widetilde{\gamma}(t) \propto \frac{\text{const} + \bar{z}_1 t}{\text{const} + \|z_1\|^2 t^2} \tag{1.8}$$

and the denominator vanishes quicker than the numerator increases. Likewise, for precisely the same reason, we have

$$\lim_{t\to-\infty} \widetilde{\gamma}(t) = 0 \tag{1.9}$$

and

$$\widetilde{\gamma}(0) = z_0^{-1} \tag{1.10}$$

Is this convincing? Well, yes and no.

Consider the stereographic projection to the sphere. We will consider longitudinal lines on the sphere. What happens when we consider the inverse to the stereographic projection? We leave this to the reader...

1.1 Riemann Mapping Theorem

Suppose that you have two domains in the plane that are not empty and not the whole plane. Is it possible to find some conformal map from one to the other? If so, how many are there?

Theorem 1.1 (Incomplete Version Riemann Mapping)**.** *If we have two nonempty, simply connected subsets of the plane (which are not the whole plane), if we map $z_0 \mapsto f(z_0)$ and preserve orientation, then there exists a unique conformal map from one to the other.*

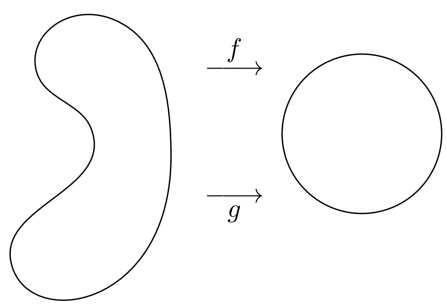
For existence, we need some extra conditions.

Lecture 2

We stated previously the first nontrivial theorem of the course:

Riemann Mapping Theorem. *If $\mathcal{U} \subset \mathbb{C}$ is a simply connected domain, and $\mathcal{U} \neq \emptyset, \mathbb{C}$, then there exists a conformal map $f \colon \mathcal{U} \to \mathcal{D}$ where $\mathcal{D} = \{z \in \mathbb{C} \mid \|z\| < 1\}$ is the unit disc. Additionally, if we fix some point $u \in \mathcal{U}$, demand it be mapped to the origin $0 \in \mathcal{D}$, and some direction specified in $\mathcal{U}$ be mapped to some direction in $\mathcal{D}$, then f is unique.*

Existence is a more delicate matter than uniqueness. The domains need not be finite (e.g., the upper half plane of $\mathbb{C}$). There are some particular cases we need to consider, e.g. when $\mathcal{U} = \mathcal{D}$.



Uniqueness can be constructed in two different forms. Consider two maps $f, g \colon \mathcal{U} \to \mathcal{D}$. We see that $h = g \circ f^{-1} \colon \mathcal{D} \to \mathcal{D}$. We also can see that together this map takes $0 \mapsto 0$. We have the Schwarz inequality $\|h(z)\| \leq \|z\|$. We can deduce this from the Cauchy integral formula. We can apply the same process to $h^{-1} = f \circ g^{-1} \colon \mathcal{D} \to \mathcal{D}$. We have the inequality $\|h^{-1}(\omega)\| \leq \|\omega\|$ where $\omega = h(z)$ and $z = h^{-1}(\omega)$. We see that this implies $\|z\| \leq \|h(z)\| \leq \|z\|$ which implies $\|h(z)\| = \|z\|$. We see then that this implies $h(z) = \lambda z$ where $\|\lambda\| = 1$ is a root of unity. In other words, h is just a rotation. By fixing the orientation we have $g \circ f^{-1} = h = \text{id}$ imply $f = g$. This is simply just from the Schwarz inequality.

There is a notion of a "**Fractional Linear Map**" where, given some $a, b, c, d \in \mathbb{C}$, we have

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = |ad - bc| \neq 0, \tag{2.1}$$

we are interested in the domain and the range of the map

$$f_A(z) = \frac{az + b}{cz + d}. \tag{2.2}$$

Observe that it is singular at $z = -d/c$, and we also see that $f_A(z) \neq a/c$.

We see that the composition of two linear fractional maps is itself a fractional linear map given by the product of the matrices.

We can consider the matrices as being unique (up to multiplication by a complex number), so lets think of matrices with determinant of 1. We should make them up to a sign, this is

$$\mathrm{SL}(2, \mathbb{C})/\{\pm 1\} \cong \mathrm{PSU}(2). \tag{2.3}$$

We want to say now that every fractional linear transformation maps circles and straight lines into circles and straight lines.

Theorem 2.1. *Fractional linear transformations preserves circles and lines.*

We can explicitly write

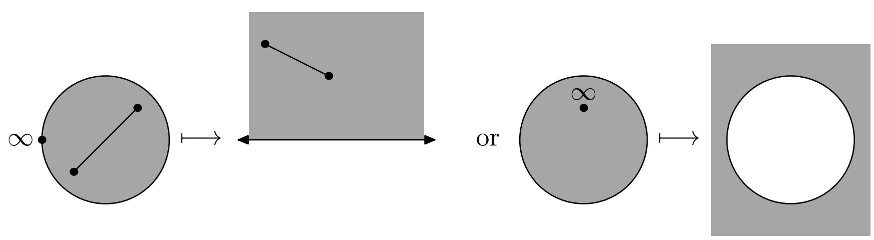
$$f(z) = \frac{az+b}{cz+d} = \frac{1}{c}\left(a + \frac{bc-ad}{cz+d}\right) \tag{2.4}$$

by considering the following:

$$\begin{aligned}
f_1(z) &= cz \\
f_2(z) &= z + d \\
f_3(z) &= 1/z \\
f_4(z) &= (bc - ad)z \\
f_5(z) &= a + z \\
f_6(z) &= \frac{1}{c} z \quad \text{assuming } c \neq 0.
\end{aligned}$$

Each of these maps—with one exception—are just translations and multiplication by complex numbers. These preserves circles and straight lines. The inversion operation, $f_3(z)$, after some thinking can be shown to preserve "generalized circles"...that is to say, straight lines become circles and circles become straight lines.

Consider the unit disc. What does a "fractional linear map" map it to? Well, there are several possibilities, doodled below. The boundary can be mapped to a line or a unit circle, and the contents may be mapped to several places.



It depends on where the point mapped to infinity lives. It can live on the boundary. If the singularity of the map is on the boundary, the circle is mapped to a line, and two distinct points on the interior of the disc is mapped to the same side of the line, so the disc is mapped to the plane. If the singularity is on the interior of the disc, the disc is mapped to the exterior of the boundary's mapping in the range. The last case of interest is if the point mapped to infinity is outside the disc, then the disc is mapped to a disc.

Take the map

$$f(z) = \frac{z-a}{1-z\bar{a}}\lambda \tag{2.5}$$

where $\|a\| < 1$ and $\|\lambda\| = 1$. This maps $D \to D$, and covers all we need for the map to exist by the Riemann mapping theorem. We see first that $f(a) = 0$. We see that

$$f'(z) = \lambda\left(\frac{1 - \bar{a}z + \bar{a}(z-a)}{(1-\bar{a}z)^2}\right) \tag{2.6a}$$

$$= \lambda \left(\frac{1 - \bar{a}a}{(1 - \bar{a}z)^2} \right) \tag{2.6b}$$

Observe that

$$f'(0) = \lambda(1 - a\bar{a}) \tag{2.7}$$

So for some orientation $+\bar{\lambda}$, we have it be mapped to the positive real direction.

If $\|z\| = 1$, then

$$\begin{aligned}
\|f(z)\| &= 1 && (2.8a)\\
&= \|f(z)\bar{z}\| && (2.8b)\\
&= \|\lambda\| \cdot \left\| \frac{z\bar{z} - \bar{z}a}{1 - \bar{a}z} \right\| && (2.8c)\\
&= \|\lambda\| \cdot \left\| \frac{1 - \bar{z}a}{1 - \bar{a}z} \right\| && (2.8d)\\
&= \|\lambda\| = 1 && (2.8e)
\end{aligned}$$

How interesting.

Lecture 3

We will discuss the scheme of two proofs of existence. Given some $\mathcal{U} \subset \mathbb{R}^2$ that is simply connected, $\mathcal{U} \neq \mathbb{R}^2$ and $\mathcal{U} \neq \emptyset$, then there exists some function

$$f : \mathcal{U} \xrightarrow{\cong} D \tag{3.1}$$

is conformal (where D is the open unit disc in $\mathbb{C}$), or holomorphic with f' nonvanishing. We fix some $a \in \mathcal{U}$.

Consider the set of maps

$$S = \{f : \mathcal{U} \to D \text{ is conformal } \mid f(a) = 0, f'(a) > 0\} \tag{3.2}$$

First we see that S is nonempty.

We suppose that $0 \notin \mathcal{U}$. Since $\mathcal{U}$ is simply connected, for each circle around o we have some point on the circle not in $\mathcal{U}$ since $\mathcal{U}$ is simply connected we can contract it.

We consider a branch of $\sqrt{-} : \mathcal{U} \to \mathbb{R}^2$, namely we see that $\sqrt{-}(\mathcal{U})$ contains some disc. If $\omega \in \sqrt{-}(\mathcal{U})$, then $-\omega \notin \sqrt{-}(\mathcal{U})$.

For every $f \in S$, $f'(a) \in \mathbb{R}_+$ is bounded. There exists some $M > 0$ such that $f(a) < M$ for all $f \in S$ (i.e., it's bounded).

Now we use a standard trick in analysis. Let

$$M = \sup\{f'(0) \mid f \in S\} \tag{3.3}$$

If we have some sequence $\{f_i\} \subset S$, we can find some convergent subsequence. We wish to deduce that $\exists f \in S$ such that $f'(a) = M$. This is our f that maps $\mathcal{U}$ to D injectively.

Suppose that $B \subset D$, $0 \in B$ and $B \neq D$ is simply connected. We claim that there is a conformal map $g : B \to D$, where $f(A) = B$. Then $g \circ f : A \to D$ is itself conformal.

Suppose we have a domain $\mathcal{U}$ which we do not demand to be simply connected. For every continuous function on $\partial\mathcal{U}$, there is an analytic function in $\mathcal{U}$; i.e.,

$$\begin{aligned}
&\forall h : \partial\mathcal{U} \to \mathbb{R}\\
&\exists u : \bar{\mathcal{U}} \to \mathbb{R} \quad \text{such that} \quad h = u|_{\partial\mathcal{U}}
\end{aligned} \tag{3.4}$$

We have

$$h(a) = \ln|z - a|. \tag{3.5}$$

Now this u may be supplemented with v such that $u + \mathrm{i}v$ is holomorphic. Then we take

$$f(z) = (z-a)\exp\big(-(u+\mathrm{i}v)\big) \tag{3.6}$$

Our claim is that this is our function, it bijectively maps $\mathcal{U}$ to D.

If $z \in \partial\mathcal{U}$, then $\|f(z)\| = 1$. We see this by direct computation:

$$\begin{aligned} \|f(z)\| &= \|z-a\| \cdot \|e^{-u(a)}\| && \text{(3.7a)} \\ &= \frac{\|z-a\|}{\|z-a\|} && \text{(3.7b)} \\ &= 1 && \text{(3.7c)} \end{aligned}$$

The only thing that requires work is that $f'(z) > 0$, or at least nonzero. We cannot use any information of the boundary of $\mathcal{U}$.

Is it possible to extend the Riemann mapping theorem from $\mathcal{U} \xrightarrow{\cong} D$ to $\bar{\mathcal{U}} \xrightarrow{\cong} \bar{D}$? Not necessarily. It is possible if $\partial\mathcal{U}$ is a continuous closed curved.

If $\mathcal{U}$, $\mathcal{V}$ are simply connected and bounded domains, then what? Well, suppose we have a conformal map

$$f\colon \bar{\mathcal{U}} \to \mathbb{R}^2 \tag{3.8}$$

such that $f|_{\mathcal{U}}$ is conformal, and additionally that $f(\partial\mathcal{U}) = \partial\mathcal{V}$. Then

$$f\colon \mathcal{U} \to \mathcal{V}. \tag{3.9}$$

It is sufficient to let $\mathcal{V} = D$, then

$$f\colon \bar{\mathcal{U}} \to D \tag{3.10}$$

and $z \in \bar{\mathcal{U}}$ implies $\|f(z)\| \leq 1$.

Let us consider conformal maps between D and the upper half plane. Consider

$$z \mapsto \mathrm{i}\left(\frac{-z+1}{z+1}\right) \tag{3.11}$$

which maps $1 \mapsto 0$, $-1 \mapsto \infty$, $\mathrm{i} \mapsto -1$, and indeed we can note

$$\mathrm{i}\left(\frac{-z+1}{z+1}\right) = \mathrm{i}\frac{1-\|z\|^2}{\|1+z\|^2} \tag{3.12}$$

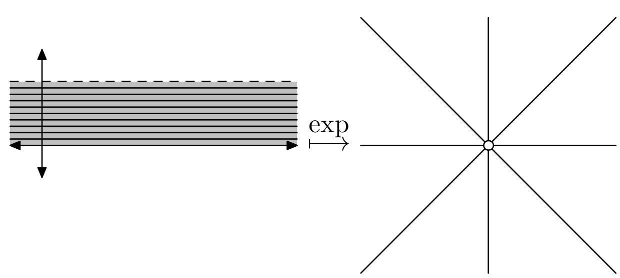
Lecture 4

It is sometimes important to give conformal equivalence between two domains. There are some books on the subject of equivalence of conformal domains.

Example 4.1. One of the important functions that is a conformal mapping is the exponential mapping

$$f(z) = \mathrm{e}^z. \tag{4.1}$$

It maps $\mathbb{R} \times [0, 2\pi) \subset \mathbb{C}$ to $\mathbb{C} - \{0\}$. We shade the domain in gray, and consider how horizontal lines behave under this mapping:



Note that horizontal lines are mapped to lines, and vertical lines are mapped to circles.

The line yi (that is, $z(t) = t$i which is purely imaginary) is mapped to the unit circle, the lines with $x < 0$ are mapped to concentric circles. The vertical lines (to the right) are mapped to circles with radius $\exp(x)$. So the region

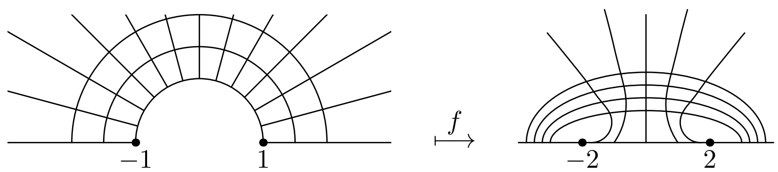
$$\{(x, y) \in \mathbb{C} \mid x \geq 0, \quad 0 \leq y \leq 2\pi\}$$

is mapped to the region *outside* the unit circle, since $\exp(x) \geq \exp(0) = 1$.

Example 4.2. Consider

$$f(z) = z + \frac{1}{z} \tag{4.2}$$

How does this behave? Lets consider a nice subdomain of the upper half plane:



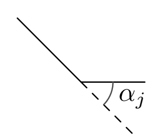
Wee see that $f(1) = 2$ and $f(-1) = -2$ by direct computation; when $\|z\| = 1$, we see that $1/z = \exp(-i\theta)$ and we have more generally $1/z = \bar{z}$. Thus $f(z) = z + \bar{z} = 2\,\mathrm{Re}(z)$. So we see that this boundary is mapped to the real line. We see that this maps the domain to the upper half-plane. But to make things interesting, we cut up the domain in the manner we have doodled. The circular arcs are mapped to elliptical arcs. The lines are mapped to hyperbolas. We have a family of ellipses given by the relation

$$\frac{x^2}{a + t} + \frac{y^2}{b + t} = 1 \tag{4.3}$$

where $t \in \mathbb{R}$ is "some parameter". There are places where it is badly behaved, but that's okay. We also have imaginary ellipses when $t < a$ or $t < b$.

There are many beautiful properties of confocal family (conformal foci family) which we will not pursue here. I believe my good friend, Dmitry B. Fuchs, has beautifully examined it in *Mathematical Omnibus: Thirty Lectures on Classic Mathematics*. Additionally, Serge Tabachnikov's *Geometry and Billiards* (American Mathematical Society, 2005) is a good resource.

There is one more transformation to be considered which is fairly beautiful, so we will consider it. Suppose we have a convex n-gon (n sided polygon), there is a conformal map from the upper half plane to this n-gon.



Let p be a convex n-gon with exterior angle $\alpha_1\pi, \ldots, \alpha_n\pi$ as doodled to the right. We see that $\alpha_1 + \cdots + \alpha_n = 2$, and $0 < \alpha_j < 1$. Consider the following: fix $n - 1$ points (on the real line) $x_1, \ldots, x_{n-1}$ ordered from left to right with distances left unspecified.

Consider

$$f(z) = b + a \int_{z_0}^{z} (\xi - x_1)^{-\alpha_1} (\ldots)(\xi - x_{n-1})^{-\alpha_{n-1}} \,\mathrm{d}\xi \tag{4.4}$$

where $a, b \in \mathbb{C}$. Why consider only $(n - 1)$ such α's? Well, the n^{th} is determined completely by our relations above.

When we consider $f(z)$ on the real line, what happens when ξ approaches, e.g., x_1? Does this converge or diverge as an improper integral? Well, since we have

$$0 < \alpha_j < 1 \tag{4.5}$$

we see the integral converges since

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2 \tag{4.6}$$

is finite. We can change variables from ξ to $\xi - x_j$, and we get something like our integral in Eq (4.6). Now, it really looks like

$$f(z) \propto \int_{z_0}^{z} \xi^{-2+\alpha_n}\mathrm{d}\xi \tag{4.7}$$

and this converges. So this function $f(z)$ is defined at the dangerous points $x_1, \dots, x_n$ and $\pm\infty$. Lets set $b = 0$ and $a = 1$ for now.

The fundamental theorem fo calculus says

$$f'(z) = (\xi - x_1)^{-\alpha_1}(\dots)(\xi - x_{n-1})^{-\alpha_{n-1}} \tag{4.8}$$

Let $x_i < z < x_{i+1}$. We are interested in

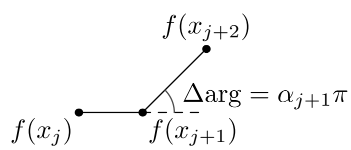
$$\arg\big(f'(z)\big) = ? \tag{4.9}$$

We see for real numbers $\arg(x) = 0$. By direct computation we find

$$\arg\left(f'(z)\right) = (-\alpha_{j+1} - \dots - \alpha_{n-1})\pi \tag{4.10}$$

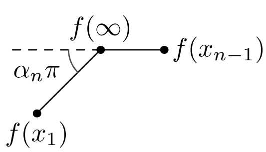
This means that the direction is constant.

We see that as we move along the real axis, the argument changes in "discrete chunks". If $z < x_1$, then what may we say about $\arg(f'(z))$? Well well well, we see that



$$\arg(f'(z)) = (-\alpha_1 - \dots - \alpha_{n-1})\pi = (2 - \alpha_n)\pi = -\alpha_n\pi \tag{4.11}$$

The last step is because $2\pi \equiv 0$ since we mod out by 2π.



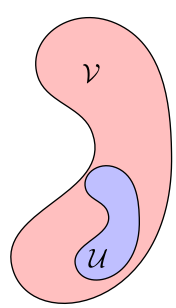
We have no x_n on the real axis specified because we have $f(\infty)$ "=" $f(x_n)$, so we have in effect what is doodled on the left. We have the identification $f(\infty) = f(-\infty)$.

What did we just do? We verified that the boundary is mapped to the boundary, but what about the *interior*?

Consider the rectangle. As a holomorphic function (obtained by our construction) we get the elliptic integral.

Remark 4.3 (Further Reading). For more on this, see Stein [5], chapter 8 §4 "Conformal mappings onto polygons."

Lecture 5: Analytic Continuation



Applications of conformal maps to partial differential equations will not be covered. Instead we will skip ahead to ANALYTIC CONTINUATION! We have a domain $\mathcal{U}$ and a domain $\mathcal{V}$. We have a function $f\colon \mathcal{U} \to \mathbb{C}$.

We wish to extend f to $g\colon \mathcal{V} \to \mathbb{C}$. That is, when we restrict g to $\mathcal{U}$ we recover f. The natural questions that arise are: does such an extension exist, and if so is it unique?

The famous example is the Riemann zeta conjecture: the real part of the nontrivial zeroes of the zeta function is 1/2, where the zeta function is

ζ function; Riemann conjecture

$$\zeta(z) = \sum_{n=1} n^{-z}. \tag{5.1}$$

Note it converges for $\mathrm{Re}(z) > 1$. Also observe for $z = 1$ we have the divergent Harmonic series.

Consider the Gamma function *Gamma function*

$$\Gamma(\mu) = \int_0^\infty x^{\mu-1}\mathrm{e}^{-x}\mathrm{d}x, \tag{5.2}$$

if $\mu < 0$ we are in trouble. We will consider $\mathrm{Re}(\mu) > 0$. Using this, we can express the zeta function as

$$\zeta(\mu) = \frac{1}{\Gamma(\mu)}\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1}\mathrm{d}x \tag{5.3}$$

Nifty!

Lemma 5.1. *Let $f, g\colon \mathcal{U} \to \mathbb{C}$ (where $\mathcal{U}$ is connected) be analytic. Let $\mathcal{V} \subset \mathcal{U}$, $f|_{\mathcal{V}} = g|_{\mathcal{V}}$, and $\mathcal{V}$ be open and nonempty. Then $f = g$.*

Lemma 5.2 ("Sublemma")**.** *For the same $f, g, \mathcal{U}$. Suppose that we have a sequence $z_i \in \mathcal{U}$, $z_i \neq z_j$ if $i \neq j$, suppose*

$$\lim_{i\to\infty} z_i = z_0 \in \mathcal{U}. \tag{5.4}$$

If $f(z_i) = g(z_i)$ for all $i \in \mathbb{N}$, then $f = g$.

Proof of Sublemma. Let $f - g =: h$. Then h is also analytic. We take

$$h(z) = a(z - z_0)^k + \dots \tag{5.5}$$

then it follows

$$|a - \varepsilon| \cdot \|z - z_0\|^k < \|h(z)\| < |a + \varepsilon| \cdot \|z - z_0\|^k \tag{5.6}$$

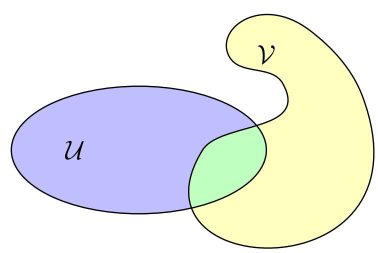
in a small neighborhood of z_0. So $\|h(z)\| > 0$, a contradiction, $h(z) \neq 0$ in the neighborhood with the point z_0 removed. □

Proof of Lemma. Let

$$\mathcal{V} = \{z_0 \in \mathcal{U} \mid f(z) = g(z) \text{ in some neighborhood of } z_0\}. \tag{5.7}$$

Well then, $\mathcal{V}$ is open (it's obvious), and it follows from the sublemma that $\mathcal{V}$ is closed: it contains its boundary points. So either $\mathcal{V}$ is $\mathcal{U}$ or $\emptyset$, but by hypothesis $\mathcal{V} \neq \emptyset$. □

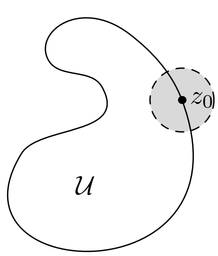
Suppose we have two functions $f\colon \mathcal{U} \to \mathbb{C}$ and $g\colon \mathcal{V} \to \mathbb{C}$. We demand that $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, we do not demand simply connectedness. Suppose also that the functions agree on the overlap, i.e. $f|_{\mathcal{U}\cap\mathcal{V}} = g|_{\mathcal{U}\cap\mathcal{V}}$. Now if we "combine" these two functions, we get an analytic function in $\mathcal{U} \cap \mathcal{V}$.



Let

$$h(z) = \begin{cases} f(z) & \text{if } z \in \mathcal{U} \\ g(z) & z \in \mathcal{V} \end{cases} \tag{5.8}$$

Our lemma implies if $\mathcal{U}$, $\mathcal{V}$ are connected, then f determines g.



Suppose again we have some function $f\colon \mathcal{U} \to \mathbb{C}$ and let $z_0 \in \partial\mathcal{U}$. We wish to define a function on a small neighborhood $B_\varepsilon(z_0)$ of z_0 such that $g\colon B_\varepsilon(z_0) \to \mathbb{C}$ such that it agrees with f on the overlap: $f|_{B\cap\mathcal{U}} = g|_{B\cap\mathcal{U}}$. If no such g exists, it is impossible to extend f to any $\mathcal{V} = \mathcal{U} \cup B_\varepsilon(z_0)$. On the other hand, if g exists, we can extend f to this new domain.

If f is so badly behaved that f is not defined on the boundary of $\mathcal{U}$, it cannot be extended at all.

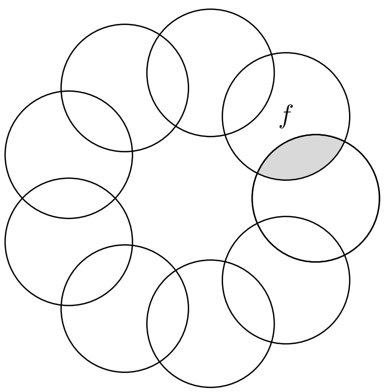
Example 5.3. Consider the function

$$f(z) = \sum_{n=0}^{\infty} z^{n!} \tag{5.9}$$

it is defined on the unit disc in $\mathbb{C}$, it is analytic, but it cannot be extended and it is ill defined on the boundary.

Remark 5.4. For several complex variables, we cannot abuse any knowledge of a single complex variable concept.

Of course any domain of interest is conformally equivalent to the unit disc, and this interesting open domain will (by the Riemann mapping theorem) always have a nonextendable function on the boundary.



Consider the function $f(z) = \sqrt{z}$, then $f(z)^2 = z$. If we extend it on some patchwork, we go around this patch work and end up in the shaded region on our original domain. This may be seen as doodled on the left. However, the resulting extension *is not our beloved f!* We do not end up with a function with a domain in the plane, no! We see this is a Riemann surface!

Aside. Riemann surfaces are fairly interesting as a subject. Note that for the most part, we will be working with discs (which are "the same" as the upper half complex plane). They are homotopically equivalent to spheres with finitely many punctures. So, in short, all we care about is how do we glue together the discs from analytically continuing a given function? Provided, of course, that it does not "close" under continuation.

Lecture 6

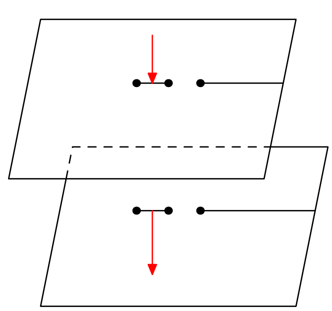
I was sick for this lecture and missed it. It covered Riemann surfaces, and introduced terminology (the family of discs which constitute a Riemann surface, we will call them "**Relatives**" and if they overlap they are "**Close Relatives**" —note this is our own terminology, not found in the literature). For a good reference, see Teleman's notes: `http://math.berkeley.edu/~teleman/math/Riemann.pdf`

Lecture 7

We discussed the Riemann surface of

$$h(z) = \sqrt{(z-1)z(z+1)}. \tag{7.1}$$

This is doodled thus:



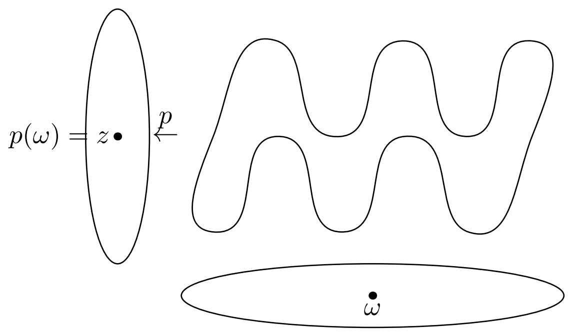
Consider

$$\omega^3 - \omega + z = 0, \tag{7.2}$$

we see that

$$z = \omega - \omega^3, \tag{7.3}$$

so we may write $z = z(\omega)$. We can invert this to find $\omega = \omega(z)$. More generally we can suppose that $p(\omega) = z$ and its inverse is $q(z) = \omega$. We can now consider the Riemann surface of this function. Consider the graph of this function (roughly doodled below to the left).



The inverse function has several values, so we get in this complex analogue a Riemann surface. From the projection, which is multivalued, we have the Riemann surface. In this case for $h(z)$, the Riemann surface is homeomorphic to a torus.

The reader should make a mental note on the importance of branch cuts in this method of constructing Riemann surfaces. Also note that we are projecting onto *Riemann spheres* which are distinguished from the notion of *Riemann surfaces.* The Riemann sphere is $\mathbb{C}$ as a sphere, obtained from Stereographic projection.

Proposition 7.1 (Fact from geometry)**.** *An orientable, closed, compact surface is homeomorphic to a torus (or more generally a sphere with p handles).*

How do we find the genus of a Riemann surface? (I.e., how do we find the value of p?) We have n roots, we count how many times we glue points together. We subtract the number of boundary points of the cuts. So we have

$$\chi = 2n - \#(\text{boundary points of the cuts}) \tag{7.4}$$

which is precisely the Euler characteristic. The genus is

Euler Characteristic of Riemann Surface

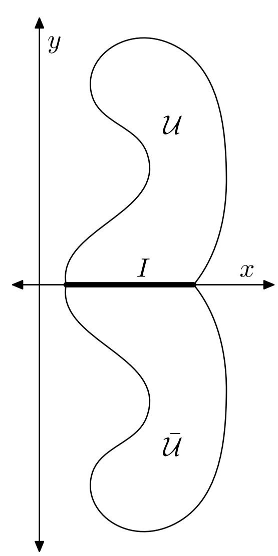
$$\frac{2-\chi}{2} = \text{genus}. \tag{7.5}$$

This is for polynomials, however.

Riemann surfaces are defined for algebraic functions. Consider the famous example of the logarithm function, it covers the complex plane infinitely many times. When we consider the Riemann surface, it's like an infinite Helix. This is the logarithmic staircase. See Penrose's *Road to Reality* for a good doodle.

7.1 Reflection Principle

We are nonetheless interested in extending functions. We have the Reflection principle. We have some domain which contains on the boundary part of the x (real) axis. We consider some function $f\colon \mathcal{U} \to \mathbb{C}$, we extend f to another function $\widetilde{f}$ on $\mathcal{U} \cup I$ where I is the real part of $\partial\mathcal{U}$, i.e., $I = \mathbb{R} \cap \partial\mathcal{U}$. We demand that $\widetilde{f}$ be continuous, and demand that $\widetilde{f}|_I$ be real. We consider the complex conjugation of $\mathcal{U}$, doodled to the right, which is $\bar{\mathcal{U}} = \{\bar{z} \mid z \in \mathcal{U}\}$. We introduce a function $\widehat{f}$ such that



$$\widehat{f}(z) = \begin{cases} f(z) & \text{if } z \in \mathcal{U} \\ \overline{f(\bar{z})} & \text{if } z \in \bar{\mathcal{U}} \\ \widetilde{f}(z) & \text{if } z \in I. \end{cases}$$

We have then $\mathcal{V} = \mathcal{U} \cup I \cup \bar{\mathcal{U}}$, and we see that $\widehat{f}$ is continuous on $\mathcal{V}$. We see that if $\widehat{f}$ is analytic on $\mathcal{U}$, then $\widehat{f}$ is analytic on $\bar{\mathcal{U}}$.

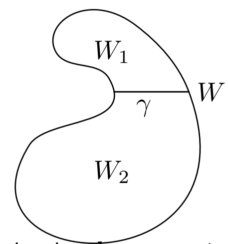
Let $z = x + \mathrm{i}y \in \mathcal{U}$,

$$\widehat{f}(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y) \tag{7.6}$$

and for $\overline{z} \in \bar{\mathcal{U}}$ we see we have

$$\overline{f(\overline{z})} = u(x, -y) - \mathrm{i}v(x, -y) \tag{7.7}$$

By the chain rule we see that $\overline{f(\overline{z})}$ satisfies the Cauchy-Riemann equations.



A General Statement. We will use the diagrams doodled on the right for reference. Consider $\varphi\colon W \to \mathbb{C}$ such that φ is continuous on W. If $\varphi|_{W_1}$ and $\bar{\varphi}|_{W_2}$ are analytic, then φ is analytic on W.

Lemma 7.2. *Suppose we have in some domain $\mathcal{U}$ a continuous function*

$$f\colon \mathcal{U} \to \mathbb{C}.$$

Let $\gamma\colon I \to \mathcal{U}$ be a continuous, closed curve, and for any such γ that is simply connected, i.e.,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Then the function is analytic.

We see that the integral over a closed, simply connected path is zero if its contained entirely in W_1 or W_2. We just need to check for a path that crosses γ, we just treat it by breaking it up into pieces.

Homework 1

▸ **EXERCISE 1**

Prove that if a conformal map of a domain $\mathcal{U}$ in $\mathbb{R}^2$ takes straight lines parallel to the x axis into parallel lines, then it takes any straight lines into straight lines and circles into circles.

▸ **EXERCISE 2**

Prove that if a conformal map of a domain $\mathcal{U}$ in $\mathbb{R}^2$ takes straight lines into straight lines, then, at least locally, f coincides with a similarity map (a combination of a rotation and a dilation). (Hint: triangles.)

▸ **EXERCISE 3**

Prove that if a smooth map $f\colon \mathcal{U} \to \mathcal{V}$ between domains in the plane preserves perpendicularity (that is, if two curve, γ, γ' in $\mathcal{U}$ intersect each other under a right angle, then so do their images), then it is conformal.

▸ **EXERCISE 4**

Let u, v, w be three arbitrary different points in $\mathbb{C} \cup \{\infty\}$. Prove that for any three different points u, v, $w \in \mathbb{C} \cup \{\infty\}$ there exists a unique fractional linear transformation f such that $f(u) = u$, $f(v) = v$, $f(w) = w$. [Hint One: it is sufficient to consider $u = 0$, $v = 1$, $w = \infty$. Hint Two: cross-ratio (Page 331 of the book); Hint Two is helpful, but not necessary.]

▸ **EXERCISE 5**

Prove that (a) a fractional linear transformation must have at least one fixed point;
(b) if a fractional linear transformation has no finite (different from ∞) fixed points, then it is a translation, $f(z) = z + b$.

▸ **EXERCISE 6**

Let $f(z) = z + \frac{1}{z}$. Prove that f takes concentric circles $\|z\| = R$, $R \geq 1$ and straight rays $z = t\alpha$, $\|\alpha\| = 1$, $t \geq 1$ into ellipses and hyperbolas with the foci $(2, 0)$ and $(2, 0)$. See the picture. (Comment. This is an “honest” picture, the figure on the right is obtained by the given transformation from the figure on the left.)

Lecture 8

A confocal family of conics — every point lies in a hyperbola and an ellipse which are perpendicular to each other.

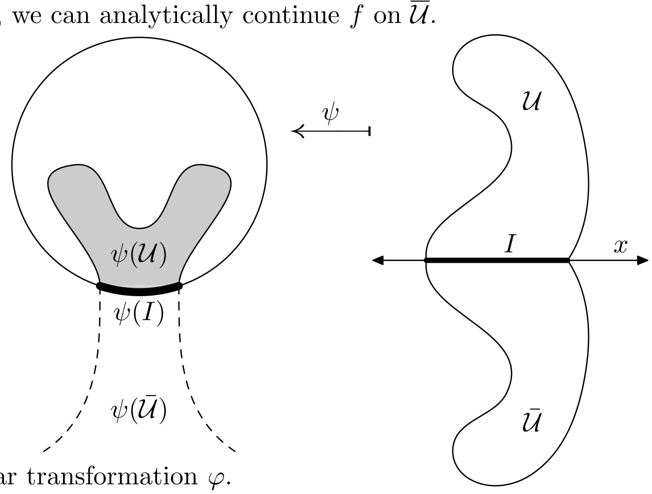
Now, let us continue considerations of the reflection principle. If $f\colon \mathcal{U} \to \mathbb{C}$ is analytic, and just to explicitly reiterate the notion of reflection we have

$$\overline{\mathcal{U}} = \{\overline{z} \mid z \in \mathcal{U}\} \tag{8.1}$$

be a reflection of $\mathcal{U}$, f be real on I, we can analytically continue f on $\overline{\mathcal{U}}$.

We can similarly do this on the unit circle, how? Well, we can use one of our beloved conformal mappings ψ which maps the region $\mathcal{U}$ to a domain $\psi(\mathcal{U})$ in the unit circle with part of its boundary $\psi(I)$ on the boundary of the unit circle. Then f can be inverted. Let the domain of the inversion be $\mathcal{V}$. Let i inverse be in the circle, j inverse be in the image. So

$$f(z) = jf(i(z)). \tag{8.2}$$

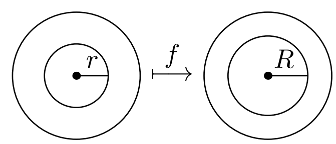


We may construct a fractional linear transformation φ.

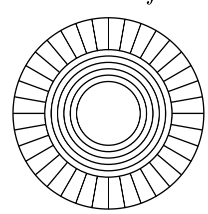
We simply replace f by $\psi^{-1} \circ f \circ \varphi$. We generalize the reflection principle to circles.

And Now, For Something Completely Different

The Riemann mapping theorem may be generalized to non-simply connected regions. We can extract it from the circle reflection principle. How to do this witchcraft? Well, suppose we have 2 discs, and inside 2 subdiscs of radii r and R. Then we have a conformal mapping from one to the other if $r = R$. The boundary circles are mapped to the boundary circles; and moreover f is a rotation.



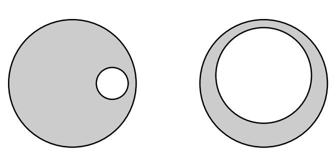
We can see from inversion as doodled to the right takes the annulus of radially shaded domain to the angularly shaded domain. We can iterate over and over again. We end up with $D - \{0\}$ — the unit disc missing the origin. Well, to be more precise, we do not invert the annulus: we invert the inner disc of radius r or R, we get the result doodled on the right. We have a holomorphic map



$$f\colon D - \{0\} \to D - \{0\} \tag{8.3}$$

which is bijective. We interpret this as mapping the center to the center, so we have a conformal map. We notice that the only undetermined property is how the mapping treats orientation, but since it is conformal then all orientation is preserved. The only such map is a rotation! Quite ingenious to say $f\colon D \to D$ such that $0 \mapsto 0$.

What happens if we have a conformal map that is not concentric (i.e., the domain is not a concentric annulus)? It becomes more of a challenge to prove that the domains are conformal. Some examples of these would look like:



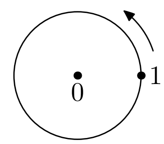
8.1 Argument Principle

We will consider several theorems whose name change but results remain the same. (I believe Led Zeppelin had a song with this title!) Suppose we have some domain, and we have some function that is not zero on the boundary of the domain and there are no poles on the boundary. So all poles are inside the domain. We have the function be meromorphic. We compute:

$$\begin{aligned} Z &= \text{number of zeroes} \\ P &= \text{number of poles} \end{aligned} \tag{8.4}$$

If we travel f along the boundary, when we return to our departure point $\|f\|$ is the same but the argument differs by $2\pi k$ ($k \in \mathbb{Z}$).

We taake the unit disc, the function $f(z) = z$. We start at 1. We travel along the boundary and when we get back to the point of departure we find



$$\Delta \arg(f(z)) = 2\pi \cdot 1. \tag{8.5}$$

In general we let γ be the boundary of the domain. We have, then, in general

$$\Delta_\gamma \arg(f(z)) = 2\pi(Z - P) \tag{8.6}$$

If we know the behavior of the function on the boundary, we have a good clue to the number of zeroes. Also note, if we were to move *clockwise* then our formula changes to

$$\Delta_\gamma \arg(f(z)) = 2\pi(P - Z) \tag{8.7}$$

instead. Note the difference between eq (8.6) and (8.7).

Theorem 8.1. *Suppose we have a meromorphic function $f\colon \mathcal{U} \to \mathbb{C}$, and $\mathcal{U}$ is our domain, and also suppose f has zeroes and poles. Consider a closed curve γ in this domain not passing through zeroes or poles. Then*

$$\Delta_\gamma \arg(f(z)) = 2\pi(Z - P) \tag{8.8}$$

holds for our oriented closed curve γ.

Theorem 8.2. *We have*

$$\int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi \mathrm{i} \left(\sum_{\textit{zeroes } a} I(\gamma, a) - \sum_{\textit{poles } b} I(\gamma, b) \right) \tag{8.9}$$

Observe that this integral is the same as

$$\int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \int_\gamma \mathrm{d}\left(\log\left(f(z)\right)\right). \tag{8.10}$$

Remark 8.3. This second theorem implies the first. The second theorem follows directly from the residue theorem.

Lecture 9

We have an analytic function $f\colon \mathcal{U} \to \mathbb{C}$ (or more precisely meromorphic), $\mathcal{U}$ is simply connected, and γ is an oriented closed curve in $\mathcal{U}$ that doesn't pass through poles or zeroes, we stated

A meromorphic map is $f\colon \mathcal{U} \to \mathbb{C} \cup \{\infty\}$, or the ratio of two holomorphic functions.

$$\int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi \mathrm{i} \left(\sum_{\text{zeroes } a} I(\gamma, a) - \sum_{\text{poles } b} I(\gamma, b) \right) \tag{9.1}$$

Remark 9.1. Remember γ does not pass through any zeroes or poles of f.

For a point z_0, we have "locally"

$$f(z) = (z - z_0)^k \varphi(z) \tag{9.2}$$

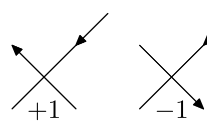
for some nonzero $k \in \mathbb{Z}$ and an analytic $\varphi(z)$ with neither pole nor zero at z_0. For $k > 0$, we see that z_0 is a zero of f, and for $-k < 0$ that f has a pole at z_0. So we can see that

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{k(z-z_0)^{k-1}\varphi(z) + (z-z_0)^k \varphi'(z)}{(z-z_0)^k \varphi(z)} && \text{(9.3a)} \\ &= \frac{k}{z - z_0} + \frac{\varphi'(z)}{\varphi(z)} && \text{(9.3b)} \end{aligned}$$

We see that k is the residue of $f'(z)/f(z)$ at z_0, so by our theorem we have

$$\int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi\mathrm{i} \left(\sum_{\substack{\text{distinct} \\ \text{zeroes} \\ a}} \mu(a) I(\gamma, a) - \sum_{\substack{\text{distinct} \\ \text{poles} \\ b}} \mu(b) I(\gamma, b) \right) \tag{9.4}$$

For the winding path, we have the following rule for each intersection of the curve:



We have this integral also be, by the fundamental theorem of calculus,

$$\int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \int_\gamma \mathrm{d} \log(z) \tag{9.5}$$

but beware! The logarithm is *multivalued*. We see that

$$\log(\omega) = \log \|\omega\| + \mathrm{i} \arg(\omega). \tag{9.6}$$

So we use a notation

$$\begin{aligned} \int_\gamma \mathrm{d}\log(f(z)) &= \Delta_\gamma \underbrace{\log \|f(z)\|}_{\text{no change if } \gamma \text{ is closed, so it's } 0} + \mathrm{i}\Delta_\gamma \arg(f(z)) \\ &= \mathrm{i}\Delta_\gamma \arg(f(z)) \end{aligned} \tag{9.7}$$

We can then write

$$\begin{aligned} \mathrm{i}\Delta_\gamma \arg(f(z)) &= 2\pi\mathrm{i}(\textstyle\sum I(\gamma, a) - \sum I(\gamma, b)) \\ \implies \Delta_\gamma \arg(f(z)) &= 2\pi(\textstyle\sum I(\gamma, a) - \sum I(\gamma, b)) \end{aligned} \tag{9.8}$$

We can write

$$\gamma \colon I \to \mathbb{C} \tag{9.9}$$

which is γ given parametrically, then write $f \circ \gamma$ or $f(\gamma(t))$. If we integrate over I we get back 2π times a number. We can write, then,

$$\begin{array}{ccc} \Delta_\gamma \arg(f(z)) & = & 2\pi(\sum I(\gamma, a) - \sum I(\gamma, b)) \\ \| & & \\ 2\pi I(f \circ \gamma, 0) & & \end{array} \tag{9.10}$$

If γ is a simple curve, we can write

$$I(f \circ \gamma, 0) = \begin{pmatrix} \text{zeroes} \\ \text{within} \\ \text{the bdry} \end{pmatrix} - \begin{pmatrix} \text{poles} \\ \text{within} \\ \text{the bdry} \end{pmatrix} \\ = Z - P \tag{9.11}$$

Recall a simple curve is any nonintersecting curve.

9.1 Rouché's Theorem

Suppose that in a simply connected domain $\mathcal{U}$ we have meromorphic functions

$$f, g \colon \mathcal{U} \to \mathbb{C} \tag{9.12}$$

let

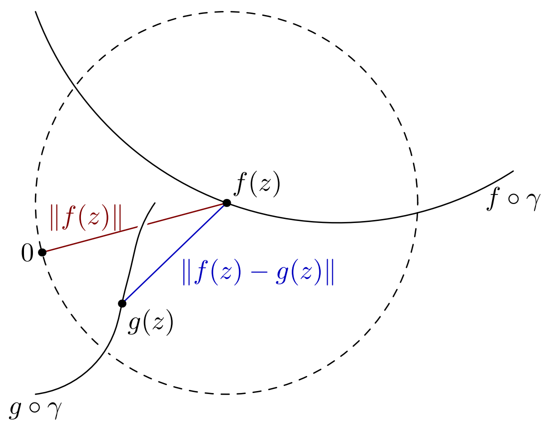
$$\|f(z) - g(z)\| < \|f(z)\| \quad \forall z \in \gamma \tag{9.13}$$

where γ is an oriented closed curve in $\mathcal{U}$. Then

$$\Delta_\gamma \arg(f) = \Delta_\gamma \arg(g) \tag{9.14}$$

is our claim.

What this means is $f \circ \gamma$ has a curve in $\mathbb{C}$ and for some $f(z) \in f \circ \gamma$, we have $g(z)$ in a disc of radius $\|f(z)\|$. We wish to argue that the distance from $f(\gamma(t))$ to $g(\gamma(t))$ is less than the distance from $f(\gamma(t))$ to the origin. So we have the inequality $\|f(\gamma(t)) - g(\gamma(t))\| < \|f(\gamma(t))\|$. This situation is doodled to the right, where the dark red line indicates the line from $f(\gamma(t))$ to the origin, and the dark blue line indicates the distance from $f(\gamma(t))$ to $g(\gamma(t))$.



Now every domain in math has a cherished proof of the fundamental theorem of algebra, we now have the tools to prove it. Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \tag{9.15}$$

and

$$g(z) = z^n. \tag{9.16}$$

Now we take

$$\|f(z) - g(z)\| = \|a_{n-1}z^{n-1} + \cdots + a_0\| \leq \|a_{n-1}\| \cdot \|z^{n-1}\| + \cdots + \|a_0\| \tag{9.17}$$

then for some radius R we consider the curve

$$\gamma = \{z \mid \|z\| = R\}. \tag{9.18}$$

We find on this curve

$$\|f(z) - g(z)\| < \|g(z)\| \tag{9.19}$$

Then the ratio

$$\frac{\|f(z) - g(z)\|}{\|g(z)\|} \leq \|a_{n-1}\|\frac{1}{R} + \|a_{n-2}\|\frac{1}{R^2} + \cdots + \|a_0\|\frac{1}{R^n} \tag{9.20}$$

is very interesting. We let

$$h(z) = \|a_{n-1}\|z + \|a_{n-2}\|z^2 + \cdots + \|a_0\|z^n \tag{9.21}$$

then we find a $\delta > 0$ such that

$$\|z\| < \delta \implies \|h(z)\| < 1 \tag{9.22}$$

We take $R = 1/\delta$.

Homework 2

▸ **EXERCISE 7**
Let $\mathcal{U}$ be the domain obtained from $\mathbb{C}$ by deleting all real x with $|x| \geq 1$. Describe (explicitly) a conformal map of $\mathcal{U}$ onto the upper half-plane.

▸ **EXERCISE 8**
Prove the following result of Karl Weierstrass. Let

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

Then f cannot be analytically continued to *any* open set properly containing

$$A = \{z \in \mathbb{C} \mid \|z\| < 1\}.$$

Hint: First consider $z = r\exp(\mathrm{i}2\pi p/q)$ where p and q are integers.

▸ **EXERCISE 9**
Describe, in terms of cutting and pasting, the Riemann surface of the function $w = w(z)$ given by the implicit formula $w^3 - w + z = 0$ (you can use the algebraic fact that a complex number a is a multiple root of an algebraic equation $p(x) = 0$ if and only if $p(a) = p'(a) = 0$). Try to generalize to $w^n - w + z = 0$.

▸ **EXERCISE 10**
Let f be a conformal map of the half-disk $D^+ = \{z \in \mathbb{C} \mid \|z\| < 1, \mathrm{Im}(z) > 0\}$ onto itself which can be extended to a continuous map of $D^+ \cup (1, 1)$ onto itself. Prove that

$$f(z) = \frac{z - a}{1 - az}$$

for some $a \in (-1, 1)$. (Hint: Schwarz's Reflection Principle.) (By the way, is the extendability condition really needed? The answer can be deduced immediately from the Riemann mapping theorem.)

▸ **EXERCISE 11**
Let d_1, d_2 be two closed discs contained in the open unit disk D. Prove that any conformal map $D - \{d_1\} \to D - \{d_2\}$ can be extended to a fractional linear map. Try to deduce a condition on d_1, d_2 under which such f can exist. (Hint: the circular Schwarz Reflection Principle, Page 370.)

Lecture 10

Today, we are going to prove a couple of theorems.

Theorem 10.1. *Suppose we have an analytic function in some domain $\mathcal{U}$ which is injective (at least locally). So $f\colon \mathcal{U} \to \mathbb{C}$ is injective, then $f'(z) \neq 0$ for all $z \in \mathcal{U}$.*

Note that this is not true for real analysis.

Proof. Suppose for contradiction this is not true. So $f'(z_0) = 0$ for some $z_0 \in \mathcal{U}$, we have $g(z) = (f(z) - f(z_0))$ be zero at z_0 and its derivative is zero. It's a double zero, at least at z_{90}. This implies there is a neighborhood around z_0, d the disc with boundary γ such that neither $g(z)$ nor $g'(z)$ is zero in the neighborhood. There exists an a such that

$$0 < a < \inf_{z \in \gamma} \|f(z) - f(z_0)\| \tag{10.1}$$

Consider

$$h(z) = f(z) - f(z_0) - a \tag{10.2}$$

By Rouché's theorem, $h(z)$ has at least two zeroes. So h has precisely k zeroes within the disc. If for a $z \in d$, $h(z) = 0$ but its derivative $h'(z) \neq 0$ is nonvanishing (we see in fact $h(z_0) = -a \neq 0$), it follows that all zeroes are different

$$h(z) = 0 \implies f(z) = f(z_0) + a = \text{constant}. \tag{10.3}$$

This contradicts the premise that $f(z)$ is injective, we reject our initial supposition. □

Theorem 10.2. *Let $f\colon \mathcal{U} \to \mathbb{C}$ be nonconstant, analytic such that for some $z_0 \in \mathcal{U}$, $f(z_0) = a$, then $f - a$ has a zero of multiplicity k at z_0.*

Consider

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{10.4}$$

We know $\zeta(2k)$ is related to the Bernoulli numbers. Try to do the computation. We will discuss it next time.

Consider infinite products $\prod a_n$, we consider the partial products

$$p_n = \prod_{k=1}^{n} a_k \tag{10.5}$$

If

$$\lim_{n\to\infty} p_n = p \tag{10.6}$$

exists and is finite, then

$$\prod_{n=1}^{\infty} a_n = p. \tag{10.7}$$

It is tempting to say this. But consider

$$\begin{aligned} \prod_{n=2}^{\infty}\left(1 - \frac{1}{n}\right) &= \lim_{N\to\infty} \frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{N-1}{N} \\ &= \lim_{N\to\infty}\frac{1}{N} = 0 \end{aligned} \tag{10.8}$$

Is this true? No! The product diverges.

Suppose

$$\prod_{n=1}^{\infty} a_n \quad \text{and} \quad a_n \neq 0 \text{ for all } n \tag{10.9}$$

and further suppose

$$\lim_{N\to\infty} \prod_{n=1}^{N} a_n \neq 0. \tag{10.10}$$

We could argue that it converges. If we extend this to include finitely many $a_n = 0$, but after some m all $a_n \neq 0$ for $n \geq m$, then

$$\lim_{N\to\infty} \prod_{n=m}^{N} a_n \neq 0. \tag{10.11}$$

We just start from where a_n will not be zero.

Consider

$$\prod_{n=1}^{\infty} (1 + z_n) \tag{10.12}$$

and suppose $z_n \neq -1$ for all n. Suppose this converges

$$\prod_{n=1}^{N} (1 + z_n) \to p \tag{10.13}$$

so it is also true that

$$\prod_{n=1}^{N} \|1 + z_n\| \to \|p\|. \tag{10.14}$$

We write

$$\sum_{n=1}^{N} \log \|1 + z_n\| \to \log \|p\| \tag{10.15}$$

so it means

$$S_N = \sum_{n=1}^{N} \log \|1 + z_n\| \tag{10.16}$$

converges. But for the sum to converge, the summands form a sequence which converge to zero, i.e.,

$$\lim_{n\to\infty} \log \|1 + z_n\| = 0. \tag{10.17}$$

But this happens if and only if

$$\|1 + z_n\| \to 1. \tag{10.18}$$

How interesting!

Euler showed that

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right), \tag{10.19}$$

the zeroes of since are 0, $\pm\pi$, $\pm 2\pi$, $\dots$, $\pm k\pi$, $\dots$; we could write this out as a product:

$$z(z-\pi)(z+\pi)(z-2\pi)(z+2\pi)(\dots) = z(z^2-\pi^2)(z^2-(2\pi)^2)(\dots) \tag{10.20a}$$

$$= z\left(\frac{z^2}{\pi^2} - 1\right)\left(\frac{z^2}{2^2\pi^2} - 1\right)(\dots) \tag{10.20b}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \tag{10.20c}$$

where the last step is justified by setting equals to equals (the product definition of sine to the Taylor series definition of sine).

Remark 10.3. Note that going from Eq (10.20a) to Eq (10.20b) is a little confusing for neophytes (read: the author). However, it is a common trick to write it in this form so the infinite product looks like

$$\sin(\dots) = z_0 \prod_{n} (1 - z_n) \tag{10.21}$$

which is the form of the infinite product we study thoroughly!

We may set terms of the same power equal to each other, we find

$$\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} = \frac{1}{3!} \tag{10.22a}$$

$$\sum_{n=1}^{\infty} \frac{1}{\pi^4 n^4} = \frac{1}{5!} \tag{10.22b}$$

and so on. How do we get this? Well, it has to do with picking the terms intelligently from the infinite product.

Lecture 11

The Riemann zeta function is investigated further in p-adic numbers. In real variables, we have Bernoulli numbers obey

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} B_k x^{2k} \tag{11.1}$$

where

$$B_k = \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{5}{66}, \frac{691}{8130}, \ldots \tag{11.2}$$

We have

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = B_k \pi^{2k} \frac{2^{2k-1}}{(2k)!} \tag{11.3}$$

So by plugging in $k = 1$ we obtain

$$\sum_{n=1} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{11.4}$$

We also assert that

$$\sum_{1 \le k_1 < \cdots < k_s} (k_1 k_2 \cdots k_s)^{-2} = \frac{\pi^{2s}}{(s+1)!} \tag{11.5}$$

This is not so obvious (which is why we *assert* it for the moment).

If we consider the following diagram

$$\begin{array}{ccc} \left(\sum \frac{1}{n^2}\right)^2 & = & \sum \frac{1}{n^4} + 2 \sum_{k_1 < k_2} \frac{1}{k_1^2 k_2^2} \\ \parallel & & \\ \frac{\pi^4}{36} & = & \left(\sum \frac{1}{n^4}\right) + \frac{\pi^4}{60} \end{array} \tag{11.6}$$

which then gives us

$$\sum \frac{1}{n^4} = \frac{\pi^4}{90} \tag{11.7}$$

We can go further

$$\left(\sum \frac{1}{n^2}\right)^3 = \sum \frac{1}{n^6} + 3 \sum_{k \neq \ell} \frac{1}{k^4 \ell^2} + 6 \sum_{k_1 < k_2 < k_3} (k_1 k_2 k_3)^{-2} \tag{11.8}$$

and take

$$\begin{array}{c} \sum \frac{1}{n^4} \sum \frac{1}{n^2} = \sum \frac{1}{n^6} + \sum_{k \neq \ell} \frac{1}{k^4 \ell^2} \\ \parallel \\ \left(\frac{\pi^4}{90}\right)\left(\frac{\pi^2}{6}\right) \end{array} \tag{11.9}$$

We subtract these results

$$\left(\sum \frac{1}{n^2}\right)^3 - 3\sum \frac{1}{n^4}\sum \frac{1}{n^2} = \left(\frac{1}{6^3} - \frac{3}{6\cdot 90}\right)\pi^6 \tag{11.10a}$$

$$= -2\sum \frac{1}{n^6} + 6\sum_{k_1<k_2<k_3}(k_1k_2k_3)^{-2} \tag{11.10b}$$

We know

$$\sum_{k_1<k_2<k_3}(k_1k_2k_3)^{-2} = 6\left(\frac{\pi^6}{7!}\right) \tag{11.11}$$

so we plug this in and we find

$$\sum \frac{1}{n^6} = \frac{\pi^6}{2}\left(\frac{6}{7!} + \frac{1}{120} - \frac{1}{6^3}\right). \tag{11.12}$$

We may continue iterating, but there is no closed form expression in general.

Now, suppose we have a sequence of complex numbers a_1, a_2, ..., and we *do not* demand they are distinct. We can construct an entire function with zeroes at a_1, ..., would this be possible? Why not? Take the product

$$f(z) = (z - a_1)(z - a_2)(\dots) \tag{11.13}$$

Why not?

Remark 11.1. If a function has singularities, we have f be continuous if it has no poles (or in a neighborhood of a pole); if we have f with an essential singularity, a function takes values (in a neighborhood of an essential singularity) all values with possibly two exceptions: 0 and ∞. This is Picard's theorem.

If we have a function without singularities, and a sequence of points a_1, ...; can we have a function $f(z)$ such that $f(a_k) = 0$ for all $k \in \mathbb{N}$? We can construct

$$f(z) = \prod_k \left(1 - \frac{z}{a_k}\right) \tag{11.14}$$

When will this work? We need to impose the condition that $\|a_k\| \to \infty$, but we need more. This will converge absolutely if

$$\sum \frac{1}{\|a_n\|} \text{ converges} \tag{11.15}$$

This is actually too strong a condition. What about $a_n = n$, the sum is harmonic and diverges!

Consider

$$f(z) = z^k \prod_j \left(1 - \frac{z}{a_j}\right) \tag{11.16}$$

we write instead

$$f(z) = \prod_n \left(1 - \frac{z}{a_n}\right)\mathrm{e}^{z/a_n} \tag{11.17}$$

which converges if

$$\sum \frac{1}{\|a_n\|^2} \text{ converges.} \tag{11.18}$$

Why? Because

$$\left(1 - \frac{z}{a_n}\right)\mathrm{e}^{z/a_n} = \left(1 - \frac{z}{a_n}\right)\left(1 + \frac{z}{a_n} + \cdots\right) \tag{11.19a}$$

$$= 1 + \frac{z^2}{a_n^2}\left(-1 + \frac{1}{2}\right) + \cdots + \frac{z^k}{a_n^k}\left(\frac{-1}{(k-1)!} + \frac{1}{k!}\right) + \cdots \tag{11.19b}$$

So we are considering evaluating products of the form

$$\prod_{n=1}^{\infty}(1+\omega_n) \tag{11.20}$$

where $\omega_n = (z/a_n)^2(-1+\frac{1}{2}) + \ldots$, we have

$$\|\omega_n\| < \sum_{k=2}^{\infty}\left\|\frac{z}{a_n}\right\|^k \underbrace{\left|\frac{1}{k!} - \frac{1}{(k-1)!}\right|}_{<1} \tag{11.21}$$

and note that the sum on the right hand side begins with $k = 2$. If $\|z\| < a_n$, then

$$\|\omega_n\| < \frac{\|z/a_n\|^2}{1 - \|z/a_n\|} \tag{11.22}$$

by geometric series.So it follows that $\prod(1+\omega_n)$ converges if the aforementioned series converges. Remember we have

$$1 + \omega_n = \left(1 - \frac{z}{a_n}\right)\mathrm{e}^{z/a_n} \tag{11.23}$$

We have our product converging, uniformly on the unit disc, with zeroes at $a_1, \ldots$.

Theorem 11.2. *If $a_1, \ldots,$ is a sequence where $a_i \neq 0$ for every i, and $\sum \|a_n\|^{-2}$ converges, then*

$$f(z) = \prod_n \left(1 - \frac{z}{a_n}\right)\mathrm{e}^{z/a_n} \tag{11.24}$$

converges and is entire with zeroes at $a_1, \ldots$.

Note that this is an existence theorem, not a uniqueness theorem. It is unique up to a factor of $\exp(g(z))$ for *arbitrary* $g(z)$.

Homework 3

▸ **EXERCISE 12**

Let f, g be two analytic functions in a domain $\mathcal{U}$ bounded by a closed curve γ, continuous in the closure of $\mathcal{U}$, and let $0 < \|g(z)\| < \|f(z)\|$ for all $z \in \gamma$. Prove that the numbers(with multiplicities) of solutions of equations $f(z) = g(z)$ and $f(z) = 0$ are the same.

▸ **EXERCISE 13**

How many zeroes does $z^6 - 4z^5 + z^2 - 1$ have in the disk $D = \{z \mid \|z\| < 1\}$?

▸ **EXERCISE 14**

Show that if $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ then there must be a point z with $\|z\| = 1$ and $p(z) \geq 1$.

▸ **EXERCISE 15**

Let $g_n = \displaystyle\sum_{k=0}^{n}\frac{z^k}{k}$. Let $D(0,R)$ be the disk of radius $R > 0$. Show that for n large enough g_n has no zeroes in $D(0;R)$.

▸ **EXERCISE 16**

Let α be a complex number.

(a) How many values at a generic z does the function $z(= \mathrm{e}^{\alpha\log z})$ have? (The answer depends on α.)

(b) Describe the Riemann surface of this function.

(c) In terms of $\|z\|$ and $\arg z$ (and standard elementary functions from Real Analysis) list all the values of z^{i}.

Lecture 12

So recall our theorem

Theorem 12.1. *If $a_1, \ldots,$ is a sequence where $a_i \neq 0$ for every i, and $\sum \|a_n\|^{-2}$ converges, then*

$$f(z) = \prod_n \left(1 - \frac{z}{a_n}\right) \mathrm{e}^{z/a_n} \tag{12.1}$$

converges and is entire with zeroes at $a_1, \ldots$.

We know

$$\sin(z) = z \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{\pi n}\right) \mathrm{e}^{z/(\pi n)} \tag{12.2}$$

We collect terms

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right)\left(1 + \frac{z}{\pi n}\right) \mathrm{e}^{z/(\pi n)} \mathrm{e}^{z/(-\pi n)} \tag{12.3a}$$

$$= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right) \tag{12.3b}$$

If we wrote

$$f(z) = \prod_{n\neq 0} \left(1 - \frac{z}{\pi n}\right) \tag{12.4}$$

instead, then $f(z)$ diverges entirely. This problem is similar to how

$$\sum (-1)^n = \left(\sum 1\right) + \left(\sum -1\right) \tag{12.5}$$

diverges but

$$\sum (-1)^n = \sum (1-1) = 0 \tag{12.6}$$

converges. So if we disregard the contribution of $\exp[z/(\pi n)]$, the series diverges very much as the harmonic series diverges. Consider

$$\sum_{n\neq 0} \frac{1}{z+n} \tag{12.7}$$

which diverges, but

$$\sum_{n\neq 0} \left(\frac{1}{z+n} - \frac{1}{n}\right) \tag{12.8}$$

converges! But

$$\sum_{n\neq 0} \frac{1}{n} = 0 \tag{12.9}$$

which changes nothing.

12.1 Gamma Function

We will begin with examining the Γ function. What do we know about it? Well,

$$\Gamma(n+1) = n! \tag{12.10}$$

and $\Gamma(-n)$ are singular (simple poles really). We also know

$$\Gamma(\mu+1) = \mu\Gamma(\mu) \tag{12.11}$$

There is a lot of information on the Γ function in Marsden [4].

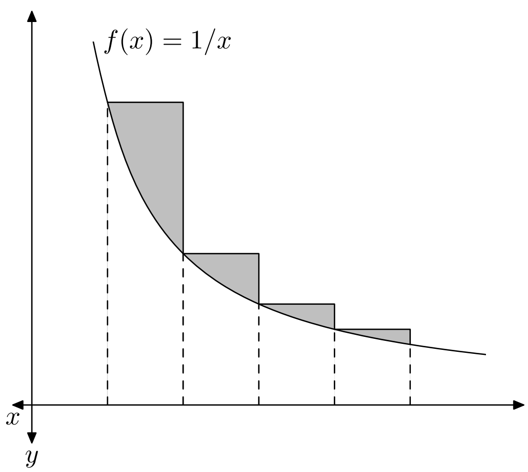
Consider its definition: the Γ function is defined by

$$\Gamma(\mu) = \int_0^\infty x^{\mu-1}\mathrm{e}^{-x}\mathrm{d}x \tag{12.12}$$

Consider some $x \in \mathbb{R}$, $\mu \in \mathbb{C}$. But are these arbitrary? When will the integral be defined? Well, lets consider

$$x^{a+\mathrm{i}b} = x^a x^{\mathrm{i}b} \tag{12.13a}$$
$$= x^a \mathrm{e}^{\mathrm{i}b\log(x)} \tag{12.13b}$$
$$= x^a \mathrm{e}^{\mathrm{i}\log(x^b)} \tag{12.13c}$$

Note that $\|x^{\mathrm{i}b}\| = 1$ for $a, b \in \mathbb{R}$. We are not interested in how our integral behaves near $x \to \infty$, since e^{-x} wins out (i.e., vanishes faster than $x^{\mu-1}$ explodes). So we are interested in the behavior of the integrand near zero. The condition is that $\mathrm{Re}(\mu) > 0$, the integral is defined.



We need to employ our favorite phrase: analytic continuation. We see that when $\mu = 0$, the integral diverges. Similarly for $\mu = -1, -2, \ldots$, the integral diverges too.

There are poles but no zeroes, and for this reason it is not entire; but its inverse $1/\Gamma(z)$ is entire.

We will consider the inverse and bring in the infinite product for the sine. Why? Well, we have zeroes for $G(z) = 1/\Gamma(z)$, which are $\mathbb{N}$. They are all multiplicity zero, so we'll use $\mathbb{N}$ to indicate the zeroes. This is a little bit sloppy, but meh, that's life. Now we know how to construct the product for $G(z)$ since we know its zeroes. We construct it explicitly as

$$G(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)\mathrm{e}^{-z/n} \tag{12.14}$$

observe $G(0) = 1$. We know the relation between $G(z)$ and $\sin(z)$ by the product series. Indeed, observe

$$\sin(\pi z) = \pi z G(z) G(-z). \tag{12.15}$$

Lets see to what extent $\Gamma(\mu+1)\mu\Gamma(\mu)$ is preserved in $G(z)$, i.e. $G(z-1) = H(z)$. We see by inspecting the zeroes of $H(z)$ that

$$H(z) = \mathrm{e}^{g(z)} z G(z) \tag{12.16}$$

we have some unknown factor $g(z)$.

Theorem 12.2. *The factor $g(z)$ is a constant denoted as γ, shorthand for*

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log(N)\right) \tag{12.17}$$

This is certainly one definition of γ. (This is drawn above to the right.)

Note that numerically,

$$\gamma \approx 0.57721\ 56649\ 01532\ 86060 \tag{12.18}$$

We will now define the Γ function as

$$\Gamma(z) = [z\mathrm{e}^{\gamma z} G(z)]^{-1} \tag{12.19}$$

We need to prove that $g(z) = \gamma$ is constant though. We do the following:

$$\frac{\mathrm{d}}{\mathrm{d}z} \log(H(z)) = \frac{\mathrm{d}}{\mathrm{d}z} \log(G(z-1)) \tag{12.20}$$

We observe

$$\begin{aligned} \log(H(z)) &= g(z) + \log(z) + \log(G(z)) && (12.21a) \\ &= g(z) + \log(z) + \sum_{n=1}^{\infty} \log\left(1 + \frac{z}{n}\right) - \frac{z}{n} && (12.21b) \end{aligned}$$

We see

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z} \log(H(z)) &= \frac{\mathrm{d}}{\mathrm{d}z} \left(g(z) + \log(z) + \sum_{n=1}^{\infty} \log\left(1 + \frac{z}{n}\right) - \frac{z}{n} \right) && (12.22a) \\ &= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right). && (12.22b) \end{aligned}$$

Now we consider calculations involving $G(z)$ thus

$$\begin{aligned} \log(G(z-1)) &= \log(\prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n}\right) \mathrm{e}^{(1-z)/n}) && (12.23a) \\ &= \sum_{n=1}^{\infty} \left(\log\left(1 + \frac{z-1}{n}\right) - \frac{z-1}{n} \right) && (12.23b) \\ \implies \frac{\mathrm{d}}{\mathrm{d}z} \log(G(z-1)) &= \sum_{n=1}^{\infty} \left(\frac{1}{z+n-1} - \frac{1}{n} \right) && (12.23c) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) && (12.23d) \end{aligned}$$

This implies that

$$g'(z) = 0 \implies g(z) = (\text{constant}) \tag{12.24}$$

which concludes the proof of the theorem.

How to find the value of $g(z)$? Take $H(z)$ and set $z = 1$, so

$$H(1) = e^{g(1)} G(1) \tag{12.25}$$

which gives us a new problem: what is $G(1)$? Well,

$$G(1) = \lim_{N \to \infty} \prod_{n=1}^{N} \left(1 + \frac{1}{n}\right) \mathrm{e}^{-1/n} \tag{12.26}$$

Observe that

$$\prod_{n=1}^{N} \left(1 + \frac{1}{n}\right) = N + 1 \tag{12.27}$$

which means

$$G(1) = \lim_{N\to\infty} (N+1)\mathrm{e}^{-\sum_{n=1}^{N}(1/n)} \tag{12.28}$$

So

$$\log(G(1)) = \lim_{N\to\infty} \log(N+1) - \sum_{n=1}^{N} \frac{1}{n} \tag{12.29a}$$

$$= -\gamma. \tag{12.29b}$$

This concludes this lecture.

Lecture 13

We will continue on the Γ function. Remember Stirling's approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \tag{13.1}$$

We see that $5! = 120$ but

$$\sqrt{10\pi}(5/\mathrm{e})^5 \approx 118.019168 \tag{13.2}$$

which is reasonably accurate. There are more accurate formulas approximating $n!$ though.

We saw that

$$\sin(\pi z) = \pi z G(z)G(-z) \tag{13.3}$$

We should regard $G(z)$ as the material for building the Γ function. Observe that

$$G(z-1) = \mathrm{e}^{\gamma} z G(z) \tag{13.4}$$

where $\gamma \approx 0.57721$ is Euler's constant. We replace $z \mapsto z+1$ which gives us

$$\begin{aligned} G(z) &= (z+1)\mathrm{e}^{\gamma}G(z+1) \\ \implies G(z+1) &= (z+1)^{-1}\mathrm{e}^{-\gamma}G(z) \end{aligned} \tag{13.5}$$

This gives us our definition

$$\Gamma(z) = \left(z\mathrm{e}^{\gamma z}G(z)\right)^{-1} \tag{13.6}$$

This is defined for all values of z. We see first of all that

$$\Gamma(1) = \left(1\mathrm{e}^{\gamma}G(1)\right)^{-1} \tag{13.7}$$

where

$$G(1) = \mathrm{e}^{-\gamma}G(0) = e^{-\gamma} \tag{13.8}$$

together both imply

$$G(1) = (1\mathrm{e}^{\gamma}\mathrm{e}^{-\gamma})^{-1} = 1 \tag{13.9}$$

We can immediately see that

$$\Gamma(z+1) = \Big((z+1)\mathrm{e}^{\gamma(z+1)}G(z+1)\Big)^{-1} \tag{13.10a}$$

$$= \Big(\cancel{(z+1)}\mathrm{e}^{\gamma(z+1)}\mathrm{e}^{-\gamma}\cancel{(z+1)^{-1}}G(z)\Big)^{-1} \tag{13.10b}$$

$$= \left(\mathrm{e}^{\gamma z}G(z)\right)^{-1} \tag{13.10c}$$

$$= z\left(z\mathrm{e}^{\gamma z}G(z)\right)^{-1} \tag{13.10d}$$

$$= z\Gamma(z) \tag{13.10e}$$

Consider the formula

$$\sin(\pi z) = \pi z G(z)G(-z) \tag{13.11}$$

We want to use $\Gamma(z)$ instead of $G(z)$, which permits us to see that

$$\sin(\pi z) = \pi(z\mathrm{e}^{\gamma z}G(z))(-z\mathrm{e}^{-\gamma z}G(-z)) \tag{13.12}$$

This allows us to calculate

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= [(z\mathrm{e}^{\gamma z}G(z))((1-z)\mathrm{e}^{\gamma(1-z)}G(1-z))]^{-1} && (13.13a)\\
&= [z\mathrm{e}^{\gamma z}\mathrm{e}^{-\gamma z}G(z)\mathrm{e}^{\gamma}(1-z)(\mathrm{e}^{-\gamma}(1-z)^{-1}G(-z))]^{-1} && (13.13b)\\
&= [zG(z)G(-z)]^{-1} = [\sin(\pi z)/\pi]^{-1} && (13.13c)\\
&= \frac{\pi}{\sin(\pi z)} && (13.13d)
\end{aligned}$$

We find by accident a cute relation

$$\Gamma(1/2)^2 = \pi \implies \Gamma(1/2) = \sqrt{\pi}. \tag{13.14}$$

So what? Well, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi} \tag{13.15}$$

which we get from all our information of $\Gamma(1/2)$, as well as exotic numbers like e and π. How is this even possible? Well we see

$$\Gamma(z) = \int_{0-}^{\infty} t^{z-1}\mathrm{e}^{-t}\mathrm{d}t \tag{13.16}$$

so we see

$$\begin{aligned}
\Gamma(1/2) &= \int_0^{\infty} t^{-1/2}\mathrm{e}^{-t}\mathrm{d}t && (13.17a)\\
&= \frac{1}{2}\int_0^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u && (13.17b)
\end{aligned}$$

by choosing $u^2 = t$. Now we can deduce the Gamma function's action on half-integral values:

$$\Gamma(3/2) = \Gamma(1+(1/2)) = \frac{\Gamma(1/2)}{2} = \frac{\sqrt{\pi}}{2} \tag{13.18}$$

and so on.

We are now interested in this integral $\int t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$. Most textbooks insist on integrating by parts, but we want to compute it. We will rewrite it as

$$\int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\mathrm{d}t = \lim_{N\to\infty}\int_0^{N} t^{z-1}\left(1-\frac{t}{N}\right)^N \mathrm{d}t \tag{13.19}$$

We just substitute in the limit definition for exponentiation. But we will change variables $t = Ns$ so we may write

$$\begin{aligned}
\lim_{N\to\infty}\int_0^{N} t^{z-1}\left(1-\frac{t}{N}\right)^N \mathrm{d}t &= \lim_{N\to\infty}\int_0^1 (Ns)^{z-1}(1-s)^N N\mathrm{d}s\\
&= \lim_{N\to\infty} N^z\int_0^1 s^{z-1}(1-s)^N\mathrm{d}s
\end{aligned} \tag{13.20}$$

We will now perform integration by parts. We do it once to find

$$\lim_{N\to\infty} N^z\int_0^1 s^{z-1}(1-s)^N\mathrm{d}s = \lim_{N\to\infty} N^z\left(\underbrace{(1-s)^N\frac{s^z}{z}\Big|_0^1}_{=0} + \frac{N}{z}\int_0^1(1-s)^{N-1}s^z\mathrm{d}s\right) \tag{13.21}$$

and again to find

$$\lim_{N\to\infty} N^z \int_0^1 s^{z-1}(1-s)^N \mathrm{d}s = \lim\dots\left(\frac{N(N-1)}{z(z+1)}\int_0^1 (1-s)^{N-2}s^{z+1}\mathrm{d}s\right) \tag{13.22}$$

and after many integration by parts we see that

$$\begin{aligned}(1-s)^N &\to N!,\\ \int s^z &\to \int_0^1 s^{z+n-1}\mathrm{d}s = \frac{1}{z+N}\end{aligned} \tag{13.23}$$

We therefore claim that

$$\Gamma(z) = \lim_{N\to\infty}\frac{N!N^z}{z(z+1)(\dots)(z+N)} \tag{13.24}$$

Let us prove this.

Consider

$$\frac{1}{\Gamma(z)} = zG(z)\mathrm{e}^{\gamma z} \tag{13.25a}$$

so we substitute in the definition of γ and $G(z)$

$$= \lim_{N\to\infty} z\left[\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)\mathrm{e}^{-z/n}\right]\exp\left[\sum^{N}(z/n) - z\ln(n)\right] \tag{13.25b}$$

we gather terms

$$= \lim_{N\to\infty} z\mathrm{e}^{-\ln(N)z}\prod_{n=1}^{N}\left(1+\frac{z}{n}\right) \tag{13.25c}$$

then by using the law of logarithms to simplify

$$= \lim_{N\to\infty} zN^{-z}\prod_{n=1}^{N}\frac{n+z}{n} \tag{13.25d}$$

and then expanding out the product yields

$$= \lim_{N\to\infty} zN^{-z}(z+1)(z+2)(\dots)(z+N)/N! \tag{13.25e}$$

$$= \lim_{N\to\infty}\frac{z(z+1)(z+2)(\dots)(z+N)}{N^zN!} \tag{13.25f}$$

Thus we are done!

Now for Stirling's approximation, we have

$$n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n \tag{13.26}$$

where

$$a_n \sim b_n \quad\text{means}\quad \lim_{n\to\infty}\frac{a_n}{b_n} = 1. \tag{13.27}$$

We can really write

$$n! = \sqrt{2\pi n}(n/\mathrm{e})^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\cdots\right) \tag{13.28}$$

if we want to use equality instead of an equivalence relation.

Homework 4

▸ **EXERCISE 17**
Show that
$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$$
Relate this to the infinite product formula for sine.

▸ **EXERCISE 18**
Show that $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{2}{n^3+1}\right)=\frac{2}{3}$.

▸ **EXERCISE 19**
It is obvious that if x is real (and not a pole of Γ), then $\Gamma(x)$ is real. What is the set $\{x\in\mathbb{R} \mid \Gamma(x)>0\}$? (The answer can be seen on the picture on page 415, but a proof is requested.)

▸ **EXERCISE 20**
Find explicitly $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$.

Lecture 14

There are several things of the Γ function which we still can discuss. There are two properties of interest: computation of its residues, and Gauss' formula.

Gauss' formula says $\Gamma(z)\Gamma(z+\frac{1}{n})\ldots\Gamma(z+\frac{n-1}{n})$ is related to $\Gamma(n+z)$. To guess the Gauss formula, choose $z=m\in\mathbb{N}$. So we want to relate $\Gamma(m)\Gamma(m+\frac{1}{n})\ldots\Gamma(m+\frac{n-1}{n})$. To begin with $n=1$ is uninteresting. So lets begin with $n=2$, we have

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=(m-1)!\cdot\Gamma(1/2)\frac{1}{2}\frac{3}{2}(\ldots)\frac{2m-1}{2} \tag{14.1a}$$
$$=\Gamma(1/2)\frac{(2m-1)!}{2^{2m-1}} \tag{14.1b}$$
$$=\frac{(2m-1)!\sqrt{\pi}}{2^{2m-1}} \tag{14.1c}$$

Now for $n=3$ what do we have? Well, we find

$$\Gamma(m)\Gamma\left(m+\frac{1}{3}\right)\Gamma\left(m+\frac{2}{3}\right)=\frac{(3m-1)!}{3^{3m-1}}\Gamma(1/3)\Gamma(2/3) \tag{14.2a}$$
$$=\frac{(3m-1)!}{3^{3m-1}}\frac{2\pi}{\sqrt{3}} \tag{14.2b}$$

We see that for some general n that

$$\Gamma(m)\Gamma\left(m+\frac{1}{n}\right)(\ldots)\Gamma\left(m+\frac{n-1}{n}\right)=\frac{(mn-1)!}{n^{mn-1}}\Gamma(1/n)\Gamma(2/n)(\ldots)\Gamma([n-1]/n) \tag{14.3}$$

We can compute the Γ terms by first squaring it and rearranging terms to read:

$$\begin{aligned}(\Gamma(1/n)(\ldots)\Gamma([n-1]/n))^2&=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right)(\ldots)\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right)\\&=\frac{\pi^{n-1}}{\sin(\pi/n)\sin(2\pi/n)(\ldots)\sin([n-1]\pi/n)}\end{aligned} \tag{14.4}$$

So we find
$$\sin(\pi/n)(\ldots)\sin\left(\frac{(n-1)}{n}\pi\right)=\frac{n}{2^{n-1}} \tag{14.5}$$

we can rewrite this as

$$\Gamma(m)\Gamma\left(m+\frac{1}{n}\right)(\dots)\Gamma\left(m+\frac{n-1}{n}\right)=\frac{\Gamma(mn)}{n^{nm-1}}\frac{\pi^{(n-1)/2}2^{(n-1)/2}}{\sqrt{n}}. \tag{14.6}$$

We then replace $m \to z$ and that is Gauss' formula.

Now to compute the residues, choose $(-m)$ where $m \in \mathbb{N}$, the residue is

$$\begin{aligned}
\lim_{z\to -m}(z+m)\Gamma(z) &= \lim_{z\to -m}(z+m)\frac{\Gamma(z+1)}{z} && (14.7a)\\
&= \lim_{z\to -m}(z+m)\frac{\Gamma(z+2)}{z(z+1)} && (14.7b)\\
&= \lim_{z\to -m}(z+m)\frac{\Gamma(z+m+1)}{z(z+1)(\dots)(z+m)} && (14.7c)\\
&= \lim_{z\to -m}\frac{\Gamma(z+m+1)}{z(z+1)(\dots)(z+m-1)} && (14.7d)\\
&= \frac{(-1)^m}{m!} && (14.7e)
\end{aligned}$$

This is the residue of the Γ function.

14.1 Asymptotics

Asymptotiocs are different than approximations, we will begin with asymptotics of factorials. Consider $N!$ for some large N, the number of digits of the number is more or less $\ln(N!)$. Lets compare $n!$ to n^n, we take

$$\frac{n!}{n^n}=\frac{1}{n}\frac{2}{n}(\dots)\frac{n}{n} \tag{14.8}$$

We take the logarithm of this to make the product into a sum

$$\ln\left(\frac{n!}{n^n}\right)=\sum_{k=1}^{n}\ln(k/n) \tag{14.9}$$

but this sum is not well behaved. To remedy the situation, we divide through by N

$$\begin{aligned}
\frac{1}{n}\ln(n!/n^n) &= \frac{1}{n}\left[\ln\left(\frac{1}{n}\right)+(\dots)+\ln\left(\frac{n}{n}\right)\right] && (14.10a)\\
&\approx \int_0^1 \ln(x)\mathrm{d}x = \lim_{\varepsilon\to 0} x\ln(x)-x\Big|_{\varepsilon}^{1} && (14.10b)\\
&= (0-0)-(1-0)=-1. && (14.10c)
\end{aligned}$$

We see that

$$\ln\left(\sqrt[n]{\frac{n!}{n^n}}\right)\approx -1 \tag{14.11}$$

so we exponentiate both sides

$$\sqrt[n]{\frac{n!}{n^n}}\approx \mathrm{e}^{-1} \tag{14.12}$$

then we raise both sides to the n^{th} power

$$\frac{n!}{n^n}\approx \mathrm{e}^{-n} \implies n!\approx\left(\frac{n}{\mathrm{e}}\right)^n \tag{14.13}$$

which is a very rough asymptotic formula, but it was the first asymptotic formula for the factorial.

There are more precisely asymptotic formulas, of which Stirling's is the most famous. It states

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \tag{14.14}$$

What about the relative error of using $\sqrt{2\pi n}(n/\mathrm{e})^n$ instead of $n!$, that is the error in terms of percents. Consider a more precise form of Stirling's approximation

$$n! \sim \sqrt{2\pi n}\left(1+\frac{1}{12n}\right)\left(\frac{n}{\mathrm{e}}\right)^n \tag{14.15}$$

There is in fact an infinite series, then next asymptotic would be

$$n! \sim \sqrt{2\pi n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}\right)\left(\frac{n}{\mathrm{e}}\right)^n \tag{14.16}$$

and so on.

Consider $n = 10$, then

$$10! = 3\ 628\ 800 \tag{14.17a}$$

$$\sqrt{2\pi 10}\left(\frac{10}{\mathrm{e}}\right)^{10} \approx 3\ 598\ 695 \tag{14.17b}$$

and

$$\sqrt{2\pi 10}\left(1+\frac{1}{120}\right)\left(\frac{10}{\mathrm{e}}\right)^{10} \approx 3\ 628\ 685 \tag{14.17c}$$

and the next expansion would be precise to one digit, probably.

Consider the asymptotics for the number of primes less than n. This is a special function denoted *Prime number function $\pi(n)$*

$$\pi(n) = \text{number of primes less than } n \tag{14.18}$$

We have two estimates: Euler's formula

$$\pi(n) \sim \log(n)/n \tag{14.19}$$

and the logarithmic integral function ("li's integral")

$$\pi(n) \sim \int_0^n (1/\ln(t))\mathrm{d}t. \tag{14.20}$$

The latter is better. Consider $n = 10^9$, what happens? Well we see (truncating to integer values) that

$$n = 10^9 \tag{14.21a}$$

$$\pi(n) = 50\ 847\ 534 \tag{14.21b}$$

$$\frac{n}{\ln(n)} \approx 48,429,482 \tag{14.21c}$$

$$\int_2^n \frac{\mathrm{d}t}{\ln(t)} = 50,849,235 \tag{14.21d}$$

Can we say anything about the error? Yes and no: yes because yes, and no because no. Most theorems about the error depends on the Riemann zeta conjecture being true.

If the Riemann hypothesis is true, then

$$|\operatorname{Li}(n) - \pi(n)| < \frac{\sqrt{n}\log(n)}{8\pi} \tag{14.22}$$

where

$$\operatorname{Li}(n) := \int_2^n \frac{\mathrm{d}t}{\log(t)} \tag{14.23}$$

The difference $\operatorname{Li}(n) - \pi(n)$ changes sign infinitely many times (John Littlewood proved this fact in 1914), the first time is at 10^{349}. (Although a more recent estimate puts this around 10^{316}.)

Partition function $p(n)$

The partition function $p(n)$ counts the number of ways to write n as a sum of positive numbers. So for example,

$$p(4) = 5 \tag{14.24}$$

since we have

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1 \tag{14.25}$$

Rademacher's formula

are the five distinct sums. We have an asymptotic formula for it

$$p(n) \sim \frac{1}{4n\sqrt{3}} \mathrm{e}^{2\pi\sqrt{n/6}} \tag{14.26}$$

This grows faster than any polynomial, but slower than any exponential.

Lecture 15

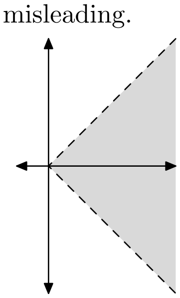
A function (for us a meromorphic function of a complex variable) may be written asymptotically as

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \dots \tag{15.1}$$

Our example from last time was

$$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{\mathrm{e}}\right)^z \left(1 + \frac{1}{12z} + \dots\right) \tag{15.2}$$

First, the series diverges in many important cases; and second, the behavior at the poles is misleading.



We make our first assumption, $\Gamma(z) \sim \sqrt{2\pi z}(\dots)$ holds *NOT* for all values of z but for a sector. That is, we confine the argument of z to satisfy

$$\alpha \leq \arg(z) \leq \beta \tag{15.3}$$

We then commit our second assumption that we deal with a finite sum

$$S_N = a_0 + \frac{a_1}{z} + \dots + \frac{a_N}{z^N} \tag{15.4}$$

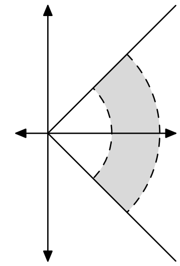
we demand that

$$\lim_{\|z\|\to\infty} \|z\|^N \cdot \|f(z) - S_N\| = 0. \tag{15.5}$$

So for each $\varepsilon > 0$ there exists a $R_N > 0$ such that if $\alpha \leq \arg(z) \leq \beta$ and $\|z\| \geq R_N$ implies

$$\|f(z) - S_N\| < \frac{\varepsilon}{\|z\|^N} \tag{15.6}$$

Within a portion of a sector the asymptotic formula is very good. Outside of it, it becomes a bad approximation. This is doodled on the right where the gray region is when S_N is very good.



Consider an example

$$f(x) = \int_x^\infty t^{-1} \mathrm{e}^{x-t} \mathrm{d}t, \tag{15.7}$$

now consider the formula

$$f(x) \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} + \cdots \tag{15.8}$$

This is the canonical example of the series with a radius of convergence being zero.

We will now show how to relate the two, or more precisely how to go from the integral, Eq (15.7), to the sum, Eq (15.8). We write

$$f(x) = \int_x^\infty t^{-1} \mathrm{e}^{x-t} \mathrm{d}t \tag{15.9}$$

and integrate by parts letting

$$u = t^{-1} \quad \text{and} \quad \mathrm{d}v = \mathrm{e}^{x-t} \mathrm{d}t \tag{15.10}$$

thus

$$\begin{aligned} f(x) &= -t^{-1} \mathrm{e}^{x-t}\big|_x^\infty - \int_x^\infty t^{-2} \mathrm{e}^{x-t} \mathrm{d}t \\ &= x^{-1} \mathrm{e}^{x-x} - \int_x^\infty t^{-2} \mathrm{e}^{x-t} \mathrm{d}t \end{aligned} \tag{15.11}$$

So we can simplify

$$f(x) = \frac{1}{x} - \int_x^\infty t^{-2} \mathrm{e}^{x-t} \mathrm{d}t \tag{15.12}$$

and by iterating we obtain

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n} + \underbrace{(-1)^n n! \int_x^\infty t^{-(n+1)} \mathrm{e}^{x-t} \mathrm{d}t}_{\text{remainder term}} \tag{15.13}$$

We will not show, but writing $(t/x)^{-1}$ instead of t, we get the remainder term being bounded by $n!/x$. So if x is really big, written

$$x \gg n! \tag{15.14}$$

we have a fairly good approximation within a sector. So fro more terms in this asymptote, the smaller the domain for convergence.

We have two things left to discuss today. First the method of steepest descent. This may be considered a generalization of Student's formula.

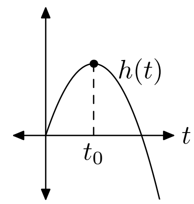
Remark 15.1. We wrote $f(x) \sim a_0 + \frac{a_1}{x} + \ldots$, we should write more generally that $f(x) \sim g(x)(a_0 + \ldots)$, so we consider

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = a_0, \tag{15.15}$$

and this first term usually contains a lot of information.

The following has nothing to do with complex analysis: it is absolutely real.

We have some function $h\colon (0,\infty) \to \mathbb{R}$, we will consider



$$f(z) = \int_0^\infty \mathrm{e}^{zh(t)}\mathrm{d}t \tag{15.16}$$

We suppose it converges. Our conditions is that $h(t)$ has a maximum at t_0, $h'(t_0) = 0$ and $h''(t_0) < 0$. We plot $h(t)$ to the right, with maximum at t_0. So then we find

$$f(x) \sim \left(\frac{\mathrm{e}^{x-h(t_0)}\sqrt{2\pi}}{\sqrt{x}\sqrt{-h''(t_0)}}\right) \tag{15.17}$$

This generalizes Stirling's formula. How? Well, observe

$$\Gamma(x+1) = \int_0^\infty t^x \mathrm{e}^{-t}\mathrm{d}t \tag{15.18a}$$

change variables $xu = t$ so $x\mathrm{d}u = \mathrm{d}t$, thus

$$= \int_0^\infty (xu)^x \mathrm{e}^{-xu} x\mathrm{d}u \tag{15.18b}$$

$$= x^{x+1}\int_0^\infty u^x \mathrm{e}^{-xu}\mathrm{d}u \tag{15.18c}$$

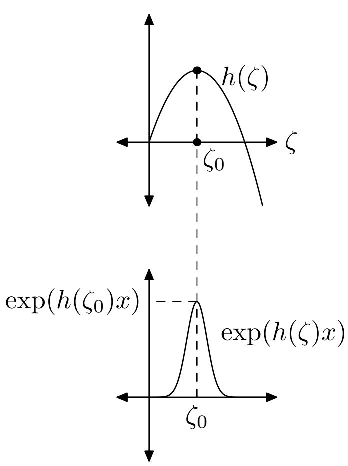
$$= x^{x+1}\int_0^\infty \exp\big(x\big(\ln(u) - u\big)\big)\,\mathrm{d}u \tag{15.18d}$$

where we let $h(u) := \ln(u) - u$. We end up with

$$\begin{aligned}\frac{\Gamma(x+1)}{x^{x+1}} &= \int_0^\infty \mathrm{e}^{x(\ln(u)-u)}\mathrm{d}u \\ &\sim \frac{\mathrm{e}^{-(x+1)}\sqrt{2\pi}}{\sqrt{x}\sqrt{1}}\end{aligned} \tag{15.19}$$

thus we get Stirling's formula back entirely.

Lecture 16



We consider the function

$$f(x) = \int_0^\infty \mathrm{e}^{xh(\zeta)}\mathrm{d}\zeta \tag{16.1}$$

and we want to consider it for big values of x. We see that for any $\zeta \neq \zeta_0$ that

$$\|h(\zeta_0) - h(\zeta)\| = \alpha \tag{16.2}$$

and since $h(\zeta_0)$ is the location of the maxima of h we see

$$h(\zeta_0) - h(\zeta) = -\alpha. \tag{16.3}$$

Thus we have

$$\mathrm{e}^{xh(\zeta)} \approx \mathrm{e}^{-x\alpha} \approx 0 \tag{16.4}$$

for big x. We see that

$$f(x) = \int_0^\infty \mathrm{e}^{xh(\zeta)}\mathrm{d}\zeta \underset{\exp}{\sim} \int_{\zeta_0-\varepsilon}^{\zeta_0+\varepsilon} \mathrm{e}^{xh(\zeta)}\mathrm{d}\zeta \tag{16.5}$$

which is an exponential equivalence, *not* an "asymptotically behaves as". The "**Exponential Equivalence**" of f and g means that they have the same asymptotics, it is considered to be "very strong".

Notation. We will indicate f is exponentially equivalent to g by the notation

$$f \underset{\exp}{\sim} g. \tag{16.6}$$

It is odd notation, but it is *our* odd notation!

Now, we have assumed that h has an extrema at ζ_0, so let us Taylor expand about ζ_0

$$h(\zeta) = h(\zeta_0) + \frac{1}{2}h''(\zeta_0)\cdot(\zeta-\zeta_0)^2 + \dots, \tag{16.7}$$

from just our simple Taylor expansion. We then have

$$f \underset{\exp}{\sim} \int_{\zeta_0-\varepsilon}^{\zeta_0+\varepsilon} \mathrm{e}^{xh(\zeta)}\mathrm{d}\zeta \tag{16.8a}$$

as we have discussed, so we change variables $\zeta = \zeta_0 + z$

$$\int_{\zeta_0-\varepsilon}^{\zeta_0+\varepsilon} \mathrm{e}^{xh(\zeta)}\mathrm{d}\zeta \sim \int_{\varepsilon}^{\varepsilon} \mathrm{e}^{xh(\zeta_0+z)}\mathrm{d}z \tag{16.8b}$$

and then plug in the Taylor expansion of h about ζ_0 (up to second order — additional orders would be negligible):

$$\int_{\varepsilon}^{\varepsilon} \mathrm{e}^{xh(\zeta_0+z)}\mathrm{d}z \sim \int_{\varepsilon}^{\varepsilon} \mathrm{e}^{x[h(\zeta_0)+h''(\zeta_0)z^2/2]}\mathrm{d}z \tag{16.8c}$$

and we observe that

$$\int_{\varepsilon}^{\varepsilon} \mathrm{e}^{x[h(\zeta_0)+h''(\zeta_0)z^2/2]}\mathrm{d}z = \mathrm{e}^{xh(\zeta_0)}\int_{\varepsilon}^{\varepsilon} \mathrm{e}^{xh''(\zeta_0)z^2/2}\mathrm{d}z \tag{16.8d}$$

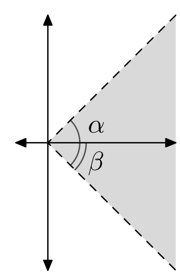
$$= \exp[xh(\zeta_0)]\sqrt{\pi}/\sqrt{-xh''(\zeta_0)/2} \tag{16.8e}$$

Thus we conclude

$$f(x) \sim \frac{\mathrm{e}^{xh(\zeta_0)}\sqrt{2\pi}}{\sqrt{x}\sqrt{-h''(\zeta_0)}} \tag{16.9}$$

as the asymptotic behavior for $f(x)$.

This steepest descent method holds for $f(z)$ provided that z lies in the sector of $\mathbb{C}$ where $x > 0$ and y is on the right hand side. See the shaded part of the diagram to the right for details.



Now we should remember

$$h(\zeta) = h(\zeta_0) + \underbrace{\frac{1}{2}h''(\zeta_0)(\zeta-\zeta_0)^2 + \dots}_{=-\omega(z)^2} \tag{16.10}$$

In a reasonable neighborhood of 0, $\omega(z)$ is invertible. We have from our computations

$$f(x) \underset{\exp}{\sim} \int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{xh(\zeta_0+z)}\mathrm{d}z \sim \int_{-\infty}^{\infty} \mathrm{e}^{xh(\zeta)}\mathrm{d}\zeta \tag{16.11a}$$

$$= \mathrm{e}^{xh(\zeta_0)}\int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{-\omega^2 x}\mathrm{d}z \tag{16.11b}$$

where we have used the fact that $\omega(z)$ is invertible in a neighborhood of 0, so we get $z = z(\omega)$ and $\mathrm{d}z = z'(\omega)\mathrm{d}\omega$, thus

$$f(x) \sim \mathrm{e}^{xh(\zeta_0)}\int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{-\omega^2 x}\mathrm{d}z$$

$$= \mathrm{e}^{xh(\zeta_0)} \int_{\omega(-\varepsilon)}^{\omega(\varepsilon)} \mathrm{e}^{-\omega^2 x} z'(\omega)\mathrm{d}\omega \tag{16.11c}$$

Now we suppose that we can power-series expand $z(\omega) = \sum a_n \omega^n$, and thus we obtain

$$f(x) \sim \mathrm{e}^{xh(\zeta_0)} \int_{\omega(-\varepsilon)}^{\omega(\varepsilon)} \mathrm{e}^{-\omega^2 x} z'(\omega)\mathrm{d}\omega$$

$$= \mathrm{e}^{xh(\zeta_0)} \sum_{n=1}^{\infty} \int_{\omega(-\varepsilon)}^{\omega(\varepsilon)} a_n n \omega^{n-1} \mathrm{e}^{-\omega^2 x}\mathrm{d}\omega \tag{16.11d}$$

$$\sim \mathrm{e}^{xh(\zeta_0)} \sum_{n=1}^{\infty} a_n n \int_{-\infty}^{\infty} \omega^{n-1} \mathrm{e}^{-\omega^2 x}\mathrm{d}\omega \tag{16.11e}$$

and only the even integrands survive, so we get

$$= \mathrm{e}^{xh(\zeta_0)} \sum_{n=1}^{\infty} a_{2n-1}(2n-1) \int_{-\infty}^{\infty} \omega^{(2n-1)-1} \mathrm{e}^{-\omega^2 x}\mathrm{d}\omega \tag{16.11f}$$

We recall that

$$\int_{-\infty}^{\infty} t^n \mathrm{e}^{-t^2}\mathrm{d}t = \underbrace{\left(\frac{t^{n+1}}{n+1}\right)\mathrm{e}^{-t^2}\Big|_{-\infty}^{+\infty}}_{=0} + \frac{2}{n+1}\int_{-\infty}^{\infty} \mathrm{e}^{-t^2} t^{n+2}\mathrm{d}t \tag{16.12}$$

and thus inductively we find

$$\int_{-\infty}^{\infty} t^{2m} \mathrm{e}^{-t^2}\mathrm{d}t = \frac{(2m)!\sqrt{\pi}}{m!2^{m-1}}. \tag{16.13}$$

When we plug this result into the integral, we find precisely

$$f(x) \sim \mathrm{e}^{xh(\zeta_0)} \sum_{m=0}^{\infty} a_{2m+1}(2m+1)\frac{(2m)!\sqrt{\pi}}{m!2^{m+1}} \tag{16.14}$$

Note that this method is useful when considering the classical limit in path integral quantization.

Homework 5

▸ **EXERCISE 21**

How many digits (in the standard decimal presentation) does the number 1000! have? Find several (as many as you can) first digits.

▸ **EXERCISE 22**

Prove that for any real a, b, $\|\Gamma(a+b\mathrm{i})\| \le \|\Gamma(a)\|$.

▸ **EXERCISE 23**

Prove that $\Gamma(x) = \dfrac{1}{x}\displaystyle\int_0^{\infty} \mathrm{e}^{-t^{1/x}}\mathrm{d}t$. (This is a generalization of the equality $\Gamma\left(\dfrac{1}{2}\right) = 2\displaystyle\int_0^{\infty} \mathrm{e}^{-t^2}\mathrm{d}t$.)

▸ **EXERCISE 24**

Let $f(x) = \displaystyle\int_x^{\infty} \frac{\mathrm{e}^{x-t}}{t}\mathrm{d}t$, $S_n(x) = \dfrac{1}{x} - \dfrac{1}{x^2} + \cdots + \dfrac{(-1)^{n-1}(n-1)!}{x^n}$. The "relative error" of the approximation $f(x) \approx S_n(x)$ is $\left|\dfrac{f(x) - S_n(x)}{f(x)}\right|$.

(a) Prove that there exists a sequence ε_n, $\lim_{n\to\infty}\varepsilon_n = 0$, such that if $x > n$, then the relative error of the approximation $f(x) \approx S_n(x)$ does not exceed ε_n.

(b) Is it true (or likely to be true) that the inequality $x > n$ in the previous statement can be replaced by $x > \sqrt{n}$?

Lecture 17

Last time, we wrote the formula

$$f(x) = \int_0^\infty e^{xh(\zeta)} d\zeta \tag{17.1}$$

where $h(\zeta_0)$ is a maximum and $h''(\zeta_0) < 0$. We know from last time that

$$f(x) \sim e^{xh(\zeta_0)} \sqrt{\frac{\pi}{x}} \left(a_1 + \frac{1 \cdot 3}{x} a_3 + \frac{1 \cdot 3 \cdot 5}{x^2} a_5 + \dots \right) \tag{17.2}$$

We wrote

$$h(\zeta) = h(\zeta_0) - \omega(\zeta)^2 \tag{17.3}$$

so

$$\omega(\zeta) = \sqrt{h(\zeta_0) - h(\zeta)} > 0 \tag{17.4}$$

since $h(\zeta_0)$ is the maximum for h. We see that $\omega(\zeta)$ is smooth, so we can express $\zeta(\omega) = \omega^{-1}(\zeta)$, thus

$$\zeta(\omega) = \zeta_0 + a_1\omega + a_2\omega^2 + \dots. \tag{17.5}$$

Certainly this could be applied to the Γ function, which is very famous.

17.1 The Laplace Transform

The Laplace transform is used to solve differential equations, it is a version of the Fourier transform. It takes sums to sums and products to convolutions.

Let

$$S = \{z \in \mathbb{C} \mid \|z\| = 1\} \tag{17.6}$$

we have characters of a group G which are maps $G \to S$, they form a group denoted

$$G^* = \mathrm{Char}(G). \tag{17.7}$$

If $G = S$, then characters are all raising to integral powers. So we have $G^* = \mathbb{Z}$. Let f be a function on G, $\widehat{f}$ be a function on G^*, we write

$$\widehat{f} = \int_G \chi(g) f(g) dg \tag{17.8}$$

What is dg? It is a Haar measure, there exists some ability to integrate functions on groups (if the group is topological, the measure is continuous). We need to consider functions that are integrable, if the group is noncompact (e.g. $\mathbb{R}$) we could get divergent integrals. We have $\widehat{f}(\chi)$ be the Fourier coefficients c_n.

Remark 17.1. What we are doing is considering the dual group $\widehat{G}$ in the sense of Pontryagin duality.

We change now to let

$$G = \mathbb{R}, \tag{17.9a}$$

$$f \colon \mathbb{R} \to \mathbb{C} \tag{17.9b}$$

and we write

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} f(x) dx \tag{17.9c}$$

It is the Fourier transform. If we have two functions

$$f_1, f_2 \colon \mathbb{R} \to \mathbb{C} \tag{17.10}$$

their convolution is

$$(f_1 * f_2)(h) = \int_G f_1(g) f_2(g^{-1}h) \mathrm{d}g \tag{17.11}$$

in a reasonable setting it is a sort of involution. Let us return to discuss the Laplace transform.

Consider $f\colon (0,\infty) \to \mathbb{C}$ (we can generalize any half line), the Laplace transform is defined as

$$\widetilde{f}(t) = \int_0^\infty \mathrm{e}^{-tx} f(x) \mathrm{d}x \tag{17.12}$$

The onyl condition we have on f is that

$$\|f(x)\| < A\mathrm{e}^{Bx} \tag{17.13}$$

where $A, B \in \mathbb{R}$ are "Big Constants".

So, after all, we are doing asymptotics, if f has two properties:

1. $f \in C^\infty(0,\infty)$, and
2. $\|f^{(n)}(x)\| < A_n \exp(B_n x)$,

then we have the following asymptotics:

$$\widetilde{f}(x) \sim \frac{f(0)}{x} + \frac{f'(0)}{x^2} + \frac{f^{(2)}(0)}{x^3} + \cdots + \frac{f^{(n-1)}(0)}{x^n} + \ldots \tag{17.14}$$

Observe that we do not demand its convergence since we do not weight by factorials.

Lecture 18

The steepest descent method may be generalized from

$$f(x) = \int_0^\infty \mathrm{e}^{-xh(\zeta)} \mathrm{d}\zeta \sim \sqrt{\frac{2\pi}{x}} \frac{\mathrm{e}^{-xh''(\zeta_0)}}{\sqrt{-h''(\zeta_0)}} \tag{18.1}$$

to

$$f(x) = \int_0^\infty \mathrm{e}^{-xh(\zeta)} g(\zeta) \mathrm{d}\zeta \tag{18.2}$$

where $g(\zeta)$ is bounded and $g(\zeta_0) \neq 0$. We see that this doesn't change anything since we're working in a small neighborhood near ζ_0, so we have essentially

$$f(x) \sim \sqrt{\frac{2\pi}{x}} \frac{g(\zeta_0)\mathrm{e}^{-xh''(\zeta_0)}}{\sqrt{-h''(\zeta_0)}} \tag{18.3}$$

We make a small change in the integrand and the limits of integration

$$f(x) = \int_a^b \mathrm{e}^{\mathrm{i}xh(\zeta)} g(\zeta) \mathrm{d}\zeta \tag{18.4}$$

We do not insist anymore that $h(\zeta_0)$ is a maximum, we relax this to

$$h'(\zeta_0) = 0 \quad \text{and} \quad h''(\zeta_0) \neq 0. \tag{18.5}$$

Here $g(\zeta)$ is not really all that important.

We recall from calculus that

$$\mathrm{e}^{\mathrm{i}xh(\zeta)} = \cos(xh(\zeta)) + \mathrm{i}\sin(xh(\zeta)) \tag{18.6}$$

How does $\cos(xh(\zeta))$ look around ζ_0? In some neighborhood of ζ_0, we have

$$\cos(xh(\zeta))\text{“}\approx\text{”}\cos(\text{constant}) = \text{constant} \tag{18.7}$$

locally in that neighborhood. A doodle to the left shows the intuitive picture. The interval where it is constant is $(\zeta_0-\varepsilon,\zeta_0+\varepsilon)$, where $\varepsilon\sim 1/\sqrt{x}$, x is huge so we can think of it as the frequency of the wave.

The contributions to the oscillations outside of this neighborhood is ≈ 0 since destructive interference sets everything to be negligible in comparison to the contribution from the $(\zeta_0-\varepsilon,\zeta_0+\varepsilon)$ neighborhood.

We then have, on the one hand

$$f(x) = \int_a^b \mathrm{e}^{\mathrm{i}xh(\zeta)}g(\zeta)\mathrm{d}\zeta \tag{18.8a}$$

$$\sim \frac{\mathrm{e}^{\mathrm{i}xh(\zeta_0)}\sqrt{2\pi}}{\sqrt{x}\sqrt{\|h''(\zeta_0)\|}}\mathrm{e}^{\mathrm{sgn}(h''(\zeta_0))\mathrm{i}\pi/4} \tag{18.8b}$$

and on the other hand we have

$$f(x) \underset{\exp}{\sim} \int_{\zeta_0-\varepsilon}^{\zeta_0+\varepsilon} \mathrm{e}^{\mathrm{i}x[h(\zeta_0)+h''(\zeta_0)(\zeta-\zeta_0)^2/2]}\mathrm{d}\zeta \tag{18.9a}$$

$$= \mathrm{e}^{\mathrm{i}xh(\zeta_0)}\int_{\zeta_0-\varepsilon}^{\zeta_0+\varepsilon} \mathrm{e}^{\mathrm{i}xh''(\zeta_0)(\zeta-\zeta_0)^2/2}\mathrm{d}\zeta \tag{18.9b}$$

change variables to $u=\zeta-\zeta_0$

$$= \mathrm{e}^{\mathrm{i}xh(\zeta_0)}\int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{\mathrm{i}xh''(\zeta_0)u^2/2}\mathrm{d}u \tag{18.9c}$$

$$= c_0\mathrm{e}^{\mathrm{i}xh(\zeta_0)}\int_{-\varepsilon}^{+\varepsilon} \mathrm{e}^{\mathrm{i}v^2}\mathrm{d}v \tag{18.9d}$$

where $v=\sqrt{xh''(\zeta_0)/2}u$. Note that we know

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}v^2}\mathrm{d}v = (1+\mathrm{i})\sqrt{\pi/2} \tag{18.10}$$

so we use it!

Lets return now to the Laplace transform. We have our function

$$f\colon (0,\infty)\to\mathbb{R} \tag{18.11}$$

and suppose it grows slower than exponentially. More precisely, there are numbers $A,B\in\mathbb{R}$ such that

$$\|f(x)\| \leq A\mathrm{e}^{Bx} \tag{18.12}$$

where $A,B\neq 0$.

We have

$$\widetilde{f}(x) = \int_0^\infty \mathrm{e}^{-xt}f(t)\mathrm{d}t \tag{18.13}$$

we require $x>B$, otherwise we don't know anything about convergence. If we allow $x\in\mathbb{C}$, then the requirement because $\mathrm{Re}(x)>B$.

Now, what about asymptotics for this function? We need $f\in C^\infty$ and $\|f^{(n)}(t)\|\leq A_n\exp(B_nt)$, we can then write

$$\widetilde{f}(x) = \int_0^\infty \mathrm{e}^{-xt}f(t)\mathrm{d}t \tag{18.14a}$$

then integrate by parts

$$= \frac{-1}{x} e^{-xt} f(t)\Big|_{t=0}^{t=\infty} + \frac{1}{x}\int_0^\infty e^{-xt} f'(t)\mathrm{d}t \tag{18.14b}$$

$$= \frac{f(0)}{x} + \frac{1}{x}\int_0^\infty e^{-xt} f'(t)\mathrm{d}t \tag{18.14c}$$

integration by parts again yields

$$= \frac{f(0)}{x} + \frac{f'(0)}{x} + \frac{1}{x^2}\int_0^\infty e^{-xt} f''(t)\mathrm{d}t \tag{18.14d}$$

and inductively we obtain

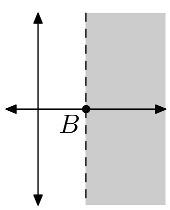
$$\widetilde{f}(x) = \frac{f(0)}{x} + \cdots + \frac{f^{(n)}(0)}{x^{n+1}} + \frac{1}{x^{n+1}}\int_0^\infty e^{-xt} f^{(n+1)}(t)\mathrm{d}t \tag{18.14e}$$

We can consider the last term

$$\frac{1}{x^{n+1}}\int_0^\infty e^{-xt} f^{(n+1)}(t)\mathrm{d}t =: R(x) \tag{18.15}$$

as the remainder term.

Lecture 19



Lets continue on the Laplace transform. Consider

$$f\colon (0,\infty) \to \mathbb{R}. \tag{19.1}$$

The expression

$$\widetilde{f}(z) = \int_0^\infty e^{-tz} f(t)\mathrm{d}t \tag{19.2}$$

it converges for mild conditions on f, namely it should satisfy

$$\|f(t)\| < A\exp(Bt) \tag{19.3}$$

where $A, B \in \mathbb{R} - \{0\}$. We demand

$$\mathrm{Re}(z) > B \tag{19.4}$$

and the imaginary part of z won't affect convergence. Then we are working with the half of the complex plane doodled to the left.

Now last time we considered for $f \in C^\infty(0,\infty)$ and

$$\|f^{(n)}(t)\| \leq A_n \exp(B_n t) \tag{19.5}$$

the asymptotics for its Laplace transform is

$$\widetilde{f}(x) \sim \frac{1}{x} f(0) + \frac{1}{x^2} f'(0) + \cdots + \frac{1}{x^{n+1}} f^{(n)}(0) + \frac{1}{x^{n+1}}\int_0^\infty e^{-xt} f^{(n+1)}(t)\mathrm{d}t. \tag{19.6}$$

If $g = f'$, and we know $\widetilde{f}$, how can we express $\widetilde{g}$? Well, we go back to the definition of the Laplace transform

$$\widetilde{g}(z) = \int_0^\infty e^{-tz} g(t)\mathrm{d}t \tag{19.7a}$$

$$= \int_0^\infty \mathrm{e}^{-tz} f'(t)\mathrm{d}t \tag{19.7b}$$

integrate by parts

$$= \mathrm{e}^{-tz} f(z)|_0^\infty - \int_0^\infty f(t) \left(\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^{-tz}\right) \mathrm{d}t \tag{19.7c}$$

$$= 0 - f(0) + z\int_0^\infty f(t)\mathrm{e}^{-tz}\mathrm{d}t \tag{19.7d}$$

$$= -f(0) + z\widetilde{f}(z) \tag{19.7e}$$

This is the relation between $\widetilde{f}(z)$ and $\widetilde{g}(z)$.

Consider

$$g(t) = f(t)\mathrm{e}^{-at} \tag{19.8}$$

for some $a \in \mathbb{C}$. This is Laplace transformed into a shift

$$\widetilde{g}(z) = \widetilde{f}(z+a) \tag{19.9a}$$

$$= \int_0^\infty \mathrm{e}^{-tz}\mathrm{e}^{-at} f(t)\mathrm{d}t \tag{19.9b}$$

$$= \int_0^\infty \mathrm{e}^{-t(z+a)} f(t)\mathrm{d}t \tag{19.9c}$$

let $\widetilde{z} = z + a$

$$= \int_0^\infty \mathrm{e}^{-t\widetilde{z}} f(t)\mathrm{d}t \tag{19.9d}$$

$$= \widetilde{f}(\widetilde{z}) = \widetilde{f}(z+a) \tag{19.9e}$$

Consider

$$g(t) = \begin{cases} 0 & \text{if } t \leq -a \\ f(t+a) & \text{if } t \geq -a \end{cases} \tag{19.10}$$

we see then that

$$\widetilde{g}(z) = \widetilde{f}(z)\mathrm{e}^{-az} \tag{19.11}$$

If we have $h(t)$ defined such that

$$\widetilde{h}(z) = \widetilde{f}(z)\widetilde{g}(z) \tag{19.12}$$

What $h(t)$ could do this? A convolution of f and g, that is

$$h(t) = (f * g)(t) = \int_0^\infty f(t-\tau)g(\tau)\mathrm{d}\tau \tag{19.13}$$

Suppose that g is bounded and

$$f(x) = \frac{1}{\varepsilon} \tag{19.14}$$

when $x \in (-\varepsilon, \varepsilon)$ for "small" $0 < \varepsilon \lll 1$. Then we have

$$\int_0^\infty f(t-\tau)g(\tau)\mathrm{d}\tau = \int_{t-\varepsilon}^{t+\varepsilon} f(t-\tau)g(\tau)\mathrm{d}\tau \tag{19.15a}$$

$$\approx \frac{1}{2\varepsilon}(g(t+\varepsilon) + g(t-\varepsilon)) \tag{19.15b}$$

$$\approx g(t) \text{ as } \varepsilon \to 0. \tag{19.15c}$$

That concludes today's lecture.

Midterm

1. Find all conformal maps of the upper half plane $\{z \mid \operatorname{Im}(z) > 0\}$ onto the lower half plane $\{z \mid \operatorname{Im}(z) < 0\}$.

2. Describe the Riemann surface of the function $f(z) = z^2 - z$.

3. Does the infinite product $\displaystyle\prod_{n=2}^{\infty}\left(1 + \frac{(-1)^n}{n}\right)$ converge? If yes, find the value of the product. Does it converge absolutely?

4. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be not identically 0, and assume that, for some k,

$$\|a_k\| > \|a_0\| + \|a_1\| + \cdots + \|a_{k-1}\| + \|a_{k+1}\| + \cdots + \|a_n\|.$$

Prove that the polynomial p has precisely k roots (counted with their multiplicities) within the disk $\{z \mid \|z\| < 1\}$.

5. Find an expression, in terms of elementary functions, for $\Gamma(5+z)\Gamma(5-z)$.

6. Find an asymptotic formula for $\binom{n}{k}$ (I mean an expression $F(n)$ in terms of elementary functions whose ratio with the given binomial coefficient has limit 1 when $n \to \infty$. But I do not mind if you get a formula like this: $\binom{n}{k} \sim [???] \cdot \left(1 \pm \frac{1}{?n} + \dots\right)$.)

Lecture 20

Remark 20.1 (On Midterm). If we wish to map the upper half of $\mathbb{C}$ to the lower half of $\mathbb{C}$, we use a fractional linear transformation with real coefficients. Also note that the product cannot be replaced by expanding into pairwise expansion, that is

$$\prod\left(1+\frac{(-1)^n}{n}\right)\neq\prod\left(1-1\frac{1}{n(n+1)}\right) \tag{20.1}$$

otherwise by this reasoning

$$\sum(-1)^n=\sum(1-1)=0 \tag{20.2}$$

is correct, which it most certainly is not.

So what do we know about the Laplace transform? It's something that looks like

$$\widetilde{f}(z)=\int_0^\infty \mathrm{e}^{-zt}f(t)\mathrm{d}t \tag{20.3}$$

What else can we say? First we fix notation, we will write

$$\mathcal{L}\left[f(t)\right](z)=\widetilde{f}(z)=\int_0^\infty \mathrm{e}^{-zt}f(t)\mathrm{d}t. \tag{20.4}$$

Let us list out the properties of the Laplace transform:

1. If f is differentiable and its derivative is not growing too fast, then $\widetilde{(f')}(z)=z\widetilde{f}(z)-f(0)$.
2. We see that $\widetilde{(tf(t))}(z)=-(\widetilde{f})'(z)$.
3. A shift corresponds to multiplication by the exponential, $\mathcal{L}\left[\mathrm{e}^{-at}f(t)\right](z)=\widetilde{f}(z+a)$ where $a\in\mathbb{C}$ is arbitrary.
4. We consider for $t\geq a$ $\mathcal{L}\left[f(t-a)\right](z)=\mathrm{e}^{-az}\widetilde{f}(z)$.
5. We have this strange correspondence between convolution and the product $\mathcal{L}\left[f*g\right](z)=\widetilde{f}(z)\widetilde{g}(z)$.
6. Consider a specific example for $a>-1$: $\mathcal{L}\left[t^a\right]=\Gamma(a+1)/z^{a+1}$.
7. An example $a\in\mathbb{C}$ is arbitrary, then $\mathcal{L}\left[\mathrm{e}^{-at}\right](z)=1/(z+a)$.
8. For cosine we have $\mathcal{L}\left[\cos(at)\right](z)=z/(z^2+a^2)$.
9. For sine we have $\mathcal{L}\left[\sin(at)\right](z)=a/(z^2+a^2)$.

20.1 Some Proofs

Let us consider some proofs of the properties just asserted.

Proposition 20.2. *For $a>-1$ we have $\mathcal{L}\left[t^a\right]=\Gamma(a+1)/z^{a+1}$.*

Proof. We see that

$$\mathcal{L}\left[t^a\right]=\int_0^\infty \mathrm{e}^{-zt}t^a\mathrm{d}t \tag{20.5}$$

we let $u=zt$, so $t=u/z$ and $\mathrm{d}t=\mathrm{d}u/z$ then

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-zt}t^a\mathrm{d}t&=\int_0^\infty \mathrm{e}^{-u}\left(\frac{u}{z}\right)^a\left(\frac{\mathrm{d}u}{z}\right)\\&=\frac{1}{z^{a+1}}\Gamma(a+1)\end{aligned} \tag{20.6}$$

where we have used the definition of the Γ function, specifically Eq (12.12). $\square$

Example 20.3. We see $\mathcal{L}[1] = z^{-1}$, $\mathcal{L}[t] = z^{-2}$, etc. Consider

$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \dots \tag{20.7}$$

then

$$\mathcal{L}[f](z) = \frac{f(0)}{z} + \frac{f'(0)}{z^2} + \dots + \frac{f^{(n)}(0)}{z^{n+1}} + \dots \tag{20.8}$$

This is a bit of a joke, the approach is unreliable. The result is correct however.

Proposition 20.4 (Transform of Cosine)**.** *For some $a \in \mathbb{C}$, we have*

$$\mathcal{L}[\cos(at)](z) = \frac{z}{z^2 + a^2} \tag{20.9}$$

Proof. By direct computation we see

$$\mathcal{L}[\cos(at)]\, z = \mathcal{L}\left[\frac{e^{iat} + e^{-iat}}{2}\right](z) \tag{20.10}$$

and by linearity we obtain

$$\mathcal{L}\left[\frac{e^{iat} + e^{-iat}}{2}\right](z) = \frac{1}{2}\mathcal{L}\left[e^{iat}\right] z + \frac{1}{2}\mathcal{L}\left[e^{-iat}\right](z). \tag{20.11}$$

We use the result from computing $\mathcal{L}[e^{-at}1]$ to find

$$\frac{1}{2}\mathcal{L}\left[e^{iat}\right] z + \frac{1}{2}\mathcal{L}\left[e^{-iat}\right](z) = \frac{1}{2}\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right) \tag{20.12}$$

and by gathering terms we have

$$\frac{1}{2}\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right) = \frac{1}{2}\left(\frac{2z}{z^2 + a^2}\right) \tag{20.13}$$

which proves the theorem. ∎

Proposition 20.5 (Transform of Sine)**.** *For some $a \in \mathbb{C}$, we have*

$$\mathcal{L}[\sin(at)](z) = \frac{a}{z^2 + a^2} \tag{20.14}$$

Proof. We may use the fact

$$\frac{1}{a}\frac{d}{dz}\sin(at) = \cos(at) \tag{20.15}$$

take the Laplace transform of both sides, and we immediately get the result. ∎

20.2 Applications

It is tempting by the first property to apply this to differential equations, we'd end up with the answer but Laplace transformed. We'd need to inverse the transform to get the answer.

Let us first consider a very simply thing. If we have, e.g.,

$$\widetilde{f}(z) = \frac{1}{z - a}, \tag{20.16}$$

what would we expect to find for $f(t)$? We expect $f(t) = \exp(at)$.

If we instead have now

$$\widetilde{f}(z) = \frac{1}{(z-1)(z-2)} \tag{20.17}$$

we'd very much like to use partial fraction decomposition

$$\widetilde{f}(z) = \frac{1}{z-2} - \frac{1}{z-1} \tag{20.18}$$

to obtain the result

$$f(t) = e^{2t} - e^t. \tag{20.19}$$

More generally, if we have a rational function

$$\widetilde{f}(z) = \frac{P(z)}{Q(z)} \tag{20.20}$$

where

$$\deg(P) < \deg(Q) \tag{20.21}$$

and we demand that all roots of Q are different (so Q has simple roots). Let $z_1, \ldots, z_n$ be the roots of Q. We write

$$f(t) = \sum_{k=1}^{m} \mathrm{e}^{z_i t}\left(\frac{P(z_i)}{Q'(z_i)}\right). \tag{20.22}$$

Let F be some function, $F = \widetilde{f}$ or in other words F is the function we will try to "invert Laplace transform"; let

$$f(t) = \sum \text{residues}(\mathrm{e}^{zt}F(z)) \tag{20.23}$$

But the sum of residues is an integral! We see that

$$f(t) = \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{e}^{zt}F(z)\mathrm{d}z \tag{20.24}$$

All the poles should be on the left of a, and there should be only finitely many poles.

Lecture 21

Now, last time we covered the inverse Laplace transform. Let $F(z)$ be analytic in $\mathbb{C}$ with possibly only finitely many poles. Then $F = \widetilde{f}$, and we obtain the original function by

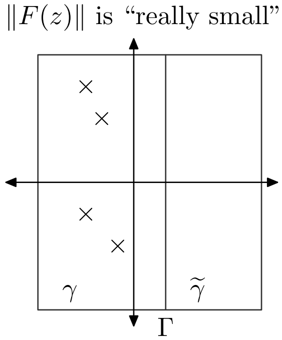
$$f(t) = \sum \big(\text{Residues of } \mathrm{e}^{tz}F(z)\big) \tag{21.1}$$

The poles of this function comes entirely from $F(z)$ since e^{tz} has no poles. We suppose that $F(z)$ is analytic on $\mathbb{C}$ except for a finite number of isolated singularities and for some $\sigma \in \mathbb{R}$ we have F be analytic on the plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > \sigma\}$.

The requirements: there are 3 positive constants M, R, $\beta > 0$ such that if $\|z\| > R$ then

$$\|F(z)\| < \frac{M}{\|z\|^{\beta}} = M(\|z\|^{-\beta}) \tag{21.2}$$

This is some contour integral with the requirement as $\|z\| \to \infty$, then on the boundary $\|F(z)\|$ is "really small".



What do we do? We create a rectangle Γ which is big enough to contain all the singularities of F. This is doodled to the left, the $\times$ indicates singularities of F. We break up Γ into two bits γ which contains all the singularities and $\widetilde{\gamma}$ which is everything else.

We see since all the singularities live inside γ that

$$\int_{\gamma} \mathrm{e}^{zt}F(z)\mathrm{d}z = 2\pi\mathrm{i}f(t) \tag{21.3}$$

How can we check that this is correct?

We take its Laplace transform

$$2\pi\mathrm{i}\widetilde{f}(z) = \int_0^\infty \mathrm{e}^{-zt}\left[\int_\gamma \mathrm{e}^{\zeta t}F(\zeta)\mathrm{d}\zeta\right]\mathrm{d}t \tag{21.4}$$

and change the order of integration

$$2\pi\mathrm{i}\widetilde{f}(z) = \int_\gamma\int_0^\infty \mathrm{e}^{-zt}\mathrm{e}^{\zeta t}F(\zeta)\mathrm{d}\zeta\mathrm{d}t.$$

This is a little bit sloppy, it is really

$$2\pi\mathrm{i}\widetilde{f}(z) = \lim_{r\to\infty}\int_\gamma\int_0^r \mathrm{e}^{-zt}\mathrm{e}^{\zeta t}F(\zeta)\mathrm{d}\zeta\mathrm{d}t \tag{21.5}$$

We then evaluate the integral and we find

$$\begin{aligned}2\pi\mathrm{i}\widetilde{f}(z) &= \lim_{r\to\infty}\int_\gamma\left(\mathrm{e}^{(\zeta-z)r}-1\right)\frac{F(\zeta)}{\zeta-z}\mathrm{d}\zeta\\ &= \int_\gamma F(\zeta)\left[\frac{-1}{\zeta-z}\right]\mathrm{d}\zeta\end{aligned} \tag{21.6}$$

We want to show that $F(z)=\widetilde{f}(z)$.

We use the fact that

$$\int_\gamma(\dots) = \int_\Gamma(\dots) + \int_{\widetilde{\gamma}}(\dots) \tag{21.7}$$

to deduce

$$-2\pi\mathrm{i}\widetilde{f}(z) = -\int_\gamma\frac{F(\omega)}{\omega-z}\mathrm{d}\omega \tag{21.8a}$$

$$= -\int_\Gamma\frac{F(\omega)}{\omega-z}\mathrm{d}\omega - \int_{\widetilde{\gamma}}\frac{F(\omega)}{\omega-z}\mathrm{d}\omega \tag{21.8b}$$

$$= -\int_\Gamma\frac{F(\omega)}{\omega-z}\mathrm{d}\omega - 2\pi\mathrm{i}F(z) \tag{21.8c}$$

and we see that

$$\int_\Gamma\frac{F(\omega)}{\omega-z}\mathrm{d}\omega \approx 0 \tag{21.9}$$

when $\|\omega-z\|\sim R$ and R becomes huge, we basically divide by "infinity". So we have $\widetilde{f}=F$.

Remark 21.1. The derivative is convolution with the derivative of the delta function.

This theorem has many corollaries. We see

$$f(t) = \sum(\text{residues } \mathrm{e}^{tz}F(z)) \tag{21.10}$$

so we can write this as an integral (thanks to the Residue theorem)

$$f(t) = \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\mathrm{e}^{zt}F(z)\mathrm{d}z \tag{21.11}$$

This formula is very close to the Laplace transform, and we derived various properties of the Laplace transform using only integration by parts (which means the inverse transform has analogous properties).

Homework 6

▸ **EXERCISE 25**
Let

$$f(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \lfloor t \rfloor + 1, & \text{if } t > 0 \end{cases} \tag{21.12}$$

(where $\lfloor t \rfloor$ denotes the maximal integer in the interval $(-\infty, t])$. Find (an explicit formula for) the Laplace transform $\widetilde{f}(z)$. What is σ?

▸ **EXERCISE 26**
The same for the "saw-function"

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t - 2n & \text{if } 2n \leq t < 2n+1 \\ 2n + 2 - t & \text{if } 2n+1 \leq t < 2n+2 \end{cases} \tag{21.13}$$

▸ **EXERCISE 27**
Let

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0 \end{cases}, \quad g(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x \geq 0. \end{cases} \tag{21.14}$$

Find $f * f$, $f * g$, $g * g$.

▸ **EXERCISE 28**
Under the appropriate assumptions (you need to formulate them), prove the identities $f * g = g * f, (f * g)' = f * g' = f' * g$.

Lecture 22

The two fields of interest were really (1) Riemann surfaces, (2) the theory of conformal mappings. Geometry merely refers to measuring lengths and angles. We will briefly discuss the Riemann zeta function. First recall the fundamental theorem of arithmetic

Theorem 22.1 (Arithmetic's Fundamental)**.** *If m is a positive integer, it can be written uniquely (up to order of factors) as a product of prime numbers*

$$m = p_1^{q_1}(\dots)p_k^{q_k} \tag{22.1}$$

where $p_1, \dots, p_k$ are all distinct primes.

Now, the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{22.2}$$

can be analytically continued to $\mathbb{C} - \{1\}$. We see that, by the fundamental theorem of arithmetic, we may group terms

$$\left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \dots\right)\left(1 + \frac{1}{3^s} + \dots\right)\left(1 + \frac{1}{5^s} + \dots\right)(\dots) = \zeta(s) \tag{22.3}$$

This is just from

$$\sum \frac{1}{n^s} = \sum \frac{1}{(p_1^{k_1} \dots p_m^{k_m})^s} \tag{22.4}$$

and thus *Euler Product*

$$\zeta(s) = \prod_{\text{prime } p} \frac{1}{\left(1 - \dfrac{1}{p^s}\right)} = \prod \frac{p^s}{p^s - 1} = \prod \left(1 + \frac{1}{p^s - 1}\right) \tag{22.5}$$

This is the "**Euler Product**". We can represent it as an integral, first note the notation $[x]$ for the integer part of x, then

$$\zeta(s) = \left[\int_0^1 \frac{x^{s-1}}{\mathrm{e}^x - 1}\mathrm{d}x\right]\frac{1}{\Gamma(s)} \tag{22.6a}$$

$$= s\int_0^\infty \frac{[x]}{x^{s+1}}\mathrm{d}x \tag{22.6b}$$

$$= \left(\frac{s}{s-1}\right) - s\int_0^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x \tag{22.6c}$$

which converges for $\mathrm{Re}(s) > 0$ which is an extension of the preceding integral. So we obtain a functional relationship

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{s\pi}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{22.7}$$

and with this functional relationship we may uniquely extend the zeta function to a larger domain.

Lecture 23

We are referred to:

- E. Titchmarsh,
  The Zeta-Function of Riemann.

for further reading on the Riemann zeta function. We will continue analytically continuing $\zeta(s)$ to $\mathbb{C} - \{1\}$. Now what happens to Eq (22.7) when we take $\zeta(1-s) = \dots$? It becomes

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{s\pi}{2}\right)\Gamma(s)\zeta(s). \tag{23.1}$$

Notice that for $s = 2n+1$, for any $n \in \mathbb{Z}$, we have

$$\zeta(1-s) = 0. \tag{23.2}$$

But due to the Γ function, we have $s > 0$, otherwise Γ is undefined. But they're killed off by $\cos(\pi s/2)$, so it's all good. There are the trivial zeroes $\zeta(-2) = \zeta(-4) = \cdots = 0$.

Now, the most famous unsolved problem in mathematics: the Riemann hypothesis. The only zeroes (in addition to the trivial zeroes) are at

$$s = \frac{1}{2} + \mathrm{i}(\text{something}). \tag{23.3}$$

Note $\zeta(s) = \zeta(\overline{s})$.

Proposition 23.1. *There are infinitely many "nontrivial zeroes" for $\zeta(s)$.*

Proposition 23.2. *If γ_n is the n^{th} nontrivial zero,* $\lim_{n\to\infty}(\gamma_n - \gamma_{n-1}) = 0$.

Conjecture 23.3. If $N(T)$ is the number of γ with $\mathrm{Im}(\gamma_n) < T$, then $N(T) \sim T\ln(T)$ is the asymptotic behavior.

The first nontrivial zero is at

$$\gamma_1 \approx \frac{1}{2} + \mathrm{i}14.1347\ 2514\ 17346. \tag{23.4}$$

So what are the distribution of these nontrivial zeroes? We see that the number $k \in \mathbb{N}$ with $\mathrm{Im}(\gamma_k) < n$ is described on the following table:

n	$\max\{k \in \mathbb{N} \mid \mathrm{Im}(\gamma_k) < n\}$
100	29
1000	649
100,000	10142
1,000,000	1,747,146

Let

$$\pi(N) := (\text{number of primes} < N) \tag{23.5}$$

We have

$$\pi(N) \sim \mathrm{Li}(N) \tag{23.6}$$

where

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}u}{\log(u)} \tag{23.7}$$

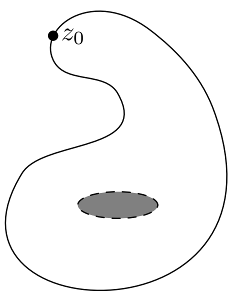
is the Logarithmic integral function. The Riemann hypothesis then states there exists constants $c, C \in \mathbb{R}$ such that $c > 0$ and $C > 0$ which obey

$$c\sqrt{N}\ln(N) < |\pi(N) - \mathrm{Li}(N)| < C\sqrt{N}\ln(N) \tag{23.8}$$

How beautiful.

Lecture 24

We will cover multiple complex variables, and analytic continuation. First the last.



We have some boundary, and a function defined inside the region. When can it not be extended beyond the region. Well, consider one point, what function cannot be extended to z_1? Well, $1/(z - z_1)$. For n points $z_1, \ldots, z_n$ we could have

$$f(z) = \sum_{k=1}^{n} \frac{1}{z - z_k} \tag{24.1}$$

We could replace it by an integral on the boundary, but that'd be hard. Why not take an infinite sum? Well, why not work with a dense (countable) set of points on the boundary? This approach is better, since it's countable. We take $\{z_k\}$ to be dense in the boundary of the region. We can write out the sum

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{z - z_k} \tag{24.2}$$

but it won't converge. We then generalize this to

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k} \tag{24.3}$$

where $\{a_k\}$ is a sequence rapidly decreasing. Although it'd converge on the interior of the region, there is no guarantee for convergence on the boundary.

We then specify

$$a_n = 2^{-n} \min_{m<n}(1, \|z_m - z_n\|) \tag{24.4}$$

and so on the boundary we have 1 singular point and a sum

$$\sum_{k \neq n}^{\infty} \frac{a_k}{z - z_k} < \sum_{k \neq n}^{\infty} 2^{-k} \tag{24.5}$$

which converges.

Lecture 25

Suppose we have a region Γ in $\mathbb{C}^2$ which we want a holomorphic function singular at $(z_0, \omega_0) \in \Gamma$. Suppose we have it be

$$f(z, \omega) = \frac{1}{z + \omega - z_0 - \omega_0} \tag{25.1}$$

This is singular at

$$\begin{aligned} z &= a + z_0 \\ \omega &= -a + \omega_0 \end{aligned} \tag{25.2}$$

for some $a \in \mathbb{C}$. This is a complex line (since a varies over all of $\mathbb{C}$). Suppose we want it to not pierce the region...then we demand the region must be convex for this to be true.

The hyperplane tangent to the point $(z_1, \ldots, z_n)$ is defined by the equation

$$a_1 z_1 + \cdots + a_n z_n + b = 0 \tag{25.3}$$

So back to the original problem in $\mathbb{C}^2$, we write $(a_1 z + a_2 \omega + b)^{-1}$ is regular in the domain and it has a pole on the surface. We do what we did last time: take a dense selection of such points, and so on. We can weaken the condition of convexity to *pseudoconvexity* (i.e., the tangent plane may possibly intersect the domain).

Consider a surface described by

$$F(z_1, \ldots, z_n) = 0 \tag{25.4}$$

We can rewrite it as

$$F(x_1, y_1, \ldots, x_n, y_n) = 0. \tag{25.5}$$

Let

$$\begin{aligned} \partial_k &= \frac{\partial}{\partial z_k} = \left(\frac{\partial}{\partial x_k}\right) + \mathrm{i}\left(\frac{\partial}{\partial y_k}\right) \\ \overline{\partial}_k &= \frac{\partial}{\partial \overline{z}_k} = \left(\frac{\partial}{\partial x_k}\right) - \mathrm{i}\left(\frac{\partial}{\partial y_k}\right) \end{aligned} \tag{25.6}$$

For the second derivatives, we have $[\partial_i \overline{\partial}_j F]$ be Hermitian, then the domain is a "**Holomorphic Domain**". They are very important!

In $\mathbb{C}^n$ (for $n \geq 2$) the region between two concentric spheres *is not* a holomorphic domain, we can extend it to the center ball though. Recall that if f is analytic inside a domain with boundary γ, then

$$f(z_0 = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} \mathrm{d}\zeta. \tag{25.7}$$

Suppose that f were continuous on γ, then

$$g(z_0) = \int_\gamma \frac{f(\zeta)}{\zeta - z_0} \mathrm{d}\zeta \tag{25.8}$$

it is a differentiable function of z_0. Very briefly, the idea is to take your domain, stack an infinite number of discs there, and perform Cauchy integration on each disc. We get a function in one variable, and this function turns out to tell us that $f = g$ in the domain. So f can be expanded. This approach can be applied to many other domains.

References

[1] John B. Conway,
Functions of one complex variable.
Springer–Verlag, 1978.

[2] John B. Conway,
Functions of one complex variable II.
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[3] Serge Lang,
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Springer–Verlag, Fourth edition, 1998.

[4] Jerrold E. Marsden and Michael J. Hoffman,
Basic Complex Anlaysis.
W. H. Freeman; Third Edition edition, 1999.

[5] Elias M. Stein, Rami Shakarchi,
Complex Analysis.
Princeton University Press, 2003.

[6] Hermann Weyl,
The Concept of a Riemann Surface.
Dover Books on Mathematics, Third edition, 2009.

Note that [4] was the "official" text for the course, although we didn't touch it in math 185B. It seems like most of the material was drawn from [3], specifically chapters 7–16.

Some other references which may be useful:

1. Walter Rudin,
Real and Complex Analysis.
Third edition, Boston: McGraw Hill, 1987.

The homological definition of integrals of the form

$$\int g(x)\mathrm{e}^{f(x)}\mathrm{d}x$$

are discussed in

2. Albert Schwarz, Ilya Shapiro,
"Twisted de Rham cohomology, homological definition of the integral and 'Physics over a ring' ".
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