Notes on Classical General Relativity

Alex Nelson*
Email: pqnelson@gmail.com

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Preface

These are my collected notes on classical general relativity. These are graduate level notes, and I have reformatted, merged, and edited them into a cohesive whole. I doubt these notes could take the place of a textbook, but may make wonderful supplement to one.

The references used are either books I own, or eprinted articles. This is the guideline I tried maintaining, but there are exceptions to the rule of "free eprinted articles." (An additional problem: some articles are so old that they are not [yet] eprinted and published online. Sad, I know, but still...)

Also be forewarned: the bibliography consists of two sections. The first consisting of books, which are recommended for the reader. The second consisting of technical articles, relevant for points made.

Strictly speaking, the math used in the first part (the pedagogical part) is not correct. We will be sloppy, as sloppy as physicists are. It's not "incorrect" per se, but it may give mathematicians indigestion.

I hope to write three texts: the first (which you are reading) is a pedagogical introduction to classical general relativity. The second concerns advanced portions of general relativity, preparing the reader for the ADM formalism, numerical relativity, treatment of spinors, and so on. The third deals with quantum gravity.

Part I
Physical Motivation

Lecture 1. Geometry and Physics.

1.1 General Relativity's Importance in Physics

The first 50 years after Einstein published his field equations, physicists held one of two opinions:

1) It was a beautiful model for how physics ought to be.

2) It was largely irrelevant unless you specialize in it.

Most people imagine it's a model for how physics ought to be, unless gravity's emergent. The second view is more or less disregarded. In high energy physics, the coupling constants converge to the same value at high enough energies where gravity is significant (perhaps it unifies with the other forces, and perhaps that's why it is significant). Trivially, General Relativity is useful in cosmology.

There exists a sizeable group of people in condensed matter physics where analog models[1] are used; e.g., an event horizon for sound as an analog to Black Holes. There are attempts to make predictions for, e.g., quantum gravity (using analogs of Hawking radiation, etc.).

In the next 5 years, there should be experimental evidence for gravitational radiation. In 15 years there will be more sensitive tests available. We can ask questions like "Does E/c^2 contribute to mass?" There are interesting anomalies, e.g., measurements[2] of Newton's constant G differ from each other by 10σ to 15σ, the Pioneer satellite feels accelerations that's still not accounted for [**39**], the predicted energy level for Dark Energy is off by 120 orders of magnitude [**35**].

1.2 Geometry and Physics

Lets recall Newton's second law

$$F = ma. \tag{1.1}$$

Initially there was some contraversy whether it's a "true natural law" or just a definition of force (see Spivak [**Spi**] for details). We understand the mass on the right hand side of Newton's second law describes *inertial mass*, the body's resistance to acting forces. But we may say a couple other things.

First, since Newton's second law involves only acceleration, we work with the second time derivative of position. Higher order time derivative models are unstable since they have energy unbounded from below, as Ostrogradski proved[3].

The second thing to say is that Newton divided the world in two: the object we are examining, and the rest of the world affecting it. But gravity is now an exception. We consider gravitational force of a body with mass M acting on another body (with mass m) in Newton's second law (1.1), writing

$$F \stackrel{\text{def}}{=} \frac{GmM}{r^2} \tag{1.2}$$

for the gravitational force, and we invoke the second law writing

$$\frac{GmM}{r^2} = ma. \tag{1.3}$$

[1] For a review, see Barcelo *et al.* [**4**].

[2] See, e.g., Gundlach's measurements [**29**], the CODATA 2002 recommended values [**36**]

[3] See Woodard [**53,** §2] for a review of Ostrogradski's theorem for classical mechanics.

We observe *mathematically* the masses m cancels out on both sides. We thus obtain

$$\frac{GM}{r^2} = a. \tag{1.4}$$

That makes gravity different from everything else, in that it makes gravity a *theory of paths*.

Equivalence Principle gives us geometry

Almost automatically, this makes gravity a theory of geometry. But first note that really we have the right hand side of (1.3) be

$$ma = m_i a \tag{1.5}$$

where m_i is the *inertial mass*, whereas the gravitational force the body with mass m feels is

$$\vec{F}_{12} = \frac{GMm_g}{r^2}\widehat{e}_r \tag{1.6}$$

where $\widehat{e}_r$ is the unit vector from the body M to the body m, m_g is the *(passive) gravitational mass*[4] which experiences the gravitational force, and M is the *(active) gravitational mass* exerting the gravitational force. We have two conceptually different masses: the inertial mass m_i, and the gravitational mass m_g. The basic ingredient for gravity is the idea

$$m_i = m_g \tag{1.7}$$

called the "**Principle of (Weak) Equivalence**".

There is a *Strong Equivalence Principle*. Using Newton's third Law, we find

$$\vec{F}_{21} = -\vec{F}_{12}. \tag{1.8}$$

The role of "active" and "passive" gravitational masses swap. Active and passive gravitational masses are "equivalent" in the sense

$$\frac{m^{(a)}}{m^{(p)}} = \frac{M^{(a)}}{M^{(p)}} \tag{1.9}$$

where m, M are gravitational masses and the superscript indicates whether they are active or passive gravitational masses. This can be checked by the Earth-Moon system. Since 1968, when NASA attached lasers and reflectors (i.e., three plane mirrors meeting mutually at right angles) to the moon, we have timed the delay of a laser pulse sent to the Moon. If the strong equivalence principle didn't hold, we'd expect the Earth–Moon system's center of mass would oscillate with the Lunar period. But we have not observed this[5].

Reiterating the main point: the equality

$$m_i = m_g \tag{1.7}$$

is what makes the geometric picture possible. The present tests (as of 2010) suggest they're equal to parts in 10^{12} or 10^{13}.

But Galileo knew the equivalence principle, did he suspect the geometrical aspects? No, because gravity determines acceleration, and paths depend on initial velocity too. Gravity doesn't determine paths in space, instead it determines *paths in spacetime*. We need to articulate our vocabulary regarding paths before we can continue discussing gravity.

First a "**Extremal Path**" between two points is referred to as a "**Geodesic**". We will set up the framework to discuss geodesics, then proceed to consider calculations.

Flat, Intrinsic, Extrinsic Geometries

We need to know a little about what it means for a space to be "curved", so we will first consider what it means for space to be "flat". We consider it to be the usual Euclidean

[4] Note that the way to think of "gravitational mass" is that it is the *"charge" gravity feels.*

[5] See Williams, et al., [**51**] for more data on this. Will [**50**] has a more broad discussion of experimental foundations underlying the equivalence principle in its various forms.

geometry. There is an important distinction between "intrinsic geometry" (the curvature of space in itself without reference to higher dimensions, e.g., the sum of angles of a triangle on Earth doesn't add up to π, but without reference to 3 dimensions) and "extrinsic geometry" (curvature as seen in higher dimensions). We mostly care about intrinsic curvature with General Relativity. There are times (e.g., in the ADM formalism) when extrinsic curvature is important.

TODO: figure out some transition motivating arclength

With paths, we really need an idea of distance. Recall for flat space, the Pythogoras' theorem gives us a path's length (more or less) as

$$s^2 = x^2 + y^2. \tag{1.10}$$

If we knew infinitesimal distances, that's enough: we can integrate to get the distance

$$s = \int \mathrm{d}s, \tag{1.11}$$

where

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2. \tag{1.12}$$

WARNING: the notation used is $\mathrm{d}s^2 = (\mathrm{d}s)^2$, which may confuse neophytes. The distance between (x_0, y_0) and (x_1, y_1) is determined by the variation

$$\delta \int_{(x_0,y_0)}^{(x_1,y_1)} \mathrm{d}s = 0. \tag{1.13}$$

Think of it like the Euler-Lagrange equation. We will now consider some special cases.

Geodesic Equation: Examples

Example 1.1 (Sphere)**.** Recall a sphere is described by

$$x^2 + y^2 + z^2 = R^2. \tag{1.14}$$

The distance is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{1.15}$$

with a constraint. In spherical coordinates we have

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ &= R^2 \end{aligned} \tag{1.16}$$

Thus r is constant. By substitution, we find

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2) \tag{1.17}$$

This is in a flat 3-dimensional space. But if we set $r = R$ and thus $\mathrm{d}r = 0$, we obtain

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2) \tag{1.18}$$

So we plug this into the variation

$$\delta \int \mathrm{d}s = 0 \tag{1.19}$$

to get the geodesic equation.

Example 1.2. Consider a surface in $\mathbb{R}^3$ defined by

$$z = f(x, y). \tag{1.20}$$

So now we have

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{1.21a}$$

$$= \mathrm{d}x^2 + \mathrm{d}y^2 + \left(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y\right)^2 \tag{1.21b}$$

$$= g_{xx}\,\mathrm{d}x^2 + 2g_{xy}\,\mathrm{d}x\,\mathrm{d}y + g_{yy}\,\mathrm{d}y^2 \tag{1.21c}$$

where we have

$$g_{xx} = 1 + \left(\frac{\partial f}{\partial x}\right)^2, \quad g_{xy} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}, \quad g_{yy} = 1 + \left(\frac{\partial f}{\partial y}\right)^2 \tag{1.22}$$

are the coefficients. These coefficients g_{ab} are called "**Components of the Metric Tensor**" These are the basic physical variables. There is one subtlety—we can have the same geometry described by *different coordinates!* For example, in Cartesian coordinates the plane $\mathbb{R}^2$ is described by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2, \tag{1.23a}$$

whereas in Polar coordinates it is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \tag{1.23b}$$

and although they describe the same geometry (a flat plane), the metric tensor is different.

Remark 1.3. Note that the way to tell there is an object experience rotation in spacetime is when the metric has a nonzero $g_{t\varphi} \neq 0$ term.

Example 1.4 (Flat $\mathbb{R}^2$)**.** Consider flat 2-dimensional space. We have

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \tag{1.24}$$

We want to describe a path, so we parametrize it:

$$(x, y) = (x(u), y(u)). \tag{1.25}$$

We want to extremize

$$\mathrm{d}s = \left[\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2\right]^{1/2}\mathrm{d}u \tag{1.26}$$

Lets call the bracketed term, say,

$$E \stackrel{\text{def}}{=} \left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2. \tag{1.27}$$

This is just an assignment of variables. Intuitively, it plays the role of "kinetic energy". We want to extremize

$$s = \int E^{1/2}\,\mathrm{d}u \tag{1.28}$$

What to do? Well,

$$\delta\int E^{1/2}\mathrm{d}u = \int \frac{1}{2}E^{-1/2}\delta E\,\mathrm{d}u \tag{1.29a}$$

$$= \int \frac{1}{2}E^{-1/2}\left[2\frac{\mathrm{d}x}{\mathrm{d}u}\delta\frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\mathrm{d}y}{\mathrm{d}u}\delta\frac{\mathrm{d}y}{\mathrm{d}u}\right]^{1/2}\mathrm{d}u \tag{1.29b}$$

$$= \int \frac{1}{2}E^{-1/2}\left[2\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}u}\delta x + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}u}\delta y\right]^{1/2}\mathrm{d}u \tag{1.29c}$$

We integrate by parts, and demand the variation vanishes at its endpoints, thus

$$\delta\int \mathrm{d}s = -\int\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}\frac{\mathrm{d}x}{\mathrm{d}u}\right)\delta x + \frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}\frac{\mathrm{d}y}{\mathrm{d}u}\right)\delta y\right]\mathrm{d}u. \tag{1.30}$$

So this is supposed to vanish, which implies for the coefficients

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}\frac{\mathrm{d}x}{\mathrm{d}u}\right) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}\frac{\mathrm{d}y}{\mathrm{d}u}\right) = 0 \tag{1.31}$$

We can integrate directly to find E =constant. There is a trick we never specified anything about u. So let us choose $u = s$, it's a perfectly kosher choice. Then

$$E = 1 \tag{1.32}$$

which makes the equations of motion

$$\frac{\mathrm{d}^2x}{\mathrm{d}u^2} = 0, \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}u^2} = 0. \tag{1.33}$$

This has its solution be

$$x(s) = as + b, \quad y(s) = \alpha s + \beta. \tag{1.34}$$

That's the geodesic for flat space in Cartesian coordinates.

Lecture 2. Geodesics.

Lets review the basic setup: gravity determines paths in spacetime, a set of preferred paths determine geometry, so we can try to go backwards and determine the geometry of spacetime from geodesics. Or given the curvature of spacetime, we can determine the geodesics.

If we consider charged bodies in the electromagnetic field, it is done in two steps: (1) use Maxwell's equations to determine the electric and magnetic fields; (2) use the Lorentz force Law to compute trajectories. If we are really careful, general relativity does the whole thing in a single step. If we have the field equations, we only get a consistent answer if everything moves (along a geodesic). In electromagnetism, we can hold something still with an uncharged body; yet for general relativity, the equivalence principle says (the analogous procedure) cannot happen.

2.1 Geodesics in Special Relativity

We were talking about deriving geodesics. Lets review spacetime in special relativity (see, e.g., Carroll [**Carroll,** §§1.1–1.4], Gibbons [**26**], or Giulini [**27, 28**] for more mathematically oriented discussions). The most basic feature: distances are relative, and time is relative. Events (things that occur at some place and time) are dependent on the observer, but the proper time s (or τ) is defined in special relativity as

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{2.1}$$

very much like Pythagoras' theorem but space and time come in different signs. We will work with units where $c = 1$. *Set $c = 1$*

If we have two events, then we can construct the geodesic from (x_0, t_0) to (x_1, t_1) as

$$s = \int_{(x_0,t_0)}^{(x_1,t_1)} \mathrm{d}s \tag{2.2}$$

then obtain from $\delta s = 0$ the equations of motion for the geodesic. These are determined, as by last time, to be the solution of the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}X^\mu(s) = 0. \tag{2.3}$$

In special relativistic spacetime, geodesics are the trajectories with the longest proper time, *not* the shortest! We will consider two examples and then the general case.

Example 2.1 ($\mathbb{R}^2$ revisited)**.** Lets consider geodesics in $\mathbb{R}^2$ using polar coordinates:

$$x = r\cos(\theta), \quad y = r\sin(\theta). \tag{2.4}$$

We then see

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \end{aligned} \tag{2.5}$$

We are interested in paths from (r_0, θ_0) to (r_1, θ_1). We label the path by some parameter u and write

$$s = \int \mathrm{d}s. \tag{2.6}$$

Note: we change the parameter $u = s$ *after* we've chosen the path. So we write

$$r = r(u), \quad \theta = \theta(u). \tag{2.7}$$

The integral becomes

$$\begin{aligned} s &= \int \frac{\mathrm{d}s}{\mathrm{d}u}\mathrm{d}u \\ &= \int \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}u}\right)^2}\mathrm{d}u \end{aligned} \tag{2.8}$$

which we extremize. We define

$$E = \left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}u}\right)^2 \tag{2.9}$$

and take the variation

$$\begin{aligned} \delta s &= \delta \int E^{1/2}\mathrm{d}u \\ &= \frac{1}{2}\int E^{-1/2}\delta E\mathrm{d}u \\ &= 0. \end{aligned} \tag{2.10}$$

First we need to compute δE, which is a triviality:

$$\delta E = 2\frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}\delta r}{\mathrm{d}u} + 2r\delta r\left(\frac{\mathrm{d}\theta}{\mathrm{d}u}\right)^2 + 2r^2\frac{\mathrm{d}\theta}{\mathrm{d}u}\frac{\mathrm{d}\delta\theta}{\mathrm{d}u}. \tag{2.11}$$

Then what? Well, plug it back into Eq (2.10) to find

$$\begin{aligned} \delta s &= \frac{1}{2}\int\left[-2\frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}\frac{\mathrm{d}r}{\mathrm{d}u}\right)\delta r + E^{-1/2}2r\left(\frac{\mathrm{d}\theta}{\mathrm{d}u}\right)^2\delta r - 2\frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}u}\right)\delta\theta\right]\mathrm{d}u \\ &= 0 \end{aligned} \tag{2.12}$$

We require the coefficients of δr, $\delta\theta$ must vanish[6]. We get a set of equations, and we have our particular path. This allows us *now* to set $u = s$, thus $E = 1$, and our equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right) = 0 \qquad (\delta\theta \text{ coefficient})$$

[6] Mathematicians may feel uneasy about this, but it is due to the fundamental lemma of variational calculus.

$$-\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = 0 \qquad (\delta r \text{ coefficient})$$

Trick #1: we have 2 second-order Ordinary Differential Equations. We can do some of the integration without even thinking about it (although this trick will give mathematicians indigestion). We have

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \implies \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = 1 \tag{2.13}$$

is the first integral of our two given equations.

The $\delta\theta$ coefficient is easy. It says

$$\begin{aligned} r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} &= (\text{constant}) \\ &\overset{\text{def}}{=} a. \end{aligned} \tag{2.14}$$

We plug this into the Eq (2.13) to find

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{a^2}{r^2} = 1. \tag{2.15}$$

Thus we have, rearranging and manipulating, the following expression

$$\begin{aligned} \mathrm{d}s &= \frac{\mathrm{d}r}{\sqrt{1-(a/r)^2}} \\ &= \frac{r\mathrm{d}r}{\sqrt{r^2-a^2}} \end{aligned} \tag{2.16}$$

and integration yields

$$s - s_0 = \sqrt{r^2-a^2} \implies r^2 = a^2 + (s-s_0)^2. \tag{2.17}$$

What to do? Well, we do the only thing we can do! We plug this expression for r into

$$r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = a$$

and we obtain

$$\left(a^2 + (s-s_0)^2\right)\frac{\mathrm{d}\theta}{\mathrm{d}s} = a. \tag{2.18}$$

We know how to solve first order differential equations, so we just integrate

$$\begin{aligned} \theta - \theta_0 &= \int \frac{a\mathrm{d}s}{a^2 + (s-s_0)^2} \\ &= \mathrm{ArcTan}\left(\frac{s-s_0}{a}\right). \end{aligned} \tag{2.19}$$

But we want to write $s - s_0$ in terms of θ, so we can plug it into Eq (2.17). What to do? Well, we can manipulate our result to obtain

$$\theta - \theta_0 = \mathrm{ArcTan}\left(\frac{s-s_0}{a}\right) \implies s - s_0 = a\tan(\theta-\theta_0). \tag{2.20}$$

So what? Well, plug this into Eq (2.17)

$$\begin{aligned} r^2 &= a^2\left(1+\tan^2(\theta-\theta_0)\right) \\ &= \frac{a^2}{\cos^2(\theta-\theta_0)}. \end{aligned} \tag{2.21}$$

So what? Well, this implies

$$r\cos(\theta-\theta_0)=a \tag{2.22}$$

is constant, which is precisely a straight line. Thus a geodesic in $\mathbb{R}^2$ using polar coordinates is precisely a straight line, the same result we obtained from considering a geodesic using Cartesian coordinates!

Example 2.2 (Hyperbolic Plane)**.** Lets consider the hyperbolic plane, where

$$\mathrm{d}s^2=\frac{1}{y^2}\left(\mathrm{d}x^2+\mathrm{d}y^2\right). \tag{2.23}$$

This is related to de Sitter space. We want to find geodesics, so we parametrize a path

$$x=x(u),\quad y=y(u) \tag{2.24}$$

then take

$$E=\frac{1}{y^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2\right]. \tag{2.25}$$

We take the variation

$$\begin{aligned}\delta s&=\delta\int E^{1/2}\mathrm{d}u\\&=\frac{1}{2}\int E^{-1/2}\delta E\mathrm{d}u\\&=0.\end{aligned} \tag{2.26}$$

First we consider the variation

$$\delta E=\frac{-2}{y^3}\delta y\left[\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2\right]+\frac{2}{y^2}\left[\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}\delta x}{\mathrm{d}u}+\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}\delta y}{\mathrm{d}u}\right] \tag{2.27}$$

Now we plugging this monstrous result into the variation of the length yields

$$\begin{aligned}\delta s&=\int E^{-1/2}\left[\frac{-1}{y}E\delta y+\frac{1}{y^2}\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}\delta x}{\mathrm{d}u}+\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}\delta y}{\mathrm{d}u}\right]\mathrm{d}u\\&=0\end{aligned} \tag{2.28}$$

Integration by parts gives the δx coefficient

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{y^2}\frac{\mathrm{d}x}{\mathrm{d}u}\right)=0 \tag{δx coefficient}$$

and the first-integral trick gives

$$\frac{1}{y^2}\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2+\frac{1}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2=1. \tag{2.29}$$

The δx coefficient may be solved explicitly as

$$\frac{\mathrm{d}x}{\mathrm{d}u}=ky^2 \tag{2.30}$$

where k is a constant. We plug this into the first-integral equation (2.29)

$$k^2y^2+\frac{1}{y}^2\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2=1. \tag{2.31}$$

There are two possible families of geodesics: when $k \neq 0$ and when $k = 0$.

If $k \neq 0$, then

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}u} \tag{2.32}$$

and we can think of y as a function

$$y = y(x). \tag{2.33}$$

The differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}u} = ky^2\frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.34}$$

which we plug into the first-integral

$$k^2y^2 + k^2y^2\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 = 1 \implies \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 = \frac{1}{(ky)^2} - 1. \tag{2.35}$$

We may solve this

$$k^2\left[y^2 + (x - x_0)^2\right] = 1 \tag{2.36}$$

which is a circle! This family of geodesics are circles centered at $(x_0, 0)$.

If, on the other hand, $k = 0$ what happens? We see that the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}u} = ky^2 = 0 \tag{2.37}$$

implies x is a constant. Thus it is a straight line.

2.2 General Geodesic Equation

We have[7]

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{2.38}$$

where g_{ab} is called the "**Metric**" and $\mathrm{d}s^2$ is called the "**Line Element**". Indices we sum over are called "**Dummy Indices**", and we may relabel them as we please

g_{ab} is metric, $\mathrm{d}s^2$ line element

$$\begin{aligned}\mathrm{d}s^2 &= g_{ab}\mathrm{d}x^a\mathrm{d}x^b\\ &= g_{cd}\mathrm{d}x^c\mathrm{d}x^d\end{aligned} \tag{2.39}$$

The "**Signature**" of the metric means the number of positive and negative eigenvalues. There are two conventions for general relativity: $(+---)$ called the "**West Coast**" convention or "**Particle Physicists**" convention; and $(-+++)$ called the "**East Coast**" convention or "**Relativists Convention**".

Now lets derive the geodesic equation. We want the path which extremizes the arc-length. This is determined by demanding the variation of the arc-length vanishes

$$\delta s = 0. \tag{2.40}$$

As usual, we define

$$\boxed{E \stackrel{\text{def}}{=} g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u}} \tag{2.41}$$

where $x^a = x^a(u)$ labels a path. Thus we have

$$s = \int E^{1/2}\mathrm{d}u \tag{2.42}$$

[7] Remember, we are using the Einstein summation convention, so $x_ay^{ab} = \sum_a x_ay^{ab}$. When the index appears both "downstairs" and "upstairs", we sum over it implicitly. But note: the indices must have the same dummy variable, and one must be downstairs while another upstairs!!!

imply

$$\delta s = \frac{1}{2}\int E^{-1/2}\delta E\mathrm{d}u \tag{2.43}$$

where

$$\delta E = (\delta g_{ab})\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u} + 2g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}\delta x^b}{\mathrm{d}u}. \tag{2.44}$$

Note that

$$\delta g_{ab} = \frac{\partial g_{ab}}{\partial x^c}\delta x^c \tag{2.45}$$

since we can "wiggle" around a path, the metric varies along that path. We can notationally write

$$\delta g_{ab} = (\partial_c g_{ab})\delta x^c \tag{2.46}$$

where

$$\partial_c = \frac{\partial}{\partial x^c} \tag{2.47}$$

and no, that is not a typo. The index on ∂_c is supposed to be downstairs provided the denominator is ∂x^c. The reason is due to how this quantity behaves when we change coordinates.

Thus

$$\delta E = (\partial_c g_{ab})\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u} + 2g_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}u}\frac{\mathrm{d}\delta x^c}{\mathrm{d}u} \tag{2.48}$$

where we intentionally relabel the dummy indices. This renders

$$\begin{aligned}\delta\int\mathrm{d}s &= \frac{1}{2}\int E^{-1/2}\left[\delta x^c\partial_c g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u} + 2g_{cb}\frac{\mathrm{d}\delta x^c}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u}\right]\mathrm{d}u\\ &= \int\left[\frac{1}{2}E^{-1/2}\partial_c g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u} - \frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}g_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}u}\right)\right]\delta x^c\mathrm{d}u\\ &= 0\end{aligned} \tag{2.49}$$

where the second line, we integrated by parts and threw away the boundary terms. This implies the bracketed term must vanish:

$$\frac{1}{2}E^{-1/2}\partial_c g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u} - \frac{\mathrm{d}}{\mathrm{d}u}\left(E^{-1/2}g_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}u}\right) = 0 \tag{2.50}$$

and this is our geodesic equation.

We can choose $u = s$ and thus $E = 1$, giving us

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}s}\right) - \frac{1}{2}\partial_c g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} = 0.} \tag{2.51}$$

This is the geodesic equation. But also note we may pick another paramter u for which

$$\frac{\mathrm{d}E}{\mathrm{d}u} = 0 \tag{2.52}$$

and u is called an "**Affine Parameter**". For light rays it is conventional to use a λ for this parameter. Also notice

$$\mathrm{d}s^2 = \begin{cases}1 & \text{for massive bodies}\\ 0 & \text{for photons}\end{cases} \tag{2.53}$$

which is our last observation for now.

Lecture 3. Massive Geodesics.

We will study the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2\right) \tag{3.1}$$

where

$$m = \frac{GM}{c^2}. \tag{3.2}$$

For the Sun, this is approximately

$$m_\odot \sim 1.5 \text{ km} \tag{3.3a}$$

whereas for the Earth

$$m \sim 1 \text{ cm}. \tag{3.3b}$$

We would like to note:
(1) as $r \to \infty$, it looks like flat spacetime;
(2) at $r = 2m$, one component vanishes while the other blows up;
(3) for $r < 2m$, the signature "changes".
This last remark means space-like curves within the region looks time-like whereas time-like curves look space-like.

We solve the geodesic equation in the time component

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{ab}\frac{\mathrm{d}x^b}{\mathrm{d}s}\right) - \frac{1}{2}(\partial_a g_{bc})\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0. \tag{3.4}$$

Consider $a = 0$, then we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{ab}\frac{\mathrm{d}x^b}{\mathrm{d}s}\right) - 0 = 0 \tag{3.5}$$

which is a constant of motion:

$$\begin{aligned} g_{tt}\frac{\mathrm{d}x^t}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(1 - \frac{2m}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}s} \\ &= -\widetilde{E} \end{aligned} \tag{3.6}$$

This notation is tradition as it vaguely reminds us of energy.

Now for $a = 3$, the φ equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{\varphi b}\frac{\mathrm{d}x^b}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(g_{\varphi\varphi}\frac{\mathrm{d}x^\varphi}{\mathrm{d}s}\right) \tag{3.7}$$

since the metric is diagonal, and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{\varphi\varphi}\frac{\mathrm{d}x^\varphi}{\mathrm{d}s}\right) = 0. \tag{3.8}$$

Thus we have

$$r^2\sin^2(\theta)\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \widetilde{L} \tag{3.9}$$

be a constant of motion, which reminds us of angular momentum.

For the $a = 2$ equation, we can set $\theta = \pi/2$ and $\mathrm{d}\theta/\mathrm{d}s = 0$ for the initial condition. This eliminates the differential equation. We are left with radial geodesics.

The trick is to write the geodesic equation's first integral as

Trick: first integral of geodesic equation

$$\begin{aligned} g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} &= \left(1 - \frac{2m}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 \\ &= 1 \end{aligned} \tag{3.10}$$

Plugging in our constants of motion:

$$1 = \left(1 - \frac{2m}{r}\right)\widetilde{E}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - \left(\frac{\widetilde{L}}{r}\right)^2. \tag{3.11}$$

This looks like the Hydrogen atom in quantum mechanics! We can solve for

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = \widetilde{E}^2 - \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\widetilde{L}^2}{r^2}\right). \tag{3.12}$$

Our equations of motion becomes

$$\boxed{\begin{aligned} \widetilde{L} &= r^2\frac{\mathrm{d}\varphi}{\mathrm{d}s} \\ -\widetilde{E} &= \left(1 - \frac{2m}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}s} \\ \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 &= \widetilde{E}^2 - \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\widetilde{L}^2}{r^2}\right) \end{aligned}} \tag{3.13}$$

with the initial conditions $\theta = \pi/2$ and $\mathrm{d}\theta/\mathrm{d}s = 0$. It turns out that

$$s = \int \mathrm{d}s \tag{3.14}$$

is a messy elliptic integral. So lets consider various perturbative techniques to approximate it.

Lets consider

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{(\mathrm{d}r/\mathrm{d}s)^2}{(\mathrm{d}\varphi/\mathrm{d}s)^2} = \left(\frac{r}{\widetilde{L}}\right)^4[\dots]. \tag{3.15}$$

Things simplify if we write everything in terms of

$$u = \frac{1}{r}. \tag{3.16}$$

We have

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{(\widetilde{E}^2 - 1)}{\widetilde{L}^2} + \frac{2mu}{\widetilde{L}^2} - u^2 + 2mu^3. \tag{3.17}$$

If it weren't for the $2mu^3$ term, we could integrate this in closed form. The extra term is precisely the relativistic corrections, so we will treat it as a perturbation.

3.1 First Approximation

We simply ignore higher order terms. So we differentiate

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} = \frac{m}{\widetilde{L}^2} - u + 3mu^2 \tag{3.18}$$

and throw away the $3mu^2$ term, obtaining

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} = \frac{m}{\widetilde{L}^2} - u. \tag{3.19}$$

This has its solution be

$$u = \frac{m}{\widetilde{L}^2} + A\cos(\varphi). \tag{3.20}$$

We can rewrite this as

$$u = \frac{1 + e\cos(\varphi)}{a(1-e^2)} \tag{3.21}$$

where e is eccentricity, and a is the semimajor axis. We see that

$$r^{-1} = \alpha + \beta\cos(\varphi) \tag{3.22}$$

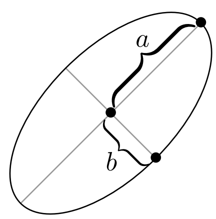
thus

$$\alpha r = 1 - \beta r\cos(\varphi) \tag{3.23}$$

and using Cartesian coordinates yields

$$x^2 + y^2 = \frac{1}{\alpha^2}(1-\beta x)^2. \tag{3.24}$$

What is this? Obviously an ellipse! This is wonderful, we recover the Newtonian solution to the Kepler problem.



3.2 Second Approximation

We write

$$u = u_0 + y \tag{3.25}$$

then our equation becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}\varphi^2} = \underbrace{\frac{m}{\widetilde{L}^2} - u_0}_{} - y + \underbrace{3mu_0^2}_{} + 6mu_0y + 3my^2 \tag{3.26}$$

where we choose u_0 to be such that the underbracketed terms vanish:

$$\frac{m}{\widetilde{L}^2} - u_0 + 3mu_0^2 = 0. \tag{3.27}$$

We ignore that $3my^2$ term as “really small.” Thus we have

$$\frac{\mathrm{d}^2y}{\mathrm{d}\varphi^2} \approx -(1-6mu_0)y \tag{3.28}$$

which has its solution be

$$y = A\cos(\sqrt{1-6mu_0}\,\varphi). \tag{3.29}$$

We see when

$$\varphi \mapsto \varphi + \frac{2\pi}{\sqrt{1-6mu_0}} \tag{3.30}$$

that $y \mapsto y$. For Mercury, this works out to be (2.7×10^{-5}) degrees per orbit, or roughly 0.1 arcseconds per orbit, or again roughly 43 arcseconds per century. Although this is small, it was observed by the end of the 19$^{\text{th}}$ century.

3.3 Remarks on Experiments

If the sun were oblate and the mass distribution followed this shape, then there is a quadrapole term in Newtonian gravity. It turns out the corrections account for roughly 4 arcseconds. After further more-precise experiments, it turns out that General Relativity is correct to one part in a thousand. The Messenger satellite is measuring the orbit of Mercury to great precision. This will theoretically give the next order term. We are also working on measuring the angular momentum of the Sun. This will contribute to extra terms in geodesic expressions.

The next planet to think about is Mars. Thanks to the Viking projects, there is precision to 100 meters of the orbit's measurements. This is fairly remarkable, if you think about it!

Another test is the binary pulsar (a pulsar is a neutron star that sends out a beam due to magnetic flux). There is a precision of roughly 17 arcseconds per year. We can do this for a binary star, but it is messy. There is a tidal distortions to the stars, so it's not a sphere. For about 27–28 binary stars their orbits agree with General Relativity (see, e.g., Kramer *et al.* [**33**]), but for a few binary stars General Relativity's predictions are really bad (most famously, DI Herculis [**16, 52**]). It's a mystery what's going on there! See Baumgarte, et al., [**8**] for simulating a binary neutron system, Laarakkers and Poisson [**34**] for rotating neutron stars.

In Scalar-Tensor theories, the extra contribution to the precession we get a scalar contribution which nearly cancel for neutron stars. For other stars, it may be observable.

EXERCISES

▸ **Exercise 1** (Geodesics on the two-sphere)**.** A two-dimensional sphere of radius R has a metric

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2) \tag{3.31}$$

Show that the geodesics of this metrics are great circles.

▸ **Exercise 2** (First integral of the geodesic equation)**.** Show that the equation

$$g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} = 1 \tag{3.32}$$

is a first integral of the geodesic equation, that is, that the s derivative of this equation vanishes whenever the geodesic equation holds.

▸ **Exercise 3** (Practice with tensors, indices, summation convention, etc.)**.** Consider the following problems.

1. Let ${\delta_a}^b$ be the Kronecker delta in an n-dimensional spacetime. Find ${\delta_a}^a$.
2. Suppose S_{ab} is symmetric (that is, $S_{ab} = S_{ba}$) and A_{ab} is antisymmetric (that is, $A^{ab} = -A^{ba}$). Show that $S_{ab}A^{ab} = 0$.
3. For A^{ab} as in part (2), and for an arbitrary tensor T_{ab}, show that

$$A^{ab}T_{ab} = A^{ab}(T_{ab} - T_{ba}) \tag{3.33}$$

▸ **Exercise 4** ([**LPPT,** 3.18])**.** Let $Y_{\alpha\beta\gamma}$ be an arbitrary tensor, show

$$Y_{\alpha\beta\gamma} \neq Y_{(\alpha\beta\gamma)} + Y_{[\alpha\beta\gamma]}. \tag{3.34}$$

Lecture 4. Lightlike Geodesics.

There are a few differences with lightlike geodesics and timelike geodesics. First we use an affine parameter λ instead of s. Second $\mathrm{d}s^2 = 0$ between events. So we have

$$g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} = 1 \qquad \text{(for a planet)}$$
$$g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0. \qquad \text{(for light)}$$

The only thing that changes is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = E^2 - \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2}. \tag{4.1}$$

The convention for light is to *not* use tildes on L and E. We also have

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{L^2}{r^4}. \tag{4.2}$$

We can see the angle as a function of distance, instead of the other way around:

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}u}\right)^2 = \frac{L^2}{E^2 - L^2u^2(1-2mu)} \tag{4.3}$$

which is the same sort of problem we've seen before. In particular, the "Newtonian Approximation" is

Newtonian Approximation

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}u} &= \frac{1}{\sqrt{(E/L)^2 - u^2}}\\ &= \frac{1}{\sqrt{b^{-2} - u^2}}.\end{aligned} \tag{4.4}$$

The solution for our differential equation is

$$\varphi - \varphi_0 = \arcsin(bu) \tag{4.5}$$

and thus

$$r\sin(\varphi - \varphi_0) = b. \tag{4.6}$$

This is a straight line! In this approximation, light moves in a straight line.

4.1 First Approximation

It is useful to use the approximation

$$u^2 - 2mu^3 \approx u^2(1-2mu)^2 - m^2u^4. \tag{4.7}$$

Let us define

$$y = u(1-mu), \tag{4.8}$$

we can ignore m^2y^4 relative to y^2. So

$$\mathrm{d}y = \mathrm{d}u(1-2mu), \tag{4.9}$$

thus

$$\begin{aligned}\mathrm{d}u &= (1-2mu)^{-1}\mathrm{d}y\\ &\approx (1+2my)\mathrm{d}y.\end{aligned} \tag{4.10}$$

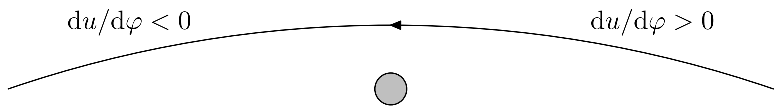
Then

$$\begin{aligned} \mathrm{d}\varphi &= \frac{\pm \mathrm{d}u}{\sqrt{(E/L)^2 - u^2(1-2mu)}} \\ &\approx \pm \frac{(1+2my)}{\sqrt{b^{-2}-y^2}}\mathrm{d}y. \end{aligned} \tag{4.11}$$

So

$$\begin{aligned} \frac{1}{2}\Delta\varphi &= \int_0^{b^{-1}} \left(\frac{1+2my}{\sqrt{b^{-2}-y^2}}\right)\mathrm{d}y \\ &= \pi + \frac{4m}{b}. \end{aligned} \tag{4.12}$$

This means light is bent, taking a trajectory roughly doodled thus:



Remark 4.1. Please avoid the temptation to Taylor expand in

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \frac{\pm 1}{\sqrt{b^{-2} - u^2 + 2mu^3}} \tag{4.13}$$

Do not Taylor expand the right hand side, specifically involving the mu^3 term, about 0. We get something circuitous if we try.

4.2 Second Approximation

An older technique no longer taught, perhaps the most straightforward, is

$$\omega^2 = u^2 - 2mu^3 \tag{4.14}$$

so we have

$$\varphi = \int_0^1 \frac{\mathrm{d}u}{\sqrt{1-\omega^2}}. \tag{4.15}$$

By expanding

$$u = \omega + \alpha_1\omega^2 + \alpha_2\omega^3 \tag{4.16}$$

we get a nice systematic perturbation. Although it is possible to solve equation (4.14), that is not the point! No, what we do is rewrite it as

$$\omega = u\sqrt{1-2mu} \tag{4.17}$$

then Taylor expand the squareroot on the right hand side up to some term. This is when we make our approximation:

$$\omega \approx u - mu^2. \tag{4.18}$$

We consider

$$m^2u^4 \lll u \tag{4.19}$$

as our approximation, so squaring equation (4.18) recovers equation (4.14).

Observe equation (4.18) is a quadratic equation in u, which has its solution

$$u_\pm = \frac{1}{2m}(1 \pm \sqrt{1-4m\omega}). \tag{4.20}$$

We Taylor expand this to third order in ω, taking the physically meaningful root $u = u_-$

$$\begin{aligned} u &\approx \frac{1}{2m}\left(2m\omega + \frac{1}{8}(4m\omega)^2 + \frac{1}{16}(4m\omega)^3\right) \\ &\approx \omega + m\omega^2 + \frac{1}{2}m^2\omega^3. \end{aligned} \tag{4.21}$$

Thus

$$\mathrm{d}u = \mathrm{d}\omega + 2m\omega\mathrm{d}\omega + \frac{3}{2}m^2\omega^2\mathrm{d}\omega, \tag{4.22}$$

and our integral becomes

$$\begin{aligned} \Delta\varphi &= \int_0^1 \frac{(1 + 2m\omega + \frac{3}{2}m^2\omega^2)}{\sqrt{1-\omega^2}}\mathrm{d}\omega \\ &= 2m + \left(1 + \frac{3m^2}{4}\right)\pi \end{aligned} \tag{4.23}$$

Observe we have an additional term involving m^2 in this approximation.

These calculations should be carefully double checked, and re-examined to make certain we did everything consistently. This is left as an exercise to you, gentle reader!

The astute reader probably feels discomfort at b disappearing. Observe that half the angle of deflection is

$$\begin{aligned} \frac{\Delta\varphi}{2} &= \int_0^{1/b} \frac{(1 + 2m\omega + \frac{3}{2}m^2\omega^2)}{\sqrt{b^{-2}-\omega^2}}\mathrm{d}\omega \\ &= \frac{\pi}{2} + \frac{2m}{b} + \frac{3\pi m^2}{8b^2}. \end{aligned} \tag{4.24}$$

Thus the total angle of deflection is

$$\Delta\varphi = \pi + \frac{4m}{b} + \frac{3\pi m^2}{4b^2}. \tag{4.25}$$

Notice this agrees, to first order in m, with the first approximation we made.

4.3 Third Approximation

Most introductory texts perform the following approximation

$$\begin{aligned} u^2 - 2mu^3 &= u^2(1-2mu) \\ &\approx u^2(1-mu)^2 \end{aligned} \tag{4.26}$$

Choose a new variable

$$y = u(1-mu), \tag{4.27}$$

and then our integral becomes

$$\varphi = \int \frac{\mathrm{d}u}{\sqrt{b^{-2} - y^2 + (\text{small factor})}} \tag{4.28}$$

To lowest order, this is the same trick as the first approximation. Higher order terms needs Newtonian corrections. We find

$$\mathrm{d}y = (1-2mu)\mathrm{d}u \tag{4.29}$$

and so

$$\mathrm{d}u \approx \frac{\mathrm{d}y}{1-2mu} \tag{4.30a}$$

$$\approx (1+2mu)\mathrm{d}y \tag{4.30b}$$
$$\approx (1+2my)\mathrm{d}y \tag{4.30c}$$

thus

$$\varphi = \pm \int \frac{(1+2my)\mathrm{d}y}{\sqrt{b^{-2}-y^2}}. \tag{4.31}$$

The first term is the Newtonian integral, and the second term is straightforward. Consider half of the path

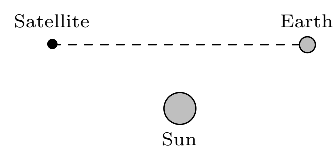
$$\begin{aligned}\varphi &= \int_{y=0}^{y=1/b} \frac{(1+2my)\mathrm{d}y}{\sqrt{b^{-2}-y^2}} \\ &= \frac{\pi}{2} + \frac{2m}{b}.\end{aligned} \tag{4.32}$$

So the total deflection $\Delta\varphi$ is twice this:

$$\Delta\varphi = \pi + 4\frac{m}{b}. \tag{4.33}$$

This is the first order correction.

4.4 Shapiro Time Delay



Here's the idea: send a radio signal from the Earth to the satellite. There is a time delay from receiving the reflection. The physical problem is doodled on the right, with the light's trajectory as the dashed line.

Lets assess the problem. Since this is light, we have

$$\begin{aligned}\mathrm{d}s^2 &= 0 \\ &= \left(1-\frac{2m}{r}\right)\mathrm{d}t^2 - \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2\end{aligned} \tag{4.34}$$

With a particular choice of coordinates we use the Newtonian approximation

$$r\sin(\varphi) = b. \tag{4.35}$$

What to do? Well, we can derive a geodesic equation for this approximation:

$$\sin(\varphi)\mathrm{d}r + r\cos(\varphi)\mathrm{d}\varphi = 0 \tag{4.36}$$

which is rearranged to become

$$\mathrm{d}\varphi = \frac{-1}{r}\tan(\varphi)\mathrm{d}r. \tag{4.37}$$

We square both sides and use basic trigonometry

$$\begin{aligned}r^2(\mathrm{d}\varphi)^2 &= \tan^2(\varphi)\,(\mathrm{d}r)^2 \\ &= \frac{b^2}{r^2-b^2}\mathrm{d}r^2.\end{aligned} \tag{4.38}$$

Why do this? Because we can replace the $r^2\mathrm{d}\varphi^2$ term in the $\mathrm{d}s^2$ expression:

$$\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 = \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + \frac{b^2}{r^2-b^2}\mathrm{d}r^2. \tag{4.39}$$

Remember we want to find the *time delay*, so we get rid of $\mathrm{d}t^2$ coefficient:

$$\mathrm{d}t^2 = \left(1-\frac{2m}{r}\right)^{-2}\mathrm{d}r^2 + \left(1-\frac{2m}{r}\right)^{-1}\frac{b^2}{r^2-b^2}\mathrm{d}r^2. \tag{4.40}$$

We make the approximation

$$\begin{aligned}\left(1-\frac{2m}{r}\right)^{-2} &\approx \left(1+\frac{2m}{r}\right)^{2}\\ &\approx 1+\frac{4m}{r}+\underbrace{\mathcal{O}(m^2/r^2)}_{\text{negligible}}\end{aligned} \tag{4.41}$$

which simplifies our expression to be

$$\mathrm{d}t^2 \approx \left[1+\frac{4m}{r}+\left(1+\frac{2m}{r}\right)\frac{b^2}{r^2-b^2}\right]\mathrm{d}r^2. \tag{4.42}$$

Now, we will perform a long and tedious calculation. The uninterested reader may skip its proof.

Proposition 4.2. *We have*

$$\left[1+\frac{4m}{r}+\left(1+\frac{2m}{r}\right)\frac{b^2}{r^2-b^2}\right] = \frac{r^2}{r^2-b^2}\left[1+\frac{4m}{r}-\frac{2m}{r}\frac{b^2}{r^2}\right] \tag{4.43}$$

Proof. We see that

$$\left(1+\frac{2m}{r}\right)\left(\frac{b^2}{r^2-b^2}\right) = \left(\frac{b^2}{r^2-b^2}+\frac{2m}{r}\frac{b^2}{r^2-b^2}\right) \tag{4.44}$$

Adding $(1+4m/r)$ to this yields

$$\begin{aligned}\left(1+\frac{4m}{r}\right)+\left(\frac{b^2}{r^2-b^2}+\frac{2m}{r}\frac{b^2}{r^2-b^2}\right) &= \left(1+\frac{4m}{r}+\frac{b^2}{r^2-b^2}+\frac{2m}{r}\frac{b^2}{r^2-b^2}\right)\\ &= \left(\frac{r^2}{r^2-b^2}+\frac{4m}{r}+\frac{2m}{r}\frac{b^2}{r^2-b^2}\right)\end{aligned} \tag{4.45}$$

Factoring out $r^2/(r^2-b^2)$ gives us

$$\left(\frac{r^2}{r^2-b^2}+\frac{4m}{r}+\frac{2m}{r}\frac{b^2}{r^2-b^2}\right) = \frac{r^2}{r^2-b^2}\left(1+\frac{4m(r^2-b^2)}{r^3}+\frac{2m}{r}\frac{b^2}{r^2}\right) \tag{4.46a}$$

$$= \frac{r^2}{r^2-b^2}\left(1+\frac{4m}{r}-\frac{4mb^2)}{r^3}+\frac{2mb^2}{r^3}\right) \tag{4.46b}$$

$$= \frac{r^2}{r^2-b^2}\left(1+\frac{4m}{r}-\frac{2mb^2)}{r^3}\right). \tag{4.46c}$$

This conclude the proof. □

Proposition (4.2) yields

$$\mathrm{d}t^2 \approx \frac{r^2}{r^2-b^2}\left[1+\frac{4m}{r}-\frac{2m}{r}\frac{b^2}{r^2}\right]\mathrm{d}r^2. \tag{4.47}$$

Thus we obtain (taking the Taylor series for the square root on the bracketed terms)

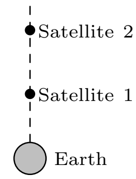
$$\mathrm{d}t \approx \frac{\pm r}{\sqrt{r^2-b^2}}\left[1+\frac{2m}{r}-\frac{mb^2}{r^3}\right]\mathrm{d}r. \tag{4.48}$$

What now?

We evaluate the integral

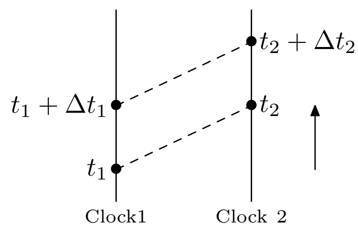
$$\int \mathrm{d}t = \pm\Bigg[\underbrace{\sqrt{r^2-b^2}}_{\substack{\text{length of line}\\ \text{in flat geometry}}} \underbrace{+2m\ln\left(\frac{r}{b}+\sqrt{\frac{r^2}{b^2}-1}\right)}_{\text{correction to first term}} -\frac{m}{r}\sqrt{r^2-b^2}\Bigg]. \tag{4.49}$$

4.5 Time Dilation



This is the last experiment we will consider: gravitational time dilation, or gravitational redshifting. Most books for the lay person describe it as “time runs more slowly in a gravitational field” (although the immediate question we should ask is: *relative to what?*).

So for that to make sense, we need to describe how we are measuring the rate of time, and how to compare these. We will work again with the Schwarzschild metric. An observer from clock 1 sends a signal to clock 2. We doodle a spacetime diagram, so it’s at the same angle (i.e., we assume $\theta = \varphi =$constant).



So the metric reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 = 0 \tag{4.50}$$

for light. After re-arranging terms we find

$$\mathrm{d}t = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r \tag{4.51}$$

Integration yields

$$t_2 - t_1 = \int_{r_1}^{r_2} \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r. \tag{4.52}$$

But look, we also have

$$(t_2 + \Delta t_2) - (t_1 + \Delta t_1) = \int_{r_1}^{r_2} \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r. \tag{4.53}$$

This implies

$$\Delta t_2 = \Delta t_1. \tag{4.54}$$

Now, an observer measures proper time, so

$$\Delta s_1 = \int \mathrm{d}s \tag{4.55a}$$

$$= \int \sqrt{1 - \frac{2m}{r_1}} \mathrm{d}t \tag{4.55b}$$

$$= \sqrt{1 - \frac{2m}{r_1}} \Delta t_1 \tag{4.55c}$$

where we consider the observer sitting at r_1 and *is not* a photon. Similarly, the observer at clock 2 will observe the interval between ticks as

$$\Delta s_2 = \sqrt{1 - \frac{2m}{r_2}} \Delta t_2 \tag{4.56a}$$

$$= \left(\frac{\sqrt{1 - \frac{2m}{r_2}}}{\sqrt{1 - \frac{2m}{r_2}}}\right) \Delta s_1 \tag{4.56b}$$

$$\approx \left(1 - \frac{m}{r_2} + \frac{m}{r_1}\right) \Delta s_1 \tag{4.56c}$$

for weak gravitational fields. We see that a photon wave is redshifted

$$\frac{\Delta\lambda}{\lambda} \approx -\frac{m}{r_2} + \frac{m}{r_1}. \tag{4.57}$$

An observer far from the black hole would see a clock on the Black Hole's horizon stop. There is nothing deep about this, however.

EXERCISES

▸ **Exercise 5** (The Newtonian Approximation)**.** In the Newtonian approximation, the spacetime metric is

$$\mathrm{d}s^2 = (1 + 2\phi)\,\mathrm{d}t^2 - (1 - 2\phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{4.58}$$

where ϕ is the Newtonian gravitational potential. This approximation holds when ϕ is small compared to 1 and velocities $v^i = \mathrm{d}x^i/\mathrm{d}t$ are also small compared to 1, with ϕ of the same order as v^2.

(Notation: Latin indices from the middle of the alphabet—i, j, k, …—are spatial indices, going from 1 to 3. Remember that we are using units $c = 1$.)

Show that to lowest order, the geodesics are the standard paths of Newtonian gravity, that is, $\mathbf{a} = -\nabla\phi$.

▸ **Exercise 6** (Geodesics and the Christoffel connection)**.** Let g^{ab} be the matrix inverse of the metric tensor, that is, $g^{ab}g_{bc} = \delta^a{}_c$. Show that the geodesic equation can be written in the form

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}s^2}\Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0 \tag{4.59}$$

where

$$\Gamma^a_{bc} = g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}) \tag{4.60}$$

Γ^a_{bc} is known as the Christoffel connection, or the "Christoffel symbols."

(Hint: you will encounter an expression of the form $\mathrm{d}g_{ab}/\mathrm{d}s$. Remember that in the geodesic equation, g_{ab} is the metric along the geodesic, and is therefore a function of $x^c(s)$. Use the chain rule.)

▸ **Exercise 7** (Deflection of a massive particle by the Sun)**.** In this problem, you will (approximately) compute the deflection of a *massive* particle in the Schwarzschild metric. Note: some of this is quite hard!

(a) Recall that for a massive particle, we defined

$$\widetilde{E} = -\left(1 - \frac{2m}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}s}. \tag{4.61}$$

Find the relationship between $\widetilde{E}$ and the particle speed

$$v^2 = \left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|^2 \tag{4.62}$$

at $r \to \infty$. (Hint: at infinity, the Schwarzschild metric reduces to the spherical coordinate form of the flat spacetime Minkowski metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$.)

(b) For a massive particle, the equation of motion we derived was

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{\widetilde{E}^2 - (1 - 2mu)(1 + \widetilde{L}^2u^2)}{\widetilde{L}^2} \tag{4.63}$$

with $u = 1/r$. Consider a particle coming in from infinity, being deflected, and returning to infinity. Find the deflection $\Delta\varphi$ in the Newtonian approximation, that is, neglecting the term mu^3. (The solution of the equation of motion is a hyperbola, and can be derived by a number of methods, but I suggest that you use the technique we saw in class, integrating $\mathrm{d}\varphi$, since this will help in part c.)

(c) Find the next order approximate expression for the deflection $\Delta\varphi$, treating the relativistic term mu^3 as a small perturbation. You can use the same method that we did in deriving the deflection of light, including the definition of a new variable $y = u(1 - mu)$, although the integral will now be somewhat different—be careful about the slightly tricky limits of integration! As in the case of light, assume that $mu \ll 1$.

(d) The impact parameter b is defined as the minimum value of r on the trajectory. You should already have worked this out in step (b) to find your integration range. (Note that b is the turning point, the value at which the derivative $\mathrm{d}u/\mathrm{d}\varphi$ changes sign.) Write $\widetilde{L}$ as a function of b, and rewrite the deflection $\Delta\varphi$ in terms of $\widetilde{E}$ and b. You may assume that $\widetilde{E}^2 - 1 \gg m/b$.

(e) Show that for speeds near the speed of light–that is, $v \lesssim 1$—the deflection is approximately

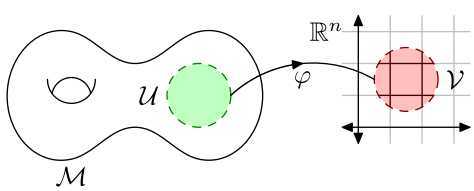
$$\Delta\varphi \approx \frac{2m}{b}\left(1 + \frac{1}{v^2}\right) \tag{4.64}$$

and agrees with our result for light when $v = 1$.

Part II
Mathematical Tools

Lecture 5. Manifolds.

A curved space is "locally like" $\mathbb{R}^n$. What does this mean? Well, we can take open discs in $\mathbb{R}^n$ and paste them together to form our curved space. The basic doodle describing this is thus:



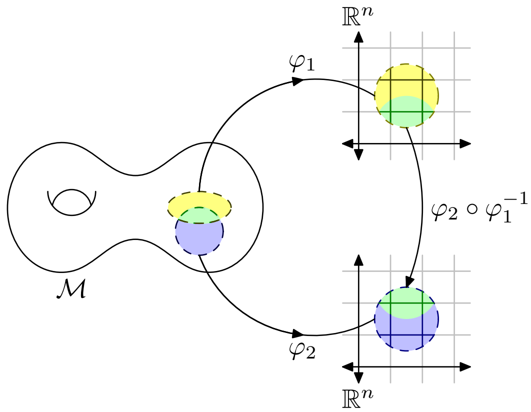
We have our curved space $\mathcal{M}$, and a neighborhood $\mathcal{U} \subset \mathcal{M}$ which is "like" a neighborhood $\mathcal{V}$ of $\mathbb{R}^n$. We make this rigorous by a mapping

$$\varphi\colon \mathcal{U} \to \mathcal{V} \tag{5.1}$$

and demand it is bijective (one-to-one and onto). The map φ is called a "**Coordinate Map**". Note that some conventions have φ going in the *opposite* direction, just a warning when reading other texts.

Note that since φ is invertible, we can express an point $p \in \mathcal{U}$ in terms of coordinates induced from $\varphi(p) \in \mathbb{R}^n$.

The question we should ask is: what happens on overlapping charts? We have two different descriptions, and we should hope that the descriptions are "the same." Lets consider the situation:



The minimal condition on $\varphi_2 \circ \varphi_1^{-1}$ is that it is continuous and has a continuous inverse (i.e., it's a "homeomorphism"). If $\varphi_2 \circ \varphi_1^{-1}$ is differentiable (or C^n or analytic or ...) and has a differentiable (C^n, analytics, ...) inverse, then $\mathcal{M}$ is a *Differentiable Manifold* (or a C^n Manifold, analytic manifold, etc.). In practice, it is sufficient using a C^2 or C^3 manifold for general relativity; however, most people prefer using C^∞ for as long as possible.

Lecture 6. Vectors.

The main lessons to take home:

1. Manifolds have coordinates and in practice the actual details of a calculation depends on using coordinates;
2. Coordinates gives a set of maps, the coordinate system are not the manifold itself. Are the properties found the properties of the coordinate systems or of the manifold?

We can answer the second point directly. Recall in $\mathbb{R}^2$ we have the line element be, in Polar coordinates,

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2. \tag{6.1}$$

But look, it has a peculiar value for θ when $r = 0$ (namely: distance is θ-independent when $r = 0$). But that's dependent on the coordinate system! On the other hand, if we change coordinate systems to write the line element as

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \theta^2\mathrm{d}r^2. \tag{6.2}$$

This is the same as equation (6.1), but θ is the radial distance and r is the angular component. Is there anything deep about $\mathbb{R}^2$ detected here? No, just faulty coordinate systems which break down at a single point.

Moral: Coordinate System's properties $\neq$ Manifold's properties.

In a manifold, there is no preferred basis, so we need to first define a basis prior to defining a vector. There are several ways to do this:

Old School: Deal with a manifold with coordinate system, and we use certain rules describing how vectors (and friends) behave under a change of coordinates.

New School: We observe the directional derivative in a particular direction is the same in any basis. So we say that a vector is $\vec{v}\cdot\nabla$, e.g., in two dimensions

$$\begin{aligned} \vec{v}\cdot\nabla &= v^x\partial_x + v^y\partial_y && (6.3a)\\ &= v^r\partial_r + v^\theta\partial_\theta && (6.3b) \end{aligned}$$

We find that

$$\begin{aligned} v^x &= v(x) \\ &= v^r \frac{\partial x}{\partial r} + v^\theta \frac{\partial x}{\partial \theta} \end{aligned} \tag{6.4}$$

We have a quantity that is independent of our choice of basis and it has a natural way to change under a change of coordinates.

In practice, we start with some manifold $\mathcal{M}$. We consider some curve

$$\gamma\colon [0,1] \to \mathcal{M}. \tag{6.5}$$

Let $p \in \mathcal{M}$, and consider a smooth function

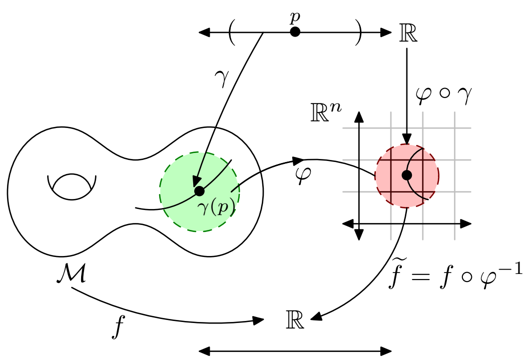
$$f\colon \mathcal{M} \to \mathbb{R} \tag{6.6}$$

We see that the mathematical definition for the vector v at p is

$$v_p(f) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\big(f \circ \gamma(\lambda)\big)\Big|_{\gamma(\lambda)=p}. \tag{6.7}$$

This is fairly abstract.

We may use local coordinates to make things a bit easier. Lets draw the doodle of the geometric situation:



We have our vector

$$\begin{aligned} v_p(f) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}(f \circ \gamma) && (6.8a) \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}(f \circ \varphi^{-1} \circ \varphi \circ \gamma) && (6.8b) \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\big(\widetilde{f} \circ (\varphi \circ \gamma)\big) && (6.8c) \\ &= \frac{\partial \widetilde{f}}{\partial \varphi^\mu} \frac{\mathrm{d}(\varphi \circ \gamma)^\mu}{\mathrm{d}\lambda} && (6.8d) \\ \text{“=”}&\, \partial_\mu \frac{\mathrm{d}\varphi^\mu}{\mathrm{d}\lambda} && (6.8e) \end{aligned}$$

where we ignore the distinction between the manifold and the coordinates on the manifold in this last step. Observe this looks like $\vec{v} \cdot \nabla \widetilde{f}$ the directional derivative!

The vector may be written as

$$v = v^\mu e_\mu, \tag{6.9}$$

with basis vectors

$$\{\partial_\mu\} = \{e_\mu\}. \tag{6.10}$$

We can have what is called an anholonomic (or "non-coordinate") basis *Anholonomic Basis*

$$f_a = f_a{}^{\mu} e_\mu \tag{6.11}$$

where $\det(f) \neq 0$ and $\mu, a = 1, \dots, n$. The coordinate basis satisfies

$$[e_\mu, e_\nu] = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0. \tag{6.12}$$

The converse is true, if given any basis f_a and

$$[f_a, f_b] = 0, \tag{6.13}$$

then there is some coordinate system

$$f_a = \partial / \partial y^a. \tag{6.14}$$

This follows from the existence theorem on partial differential equations.

Lets consider how vectors transform under coordinate changes. We see that in two different coordinate systems ∂_μ and $\partial_{\mu'}$ we can write *Vectors under change of coordinates*

$$v = v^\mu \partial_\mu = v^{\mu'} \partial_{\mu'}. \tag{6.15}$$

Observe how it acts on x^ν:

$$\begin{aligned} v(x^\nu) &= v^\mu \partial_\mu x^\nu && \text{(6.16a)} \\ &= v^\mu \delta_\mu{}^\nu && \text{(6.16b)} \\ &= v^\nu && \text{(6.16c)} \end{aligned}$$

and using the other coordinate representation

$$\begin{aligned} v(x^\nu) &= v^{\mu'} \partial_{\mu'} x^\mu && \text{(6.17a)} \\ &= v^{\mu'} \frac{\partial x^\nu}{\partial x^{\mu'}} && \text{(6.17b)} \end{aligned}$$

and setting equals to equals tells us

$$v^\nu = v^{\mu'} \frac{\partial x^\nu}{\partial x^{\mu'}}. \tag{6.18}$$

This gives us the transformation law between our two coordinate systems.

Remark 6.1. Note that all the vectors living at a single base point $p \in \mathcal{M}$ form a linear space $\mathrm{T}_p\mathcal{M}$ called the "**Tangent Space at** p".

Lecture 7. Vector Fields.

So, some things worth knowing:

1. Vectors are defined independent of coordinates.
2. We can represent a vector as a directional derivative $v = v^\mu \partial_\mu$ which is also independent of coordinates.
3. The coordinate basis ∂_μ, we can have an arbitrary basis $e_a = e_a{}^\mu \partial_\mu$ which is a linear combination of basis vectors, and called a "**Frame**" or "**Vierbein**" in 4-dimensions (or tetrad).

▸ **Exercise 8.** Prove in polar coordinates we have $(\partial_r, \partial_\theta)$ is a coordinate basis, and $(\partial_r, r^{-1}\partial_\theta)$ is a basis but not a coordinates basis.

A vector field is a map $\mathcal{M} \to \mathrm{T}\mathcal{M}$ such that at each point $p \in \mathcal{M}$ we assign to it a tangent vector v_p in a "smooth way". In other fields of physics, we may work with a "tangent spinor" or something similar. Assigning such a gadget to each point in spacetime is really a "**Section**" of a fiber bundle (and doing it in such a way that we have a "tangent spinor" requires something more, something called a "**Solder Form**").

Although it appears straightforward to generalize from tangent vector to tangent spinor to an arbitrary tangent *gadget*, there is dangerous subtlety here! There may be *obstructions* to such generalizations, we require tools from algebraic topology to study such obstructions. See Hatcher [**Hat1,** §4.3] and [**Hat2**] for topological aspects, and Sharpe [**Sharpe**] for geometric aspects. There are some very serious applications to physics (e.g., involving Dirac operators), since Lorentzian manifolds have particularly unique topology.

We should warn the reader, there are three competing notations used in general relativity. We use Greek indices when referring to components in a coordinate basis, and Latin indices give components in an arbitrary basis, but later Latin indices in the middle of the alphabet (i, j, k, …) gives spatial components in a coordinate basis.

Now what is an example of a generalization we have discussed? Well, quite simple: the notion of a "**Covector**" (a.k.a., covariant vector, dual vector, one-form, etc.). It is an element in the dual space to $\mathrm{T}_p\mathcal{M}$, denoted $\mathrm{T}^*_p\mathcal{M}$ and called the "**Cotangent Space**". We indicate the cotangent bundle as $\mathrm{T}^*\mathcal{M}$, and it consists of all the covectors on $\mathcal{M}$.

Example 7.1. Given a manifold $\mathcal{M}$ and a function $f\colon\mathcal{M}\to\mathbb{R}$, then the "**Gradient**" of f is $\mathrm{d}f\in\mathrm{T}^*\mathcal{M}$. The directional derivative is

$$\mathrm{d}f[v] = v(f) \tag{7.1}$$

where $v\in\mathrm{T}\mathcal{M}$. More generally, if

$$\mathrm{d}f = \omega_\mu \mathrm{d}x^\mu, \tag{7.2}$$

then

$$\langle \mathrm{d}f|\partial_\nu\rangle = \partial_\nu f \tag{7.3a}$$

but also

$$\begin{aligned}
\langle \mathrm{d}f|\partial_\nu\rangle &= \langle\omega_\mu \mathrm{d}x^\mu|\partial_\nu\rangle && (7.3\text{b})\\
&= \omega_\mu\langle \mathrm{d}x^\mu|\partial_\nu\rangle && (7.3\text{c})\\
&= \omega_\mu\delta^\mu{}_\nu && (7.3\text{d})\\
&= \omega_\nu && (7.3\text{e})
\end{aligned}$$

and setting equals to equals yields

$$\omega_\nu = \partial_\nu f. \tag{7.4}$$

This implies

$$\mathrm{d}f = (\partial_\mu f)\mathrm{d}x^\mu \tag{7.5}$$

as before.

Lecture 8. Tensors.

A type (k,l)-tensor T is a multilinear map from k dual vectors and l vectors to $\mathbb{R}$:

$$T\colon \underbrace{\mathrm{T}^*\mathcal{M}\times\cdots\times\mathrm{T}^*\mathcal{M}}_{k\text{ times}}\times\underbrace{\mathrm{T}\mathcal{M}\times\cdots\times\mathrm{T}\mathcal{M}}_{l\text{ times}}\to\mathbb{R}. \tag{8.1}$$

This is a linear map, so if we know what it does on the basis vectors (and covectors), we know everything. We have

$$T(\mathrm{d}x^{\mu_1},\dots,\mathrm{d}x^{\mu_k},\partial_{\nu_1},\dots,,\partial_{\nu_\ell}) = T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell} \tag{8.2}$$

are the components of T in a coordinate basis.

We have this method of constructing new tensors out of old ones: the tensor product. The idea is simple, basically multiply the components together. More formally, if we take a (k,ℓ) tensor and a (m,n) tensor, their tensor product gives us a $(k+m,\ell+n)$ tensor denoted

$$\overset{(k,\ell)}{S} \otimes \overset{(m,n)}{T} = \overset{(k+m,\ell+n)}{U} \tag{8.3}$$

and it has components given by

$$\begin{aligned}(S\otimes T)(\omega_1,\dots,\omega_{k+m},v^1,\dots,v^{\ell+n}) = \\ S(\omega_1,\dots,\omega_k,v^1,\dots,v^\ell)T(\omega_{k+1},\dots,\omega_{k+m},v^{\ell+1},\dots,v^{\ell+n}).\end{aligned} \tag{8.4}$$

So what happens in practice? Well, consider a $(1,1)$ tensor

$$T = T^\mu{}_\nu \partial_\mu \otimes \mathrm{d}x^\nu \tag{8.5}$$

we say

$$\begin{aligned} T(\omega,v) &= T(\omega_\rho \mathrm{d}x^\rho, v^\sigma\partial_\sigma) && (8.6a)\\ &= T^\mu{}_\nu \partial_\mu \otimes \mathrm{d}x^\nu(\omega_\rho \mathrm{d}x^\rho, v^\sigma\partial_\sigma) && (8.6b)\\ &= T^\mu{}_\nu (\omega_\rho\partial_\mu \mathrm{d}x^\rho)\otimes(v^\sigma \mathrm{d}x^\nu \partial_\sigma) && (8.6c)\\ &= T^\mu{}_\nu(\omega_\rho \delta_\mu{}^\rho)(v^\sigma\delta^\nu{}_\sigma) && (8.6d)\\ &= T^\mu_\nu \omega_\mu v^\nu. && (8.6e)\end{aligned}$$

Again, this is what physicists say. Mathematicians would be a little more cautious, but get the same result.

Warning: It looks like anything with an index is a tensor, in some sense this is true but a tensor is independent of what basis you're using. So lets consider a misleading non-example: a type $(2,0)$ tensor $(\partial\omega)$ which satisfies

$$(\partial\omega)(\partial_\mu,\partial_\nu) = \partial_\mu\omega_\nu. \tag{8.7}$$

Suppose we choose a different coordinate system, then

$$(\partial\omega)(\partial_\mu,\partial_\nu) \neq (\partial\omega)(\partial_{\mu'},\partial_{\nu'}) \tag{8.8}$$

In other words: it is not even linear!
Moral: Indices don't make something an tensor!

Example 8.1 (Kronecker Delta)**.** A type $(1,1)$ tensor is the Kronecker delta

$$\begin{aligned}\delta &= \delta^\mu{}_\nu\partial_\mu\otimes \mathrm{d}x^\nu && (8.9a)\\ &= \partial_\mu \otimes \mathrm{d}x^\mu. && (8.9b)\end{aligned}$$

So

$$\delta(\omega,v) = \omega_\mu v^\mu. \tag{8.10}$$

We can show this by showing δ is linear, or we can show that it's independent of coordinates.

Example 8.2 (Field Strength Tensor)**.** The field strength tensor is a $(2,0)$ tensor

$$F = F_{\mu\nu}\mathrm{d}x^\mu\otimes \mathrm{d}x^\nu \tag{8.11}$$

with components

$$F_{0i}\sim E_i, \quad \text{and} \quad F_{ij}\sim B_k \tag{8.12}$$

Usually it is written $F = \mathrm{d}A + A\wedge A$.

Example 8.3 (Metric Tensor)**.** A $(2,0)$ tensor we have seen before is the metric tensor

$$g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{8.13}$$

which behaves on vectors as

$$g(v,w) = g_{\mu\nu}v^\mu w^\nu. \tag{8.14}$$

This is a generalization of the inner product.

The metric lets us change a vector to a dual vector. Consider (in some basis) a vector $v = v^\mu\partial_\mu$, then $g(v,-)$ is an object taking a vector to a real number:

$$\begin{aligned} g(v,-)\colon \mathrm{T}\mathcal{M} &\to \mathbb{R}, \\ \omega &\mapsto g(v,\omega). \end{aligned} \tag{8.15}$$

Component-wise this looks like

$$\begin{aligned} g(v,-) &= (g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu)(v^\rho\partial_\rho, -) && \text{(8.16a)} \\ &= g_{\mu\nu}\langle \mathrm{d}x^\mu | v^\rho\partial_\rho\rangle \mathrm{d}x^\nu && \text{(8.16b)} \\ &= g_{\mu\nu}(v^\rho {\delta_\rho}^\mu)\mathrm{d}x^\nu && \text{(8.16c)} \\ &= (g_{\mu\nu}v^\mu)\mathrm{d}x^\nu. && \text{(8.16d)} \end{aligned}$$

NOTATION: $\quad g_{\mu\nu}v^\mu = v_\nu$.

Note that we need one more condition for g to be a metric: it must be nondegenerate. So

$$g(u,v) = 0 \quad \text{for all } v \tag{8.17}$$

only when $u = 0$. Equivalently, $g_{\mu\nu}$ must be invertible. Its inverse is denoted $g^{\mu\nu}$ so

$$g_{\alpha\mu}g^{\mu\beta} = {\delta_\alpha}^\beta. \tag{8.18}$$

The metric has to be symmetric. The inverse metric tensor

$$g = g^{\mu\nu}\partial_\mu \otimes \partial_\nu \tag{8.19}$$

is also an honest tensor.

Lecture 9. Tensor Densities.

9.1 Metric Signatures, Index Gymnastics

We can use the metric to go from covectors to vectors, and back again. We also use it for index gymnastics:

$$\begin{aligned} {T^{\mu\nu}}_\sigma g_{\mu\rho} &= {{T_\rho}^\nu}_\sigma && \text{(9.1a)} \\ {{T_\rho}^\nu}_\sigma g^{\rho\tau} &= {T^{\mu\nu}}_\sigma g_{\mu\rho}g^{\rho\tau} && \text{(9.1b)} \\ &= {T^{\tau\nu}}_\sigma && \text{(9.1c)} \end{aligned}$$

For $g_{\mu\nu}(x)$ at a point, we can find coordinates where this is of the form

$$g_{\mu\nu}(x) = \mathrm{diag}(\underbrace{-1,\dots,-1}_{p}, \underbrace{+1,\dots,+1}_{q}), \tag{9.2}$$

where $p + q = n$.

The signature of the metric is

$$\begin{pmatrix}\text{metric}\\ \text{signature}\end{pmatrix} = \begin{pmatrix}\text{number}\\ \text{of } +\end{pmatrix} - \begin{pmatrix}\text{number}\\ \text{of } -\end{pmatrix} \tag{9.3}$$

Physicists always write $+---$ or $-+++$ indicating the metric signature, mathematicians write $(1,3)$ or $(3,1)$. If the signature is $(+\cdots+)$, the metric is called "**Riemannian**"; and if the signature is either $(-+\cdots+)$ or $(+-\cdots-)$, the metric is "**Lorentzian**".

At any point P with coordinates $\overline{x}$, there is a coordinate system in which *Riemann Normal Coordinates*

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \mathcal{O}(x-\overline{x})^2 \tag{9.4}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric. These coordinates are called the "**Riemann Normal Coordinates**". Physically this is a freely falling frame, no first order fictitious forces felt.

9.2 Tensor Densities

Suppose we have an n-dimensional manifold, we look at a totally antisymmetric type $(0,n)$ tensor

$$\mathsf{T} = T_{\mu_1\mu_2\ldots\mu_n} \tag{9.5}$$

has only one component. We see the nonzero component is

$$T_{012\ldots(n-1)} \tag{9.6}$$

since if $\mu_i = \mu_j$, antisymmetry demands $\mathsf{T}=0$ for the component. It looks like a function, but it doesn't transform as such.

How does a totally antisymmetric tensor transform under a change of coordinates? We write out the components

$$T_{\mu_1\mu_2\ldots\mu_n}\mathrm{d}x^{\mu_1}\otimes\cdots\otimes\mathrm{d}x^{\mu_n} = T'_{\nu_1\nu_2\ldots\nu_n}\mathrm{d}y^{\nu_1}\otimes\cdots\otimes\mathrm{d}y^{\nu_n} \tag{9.7}$$

The first thing to do is act on $(\partial'_{\rho_1},\ldots,\partial'_{\rho_n})$ which obey

$$\langle \mathrm{d}y^{\nu_i}, \partial'_{\rho_j}\rangle = \delta^{\nu_i}{}_{\rho_j}, \tag{9.8}$$

so

$$\begin{aligned} T'_{\rho_1\ldots\rho_n} &= T_{\mu_1\ldots\mu_n}\langle \mathrm{d}y^{\mu_1},\partial'_{\rho_1}\rangle(\ldots)\langle \mathrm{d}y^{\mu_n},\partial'_{\rho_n}\rangle \\ &= T_{\mu_1\ldots\mu_n}\frac{\partial x^{\mu_1}}{\partial y^{\rho_1}}(\ldots)\frac{\partial x^{\mu_n}}{\partial y^{\rho_n}} \end{aligned} \tag{9.9}$$

Wonderful, lets work in a concrete situation: 2-dimensional manifolds. We see

$$T'_{01} = T_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^0}\frac{\partial x^{\nu}}{\partial y^1} \tag{9.10a}$$

$$= T_{01}\frac{\partial x^0}{\partial y^0}\frac{\partial x^1}{\partial y^1} + T_{10}\frac{\partial x^1}{\partial y^0}\frac{\partial x^0}{\partial y^1} \tag{9.10b}$$

$$= T_{01}\left(\frac{\partial x^0}{\partial y^0}\frac{\partial x^1}{\partial y^1} - \frac{\partial x^1}{\partial y^0}\frac{\partial x^0}{\partial y^1}\right) \tag{9.10c}$$

$$= T_{01}\det\left|\frac{\partial x^{\mu}}{\partial y^{\nu}}\right| \tag{9.10d}$$

We can generalize this result

$$T'_{01\ldots(n-1)} = T_{0\ldots(n-1)}\det|\partial x/\partial y|. \tag{9.11}$$

An object which transforms this way is called a scalar density of weight -1.

Again, the general notion is a tensor density of weight ω is *Tensor Density of Weight ω*

$$T'_{\mu_1\ldots\mu_n} = T_{\nu_1\ldots\nu_n}\frac{\partial x^{\nu_1}}{\partial y^{\mu_1}}(\ldots)\frac{\partial x^{\nu_n}}{\partial y^{\mu_n}}\det|\partial y/\partial x|^{\omega}. \tag{9.12}$$

A tensor density is more general than a pseudotensor (recall: a pseudotensor is just some quantity with indices).

Example 9.1 (Determinant of Metric Tensor)**.** Consider

$$g = \det|g_{\mu\nu}| \tag{9.13}$$

Under a coordinate transformation $x^{\mu} \to y^{\nu}(x)$, we have

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial y^{\mu}}\frac{\partial x^{\beta}}{\partial y^{\nu}} \tag{9.14}$$

thus

$$g' = g\det|\partial x/\partial y|^{2} \tag{9.15}$$

So g is a scalar density of weight -2, and moreover this implies $\sqrt{|g|}$ is a scalar density of weight -1.

Example 9.2 (Levi–Civita Symbol)**.** Consider the alternating symbol

$$\widetilde{\varepsilon}_{\mu_1\dots\mu_n} = \begin{cases} +1 & \text{if even permutation}\\ -1 & \text{if odd permutation}\\ 0 & \text{if any two indices equal.}\end{cases} \tag{9.16}$$

It's a totally antisymmetric matrix, but it is not a tensor: the alternating symbol is a tensor density of weight $+1$. We can define a genuine tensor by

$$\varepsilon_{\mu_1\dots\mu_n} = \sqrt{|g|}\widetilde{\varepsilon}_{\mu_1\dots\mu_n} \tag{9.17}$$

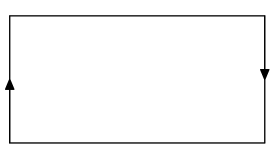
which is the Levi–Civita symbol. We should note, using abstract index notation, the cross product is

$$(A\times B)^{i} = g^{ij}\widetilde{\varepsilon}_{jk\ell}A^{k}B^{\ell}. \tag{9.18}$$

EXERCISES

▸ **Exercise 9** (Manifolds)**.** Give one example of a one-dimensional space that is a manifold, and one example of a one-dimensional space that is *not* a manifold. You can draw sketches to answer this question, but be sure to specify whether the end points of any line segments are or are not included.

▸ **Exercise 10** (The Möbius Strip)**.** The Möbius strip is the space formed by joining two ends of a strip with a 180° twist:



Show that this is a manifold, by constructing two coordinate charts and a transition function. Show all of the details—give the coordinate maps, etc. as explicitly as possible.

(Technically, the Möbius strip is a "manifold with boundary," since the top and bottom edges of the strip are boundaries that are not joined to anything.)

▸ **Exercise 11** (Derivations)**.** Consider the manifold $M = \mathbb{R}$ (the real line). A vector field is a differential operator,

$$v_x = v^{1}(x)\frac{\mathrm{d}}{\mathrm{d}x} \tag{9.19}$$

where the subscript x in v_x means we are evaluating v at point x, and the component $v^{1}(x)$ is an ordinary function. As a derivative, v obeys two rules:

1. linearity: $v_x(af+bg) = av_x(f)+bv_x(g)$

2. Leibniz rule (product rule): $v_x(fg) = g(x)v_x(f) + f(x)v_x(g)$.

Any operator obeying these two rules is called a "derivation."

Show that the converse is true: if v is a derivation on $\mathbb{R}$, then v *necessarily* has the form of Equation (9.19). Hint:

1. Show that $v(1) = 0$, where 1 means the constant function $f(x) = 1$.
2. Show that $v(c) = 0$ for any constant function $f(x) = c$.
3. Find $v(f)$ for functions $f(x) = x^n$.
4. Let $f(x)$ be an arbitrary function with a Taylor expansion around $x = 0$. Show that the desired relation holds for the Taylor expansion.

(Technically, this isn't quite enough for the proof—you should also consider functions with no Taylor expansion around $x = 0$—but it will do for this course.) In some mathematical approaches, a vector field on a manifold is *defined* as a derivation.

▸ **Exercise 12** (Commutators)**.** Let u and v be two tangent vectors, in the somewhat careful mathematical sense described in class and in, e.g., section 2.3 of Carroll [**Carroll**].

1. Show that the commutator $[u, v]$, defined by

$$[u, v](f) = u(v(f)) - v(u(f)) \tag{9.20}$$

is a derivation (see Exercise 11). This is enough to show that it is a tangent vector.

2. Find an expression for the commutator $[fu, v]$, where f is an arbitrary differentiable function.

▸ **Exercise 13** (Exterior derivative)**.** Let ω be a one-form, that is, a cotangent vector, and let $\langle \omega, u\rangle$ be the pairing between one-forms and vectors. Define a map $\mathrm{d}\omega$ from $TM \times TM$ to $\mathbb{R}$—that is, a function that takes two tangent vectors and gives a real number—by

$$\mathrm{d}\omega(u, v) = \frac{1}{2}\Big(u(\langle \omega, v\rangle) - v(\langle \omega, u\rangle) - \langle \omega, [u, v]\rangle\Big) \tag{9.21}$$

where $[u, v]$ is the commutator defined in Exercise 12.

1. Show that $\mathrm{d}\omega$ is a tensor, that is, that it is a bilinear map—in other words, that

$$\mathrm{d}\omega(fu + gv, w) = f\,\mathrm{d}\omega(u, w) + g\,\mathrm{d}\omega(v, w) \tag{9.22}$$

(and similarly for the second argument). You will need the answer from part 2 of Exercise 12 to do this.

2. Find the components $\mathrm{d}\omega(\partial_\mu, \partial_\nu)$ in a coordinate basis.

▸ **Exercise 14** (Tensors and coordinate transformations)**.**

1. Show that under a change of coordinates, the Kronecker delta $\delta^a{}_b$ transforms as a tensor.
2. Show that the derivative $\partial_a v^b$ of the components of a vector does not transform as a tensor under coordinate changes.
3. Does the antisymmetrized derivative $\partial_a v^b - \partial_b v^a$ of the components of a one-form (covariant vector) transform as a tensor under coordinate changes? Show how you reach your conclusion.

Lecture 10. Exterior Algebra.

We want differentiate quantities. Why not just differentiate "in the obvious way" (i.e., take their derivatives!)? The problem with just taking derivatives is we get under a change of coordinates

$$\partial_\mu v^\nu \neq \partial_{\mu'} v^{\nu'}. \tag{10.1}$$

What to do? Well, we should recall that a "***p*-Form**" is a totally antisymmetric $(0,p)$-tensor

p-form

$$\omega = \omega_{\mu_1\ldots\mu_p}\mathrm{d}x^{\mu_1}\otimes\cdots\otimes\mathrm{d}x^{\mu_p} \tag{10.2}$$

What do we mean by totally antisymmetric? Well, it obeys

$$\omega_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_p} = -\omega_{\mu_1\ldots\mu_{k+1}\mu_k\ldots\mu_p}. \tag{10.3}$$

For a 2-form, the components look like

$$\partial_\mu\omega_\nu - \partial_\nu\omega_\mu \tag{10.4}$$

for example.

10.1 Exterior Calculus

We will use the notation

$$A_{[\mu_1\ldots\mu_p]} = \frac{1}{p!}(-1)^\pi A_{\pi(\mu_1)\ldots\pi(\mu_p)} \tag{10.5}$$

where π is a permutation of the indices. Given a p-form A, and a q-form B, the exterior product is defined to be

Exterior Product

$$(A\wedge B)_{\mu_1\ldots\mu_{p+q}} = \left(\frac{(p+q)!}{p!q!}\right)A_{[\mu_1\ldots\mu_p]}B_{[\mu_{p+1}\ldots\mu_{p+q}]}. \tag{10.6}$$

We also have an exterior derivative

Exterior Derivative

$$(\mathrm{d}A)_{\mu_1\ldots\mu_{p+1}} = \partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}. \tag{10.7}$$

Consider concrete cases. If A is a 1-form, then

$$\begin{aligned}(\mathrm{d}A)_{\mu\nu} &= \partial_{[\mu}A_{\nu]}\\ &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\end{aligned} \tag{10.8}$$

Let A be a p-form and B be a q-form, then

$$\mathrm{d}(A\wedge B) = (\mathrm{d}A)\wedge B + (-1)^p A\wedge(\mathrm{d}B). \tag{10.9}$$

Equivalently, in this formulation, we have for a function f

$$\mathrm{d}f = \vec{\nabla}f \tag{10.10}$$

and

$$\mathrm{d}^2 f = 0. \tag{10.11}$$

Whenever we have a p-form A such that

$$\mathrm{d}A = 0 \tag{10.12}$$

we call it a "**Closed Form**". We have an "**Exact Form**" be a p-form B such that it is of the form

Closed form: $\mathrm{d}A = 0$
Exact Form $B = \mathrm{d}C$

$$B = \mathrm{d}C \tag{10.13}$$

where C is a $(p-1)$-form. Not all closed forms are exact.

Example 10.1 (Closed Inexact Form)**.** A closed but inexact form on a circle is $\mathrm{d}\theta$.

10.2 Differentiating Tangent Vectors

Consider an arbitrary tangent vector

$$v = v^\mu \partial_\mu = v^a e_a. \tag{10.14}$$

Lets consider what differentiation would look like in this approach, we have

$$\partial_\rho v \text{ “=” } (\partial_\rho v^a) e_a + v^a (\partial_\rho e_a). \tag{10.15}$$

But this second term is ambiguous? What should we have? Well, we should write

$$\partial_\rho e_a = e_b \Gamma^b{}_{\rho a} - e_b \Gamma_\rho{}^b{}_a \tag{10.16}$$

where $\Gamma_\rho{}^b{}_a$ is called the connection's components. In general, we may say absolutely nothing about the connection as it specifies the manifold. We can now write

$$\begin{aligned} \partial_\rho v \text{“=”} &(\partial_\rho v^b) e_b + e_b \Gamma_\rho{}^b{}_a v^a \\ \text{“=”} &\underbrace{\left(\partial_\rho v^b + \Gamma_\rho{}^b{}_a v^a\right)}_{\nabla_\rho v^b} e_b \end{aligned} \tag{10.17}$$

where $\nabla_\rho v^b$ is the "**Covariant Derivative**". The intuition is

$$\begin{pmatrix}\text{Covariant}\\ \text{Derivative}\end{pmatrix} = \begin{pmatrix}\text{Derivative in}\\ \text{Flat Space}\end{pmatrix} + \begin{pmatrix}\text{Corrections to}\\ \text{stay on the}\\ \text{manifold}\end{pmatrix} \tag{10.18}$$

where the connection components are precisely these correction terms. In a coordinate basis, the components becomes $\Gamma_\rho{}^\nu{}_\mu$ which are called the "**Christoffel Symbols**".

Lecture 11. Connection on Manifold.

We ended up with an expression for the covariant derivative

$$\nabla_\mu v^a = \partial_\mu v^a + \Gamma_\mu{}^a{}_b v^b \tag{11.1}$$

where $\Gamma_\mu{}^a{}_b$ is the corrections to stay on the manifold. The connection determines the geometry of the manifold.

Covariant differentiation commutes with contraction

$$\begin{aligned} \nabla_\mu (v^a w_a) &= (\nabla_\mu v^a) w_a + v^a (\nabla_\mu w_a) \\ &= (\partial_\mu v^a) w_a + v^a (\partial_\mu w_a) \end{aligned} \tag{11.2}$$

to satisfy the Leibniz property for covariant derivatives. This implies

$$\nabla_\mu w_a = \partial_\mu w_a - \Gamma_\mu{}^b{}_a w_b \tag{11.3}$$

and we observe

$$\nabla_\mu T^{ab}{}_c = \partial_\mu T^{ab}{}_c + \Gamma_\mu{}^a{}_d T^{db}{}_c + \Gamma_\mu{}^b{}_d T^{ad}{}_c - \Gamma_\mu{}^d{}_c T^{ab}{}_d. \tag{11.4}$$

Lets consider the second covariant derivative of a function

$$\begin{aligned} \nabla_\mu \nabla_\nu f &= \nabla_\mu (\partial_\nu f) \\ &= \partial_\mu \partial_\nu f - \Gamma_\nu{}^\rho{}_\mu \partial_\rho f. \end{aligned} \tag{11.5}$$

The mixed partials cancel, yielding

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = -(\Gamma_\mu{}^\rho{}_\nu - \Gamma_\nu{}^\rho{}_\mu) \partial_\rho f \tag{11.6}$$

which is a tensor. We call

$$\Gamma_{\mu}{}^{\rho}{}_{\nu} - \Gamma_{\nu}{}^{\rho}{}_{\mu} = T^{\rho}_{\mu\nu} \tag{11.7}$$

the “**Torsion Tensor**”. In Riemannian geometry, this is equal to zero.

The next thing to look at is the covariant derivative of the metric

$$\begin{aligned}\nabla_{\rho} g_{\mu\nu} &= \partial_{\rho} g_{\mu\nu} - \Gamma_{\rho}{}^{\sigma}{}_{\nu} g_{\mu\sigma} - \Gamma_{\rho}{}^{\sigma}{}_{\mu} g_{\sigma\nu} \\ &= 0\end{aligned} \tag{11.8}$$

where we obtain the second line through using metric compatibility. Physically this means if we take the inner product of two vectors, then move them along a geodesic, the inner product should be invariant.

What if we drop this? We get *Nonmetricity*

$$\nabla_{\rho} g_{\mu\nu} = K_{\rho\mu\nu} \tag{11.9}$$

which is called “**Nonmetricity**”. There is one appealing version of nonmetricity that Hermann Weyl introduced. Suppose we require

$$\nabla_{\rho} g_{\mu\nu} = A_{\rho} g_{\mu\nu} \tag{11.10}$$

where A_{ρ} is the electromagnetic 4-potential. This yields a tensor that looks like the field-strength tensor. But there is a problem since the length of a tensor is history-dependent, but this is observably untrue (e.g., the frequency of the photon from a Hydrogen atom's electron changing orbitals).

Take two vectors v^{μ}, w^{μ} and define a curve x^{μ} with tangent vector

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s} = u^{\mu}. \tag{11.11}$$

Observe, since we are moving along the curve

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{\mu\nu} v^{\mu} w^{\nu}) = u^{\rho} \partial_{\rho}(g_{\mu\nu} v^{\mu} w^{\nu}) \tag{11.12}$$

and since this is the derivative of a scalar invariant, we have

$$u^{\rho} \partial_{\rho}(g_{\mu\nu} v^{\mu} w^{\nu}) = u^{\rho} \nabla_{\rho}(g_{\mu\nu} v^{\mu} w^{\nu}). \tag{11.13}$$

Using the Leibniz rule yields

$$u^{\rho} \nabla_{\rho}(g_{\mu\nu} v^{\mu} w^{\nu}) = (u^{\rho} \nabla_{\rho} g_{\mu\nu}) v^{\mu} w^{\nu} + g_{\mu\nu} u^{\rho} \big((\nabla_{\rho} v^{\mu}) w^{\nu} + v^{\mu} (\nabla_{\rho} w^{\nu})\big). \tag{11.14}$$

If we move in such a way that *Parallel Propagation*

$$u^{\rho} \nabla_{\rho} v^{\mu} = 0, \tag{11.15}$$

called “parallel propagation,” the change in the inner product comes from the $\nabla g_{\mu\nu}$ term. The Weyl inner product gives us

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{\mu\nu} v^{\mu} v^{\nu}) = (u^{\rho} A_{\rho})(g_{\mu\nu} v^{\mu} v^{\nu}) \tag{11.16}$$

Observe this is how it changes “infinitesimally” along the path. The total magnitude ℓ changes as

$$\ell^2 \to \ell^2 \exp\left(\int A_{\rho}\, \mathrm{d}x^{\rho}\right) \tag{11.17}$$

It turns out, if we stick an $\mathrm{i} = \sqrt{-1}$ into Weyl's idea, we recover some notions in quantum field theory (e.g., phase shifting the Dirac field, etc.). For more historical details, see Straumann [**45**].

Lets assume we have a Torsion-free, metric-compatible connection. If we set the torsion to zero, we uniquely get

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{11.18}$$

which is the "**Christoffel Connection**". The geodesic equation can be written as

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2} + \Gamma^{\rho}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \tag{11.19}$$

If it turns out the nonmetricity is nonzero, we can write down the equation with all the nonmetricity as well as all the metric compatible components. This gives us two sets of paths that are different.

If we set

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \tag{11.20}$$

we have the geodesic equation becoming *Autoparallel transport*

$$u^\rho \nabla_\rho u^\mu = 0 \tag{11.21}$$

which is in some sense the derivative of the tangent is zero (or, in other words, "it remains parallel to itself"). This is precisely the condition of autoparallel transport. The two definitions of autoparallel and shortest distance become inequivalent for nonmetricity situations.

Proposition 11.1. *Let* $g = \det|g_{\mu\nu}|$*, then* ${\Gamma^\rho}_{\mu\rho} = (\sqrt{-g})^{-1}\partial_\mu\sqrt{-g}$.

Proposition 11.2. *For any vector* v^μ *we have* $\sqrt{-g}\nabla_\mu v^\mu = \partial_\mu(\sqrt{-g}v^\mu)$.

Lecture 12. Spin Connection.

We will consider the slickest way to compute connections. First recall the basic conditions for a connection are:

1. Torsion free $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$
2. Metric Compatible $\nabla_\mu g_{ab} = 0$.

We start with an orthonormal basis for tangent vectors

$$e_a = {e_a}^\mu \partial_\mu. \tag{12.1}$$

This makes sense only if we already have a metric, but orthonormality demands

$$g(e_a, e_b) = \eta_{ab} \tag{12.2}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$.

The covariant derivative

$$\nabla_\mu {e_a}^\rho = {\omega_\mu}^c{}_a {e_c}^\rho \tag{12.3}$$

where the index a labels which vector we're discussing, ρ labels the component of the vector we're discussing. This ω is called the "**Spin Connection**", the term originated from trying to understand particle spin in general relativity.

Lets examine the metric compatibility condition first (torsion-free is trivial). First observe in components

$$g(e_a, e_b) = g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab} \tag{12.4}$$

so the metric compatibility condition becomes

$$\nabla_\mu g_{ab} = 0 \tag{12.5a}$$

$$= \partial_\mu g_{ab} - (\omega_\mu{}^c{}_a g_{cb} + \omega_\mu{}^c{}_b g_{ac}) \tag{12.5b}$$
$$= 0 - \omega_{\mu ba} - \omega_{\mu ab} \tag{12.5c}$$

Metric compatibility implies antisymmetry in spin connection's orthonormal basis indices; this critically depends on $\partial_\mu g_{ab} = 0$. Thus metric compatibility implies

$$\omega_{\mu ab} = -\omega_{\mu ba}. \tag{12.6}$$

Now let us consider the torsion free condition.

Remember the spin connection is a connection in an orthonormal basis satisfies

$$\nabla_\mu e^a{}_\rho = \omega_\mu{}^{ca} e_{c\rho} \tag{12.7}$$

by the fact that the metric vanishes under covariant differentiation. In a theory with nonmetricity, we would get an extra term. This is antisymmetric, so

$$\nabla_\mu e^a{}_\rho = -\omega_\mu{}^a{}_c e^c{}_\rho \tag{12.8}$$

We can write the frame as a one-form:

$$e^a = e^a{}_\rho \, \mathrm{d}x^\rho \tag{12.9}$$

Observe

$$\nabla_\mu e^a{}_\rho = \partial_\mu e^a{}_\rho - \Gamma_\mu{}^\sigma{}_\rho e^a{}_\sigma \tag{12.10}$$

where a just labels which vector we're talking about. We can rewrite this as

$$\Gamma_\mu{}^\sigma{}_\rho e^a{}_\sigma = \partial_\mu e^a{}_\rho - \nabla_\mu e^a{}_\rho. \tag{12.11}$$

This lets us translate from Γ to ω given e. Observe

$$(\Gamma^\sigma_{\mu\rho} - \Gamma^\sigma_{\rho\mu}) e^a{}_\sigma = 0 \tag{12.12}$$

is the torsion-free condition. This implies

$$\partial_\mu e^a{}_\rho - \partial_\rho e^a{}_\mu + \omega_\mu{}^a{}_c e^c{}_\rho - \omega_\rho{}^a{}_c e^c{}_\mu = 0 \tag{12.13}$$

by substitution. If we use the tetrad one-form from Equation (12.9) we get something slick:

$$\boxed{\mathrm{d}e^a + \omega^a{}_c \wedge e^c = 0.} \tag{12.14}$$

Thus we have simply the same conditions as

1. Torsion-free $\quad \mathrm{d}e^a + \omega^a{}_c \wedge e^c = 0.$
2. Metric Compatible $\quad \omega_{\mu ab} + \omega_{\mu ba} = 0.$

Remark 12.1. The "First Cartan Structure Equation" is $\quad \mathrm{d}e^a + \omega^a{}_c \wedge e^c = 0.$

Applications

Example 12.2 (2-Sphere)**.** Lets recall the usual sphere $S^2 \subseteq \mathbb{R}^3$, it has its line element be

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\, \mathrm{d}\varphi^2. \tag{12.15}$$

We interpret the "d"s here as one-forms. So we obtain

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\ &= g_{ab}(e^a{}_\mu \, \mathrm{d}x^\mu) \otimes (e^b{}_\nu \, \mathrm{d}x^\nu) \end{aligned} \tag{12.16}$$

We can immediately read off an orthonormal basis

$$e^1 = \mathrm{d}\theta, \quad e^2 = \sin(\theta)\,\mathrm{d}\varphi. \tag{12.17}$$

We can rotate and take linear combinations if we want a new orthonormal basis. Finding an orthonormal basis is very much like completing a square.

The spin connection has only one-component:

$$\omega_{12} = -\omega_{21} = \omega. \tag{12.18}$$

The Cartan structure has two components

$$\begin{aligned} \mathrm{d}e^1 + {\omega^1}_2 \wedge e^2 &= 0 && (12.19a)\\ &= \mathrm{d}(\mathrm{d}\theta) + \omega \wedge (\sin(\theta)\,\mathrm{d}\varphi) && (12.19b)\\ &= 0 + (\omega \wedge \mathrm{d}\varphi)\sin(\theta) && (12.19c) \end{aligned}$$

which implies

$$\mathrm{d}\varphi \wedge \omega = 0. \tag{12.20}$$

We expect the general solution should look like

$$\omega = A\,\mathrm{d}\varphi \tag{12.21}$$

The second structure equation yields

$$\begin{aligned} \mathrm{d}e^2 + {\omega^2}_1 \wedge e^1 &= 0 && (12.22a)\\ &= \mathrm{d}(\sin(\theta)\,\mathrm{d}\varphi) + \omega \wedge \mathrm{d}\theta && (12.22b)\\ &= \cos(\theta)\,\mathrm{d}\theta \wedge \mathrm{d}\varphi + \sin(\theta)\,\mathrm{d}\varphi \wedge \mathrm{d}\varphi - \omega \wedge \mathrm{d}\theta && (12.22c)\\ &= \cos(\theta)\,\mathrm{d}\theta \wedge \mathrm{d}\varphi + 0 - \omega \wedge \mathrm{d}\theta && (12.22d) \end{aligned}$$

So, if we gather terms together we find

$$-\cos(\theta)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta - \omega \wedge \mathrm{d}\theta = 0 \tag{12.23}$$

and thus

$$-\big(\cos(\theta)\,\mathrm{d}\varphi + \omega\big) \wedge \mathrm{d}\theta = 0. \tag{12.24}$$

This tells us

$$\omega = -\cos(\theta)\,\mathrm{d}\varphi. \tag{12.25}$$

Observe how easy it was to find these components, compared to the approach using the metric components.

Example 12.3 (Simple Cosmology)**.** Lets consider a simple metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big) \tag{12.26}$$

We can choose a basis of one forms quite simply:

$$\begin{aligned} e^0 &= \mathrm{d}t\\ e^i &= a(t)\,\mathrm{d}x^i, \end{aligned} \tag{12.27}$$

then apply Cartan's equation

$$\begin{aligned} \mathrm{d}e^0 + {\omega^0}_i \wedge e^i &= 0\\ &= a(t){\omega^0}_i \wedge e^i \end{aligned} \tag{12.28}$$

This tells us that ${\omega^0}_i$ does not have any dt's in it, so

$$ {\omega^0}_i = A_{ij}\,\mathrm{d}x^j. \tag{12.29} $$

We then have

$$ A_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j = 0 \implies A_{ij} = A_{ji}. \tag{12.30} $$

Now, we **GUESS** that

$$ A_{ij} = A\delta_{ij} \tag{12.31} $$

since this is the simplest symmetric tensor. Now we consider the other part of Cartan's structure equation

$$ \mathrm{d}e^i + {\omega^i}_j \wedge e^j + {\omega^i}_0 \wedge e^0 = 0. \tag{12.32} $$

We see

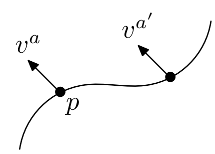
$$ \mathrm{d}e^i = \mathrm{d}a(t) \wedge \mathrm{d}x^i \tag{12.33a} $$
$$ = \dot{a}(t)\mathrm{d}t \wedge \mathrm{d}x^i \tag{12.33b} $$

and plug this back into Equation (12.32)

$$ 0 = \dot{a}\,\mathrm{d}t \wedge \mathrm{d}x^i + {\omega^i}_j \wedge (a\,\mathrm{d}x^j) + (A\,\mathrm{d}x^i \wedge \mathrm{d}t) \tag{12.34} $$

The simplest solution has ${\omega^i}_j = 0$ and $A = \dot{a}$. It turns out this is precisely the Friedmann–Lemaître–Robertson–Walker metric describing a simple universe.

Lecture 13. Curvature.



We will discuss a bit more about parallel transport, since it is the main ingredient in defining curvature.

Parallel transport a vector v^a at p along a curve through p. We find the parallel transport by considering

$$ \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\nabla_\mu v^a = 0, \tag{13.1} $$

which is a first order differential equation.

Consider flat space in Cartesian coordinates, this equation becomes

$$ \begin{aligned} \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\partial v^a}{\partial x^\mu} &= \frac{\mathrm{d}v^a}{\mathrm{d}s} \\ &= 0 \end{aligned} \tag{13.2} $$

which means we keep components constant. In flat space, using arbitrary coordinates, this is *necessarily* true too!

In curved space, using the Christoffel connection, a geodesic is given by

$$ \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\nabla_\mu\left(\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right) = 0. \tag{13.3} $$

This demands that the tangent to the geodesic remains parallel to itself. Further, if

$$ \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\nabla_\mu v^a = 0 \tag{13.4} $$

this implies the length of v is constant and the angle between v and the tangent is constant.

We can go backwards: first defining parallel transport, then obtaining the covariant derivative. Lets define Cartesian coordinates in flat space, which is trivial. We just parallel transport the orthonormal basis to each point in space.

Lets start with a curved space and an orthonormal frame at a point. We can then parallel transport it to every point. So there's a tad of a paradox. The problem is there are many ways to go from one point to another. It can yield two different bases, given two different parallel transports from one point to another. It can yield two different bases given two different parallel transports from one point to another. We can measure curvature by the difference. An equivalent procedure begins with a point, make a loop, then transport the frame around the loop. How it changes yields information about the curvature. Recall the geodesic equation

$$\frac{\mathrm{d}v^a}{\mathrm{d}s} + \Gamma_\mu{}^a{}_b \frac{\mathrm{d}x^\mu}{\mathrm{d}s} v^b = 0 \tag{13.5}$$

although for simplicity we write

$$\Gamma_\mu{}^a{}_b \frac{\mathrm{d}x^\mu}{\mathrm{d}s} = A^a{}_b. \tag{13.6}$$

Thus the geodesic equation is the familiar

$$\frac{\mathrm{d}v^a}{\mathrm{d}s} + A^a{}_b v^b = 0 \tag{13.7}$$

and $v^a(0)$ is given.

We can use standard methods for solving coupled differential equations, or we can integrate

$$\begin{aligned}
v^a(s) &= v^a(0) - \int_0^s A^a{}_b(s_1) v^b(s_1)\,\mathrm{d}s_1 && (13.8)\\
&= v^a(0) - \int_0^s A^a{}_b(s_1)\left[v^b(0) - \int_0^{s_1} A^b{}_c(s_2) v^c(s_s)\,\mathrm{d}s_2\right]\mathrm{d}s_1 && \text{(iterate)}\\
&= v^a(0) - \int_0^s A^a{}_b(s_1)\left[v^b(0) - \int_0^{s_1} A^b{}_c(s_2)\left(v^c(0) - \int_0^{s_2} A^c{}_d(s_3) v^d(s_3)\mathrm{d}s_3\right)\mathrm{d}s_2\right]\mathrm{d}s_1\\
& && \text{(iterate again)}
\end{aligned}$$

This is the origin of holonomy. This iterative process yields *Holonomy*

$$v^a(s) = \sum_{n=0}^{\infty} \int_0^s \int_0^{s_1} \cdots \int_0^{s_{n-1}} \big(A(s_1)A(s_2)\ldots A(s_n)\big)^a{}_b v^b(0)\,\mathrm{d}s_n \ldots \mathrm{d}s_1. \tag{13.9}$$

We define the path-ordering operator *Path-Ordering Operator $\mathcal{P}$*

$$\mathcal{P}\big(A(s_1)B(s_2)\big) = \begin{cases} A(s_1)B(s_2) & \text{if } s_1 > s_2 \\ B(s_2)A(s_1) & \text{if } s_2 > s_1 \end{cases} \tag{13.10}$$

The mnemonic is "later is last" (where last is the left-most).

So we have

$$\begin{aligned}
v^a(s) &= \sum \frac{(-1)^n}{n!}\,\mathcal{P}\left(\int_0^s \cdots \int_0^s \big(A(s_1)\ldots A(s_n)\big)^a{}_b v^b(0)\,\mathrm{d}s_n \ldots \mathrm{d}s_1\right)\\
&= \mathcal{P}\Bigg(\underbrace{\exp(-\int_0^s A\,\mathrm{d}s_1)^a{}_b}_{\text{Parallel Transport Matrix}}\, v^b(0)\Bigg).
\end{aligned} \tag{13.11}$$

For a closed loop, it is called the "**Holonomy**" of the loop. We can write the holonomy as

$$H = \mathcal{P}\left(\exp(-\oint \Gamma_\mu\,\mathrm{d}x^\mu)^a{}_b\, v^b(0)\right). \tag{13.12}$$

For an infinitesimal curve, we find

$$H^a{}_b = \delta^a{}_b + R_{\mu\nu}{}^a{}_b \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \begin{pmatrix}\text{Higher Order} \\ \text{Terms}\end{pmatrix} \tag{13.13}$$

where using the ordinary Stokes' theorem[8]

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \Gamma_\nu{}^a{}_b - \partial_\nu \Gamma_\mu{}^a{}_b + \Gamma_\mu{}^a{}_c \Gamma_\nu{}^c{}_b - \Gamma_\nu{}^a{}_c \Gamma_\mu{}^c{}_b \tag{13.14}$$

which is the curvature tensor. It's antisymmetric in the first two indices, and if we lower the a index the last two indices are antisymmetric in an orthonormal frame.

Remark 13.1. We have the Holonomy group $\mathrm{GL}(4,\mathbb{R})$ in general, but we can impose symmetry. The A is a Lie-algebra valued one-form.

The curvature form is thus *Curvature Form*

$$\mathcal{R}^a{}_b = \frac{1}{2} R_{\mu\nu}{}^a{}_b \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{13.15}$$

in that case. In an orthonormal basis, the equation for curvature becomes:

$$\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{13.16}$$

This is the second Cartan structure equation.

Note that Equation (13.16) holds only when we work with an orthonormal frame!

Once we have the curvature two-form (i.e., we know what $\mathcal{R}^a{}_b$ is) we can go back to Equation (13.15) to obtain $R_{\mu\nu}{}^a{}_b$ in some basis.

Using symmetries of the curvature tensor, some authors (e.g., Carroll [**Carroll**]) write $\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Just be forewarned on the different equivalent ways of defining the curvature two-form!

Lecture 14. Geodesic Deviation, Curvature Properties.

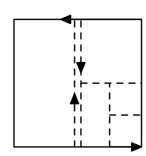
Parallel transport v^a around a closed loop, and we get

$$v^a_{\parallel} = H^a{}_b v^b \tag{14.1}$$

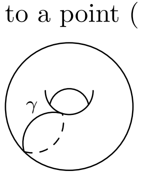
where

$$H^a{}_b = \delta^a{}_b + \int_S R_{\mu\nu}{}^a{}_b \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu + \dots \tag{14.2}$$

is the holonomy (as before).



We can get the Holonomy as a product of holonomies of arbitrarily small curves. This idea is doodled on the right, which resembles the Fibonacci rectangle. In this sense, the curvature tensor tells you everything. But we have to assume this curvature encloses a surface. It is possible this is not the case. In the case of the torus, the circle curve γ cannot be broken into smaller closed curves. A space is called "**Simply Connected**" if every loop can be continuously shrunk to a point ("contracted").



Curvature is a measure of the inability to define a (universal) Cartesian coordinate system.

If we look at the commutator of covariant derivatives,we find

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v^a = R_{\mu\nu}{}^a{}_b v^b. \tag{14.3}$$

▸ **Exercise 15.** Check this explicitly!

Recall

$$\nabla_\nu v^a = \partial_\nu v^a - \Gamma_\nu{}^a{}_b v^b. \tag{14.4}$$

We may think of this as an integrability condition, or an infinitesimal holonomy.

[8] Mathematicians call *everything* "Stokes' theorem", so be forewarned gentle physicist!

14.1 Geodesic Deviation

We would like to compare a one-parameter family of geodesics. One parameter is s, the proper time along the geodesic, and t which labels the geodesic we are on. There are two parameters

$$u^\mu = \frac{\partial x^\mu}{\partial s} \tag{14.5a}$$

$$X^\mu = \frac{\partial x^\mu}{\partial t} \tag{14.5b}$$

where u tells us the velocity, and X points from one geodesic to its neighboring geodesic. (Carroll [**Carroll**] refers to these as T^μ and S^μ, respectively.)

We can define a relative velocity

$$V^\mu = u^\rho \nabla_\rho X^\mu \tag{14.6}$$

which is the relative velocity rate the separation is changing in time. Similarly, we may define

$$A^\mu = u^\rho \nabla_\rho V^\mu \tag{14.7}$$

which is the relative acceleration.

Let us start with an identity

$$u^\rho \nabla_\rho X^\mu = X^\rho \nabla_\rho u^\mu \tag{14.8}$$

If we consider the difference between the right hand side and the left hand side

$$u^\rho \nabla_\rho X^\mu - X^\rho \nabla_\rho u^\mu = \dots \tag{14.9}$$

What happens? Well, the first thing to observe is that there are no terms involving connection components (they drop out). So we are left with:

$$u^\rho \nabla_\rho X^\mu - X^\rho \nabla_\rho u^\mu = u^\rho \partial_\rho X^\mu - X^\rho \partial_\rho u^\mu. \tag{14.10}$$

We also use the geodesic equation for autoparallel situations:

$$u^\mu \nabla_\mu u^\rho = 0. \tag{14.11}$$

Now what? Well, we will consider

$$u^\rho \nabla_\rho V^\mu = A^\mu. \tag{14.12a}$$

Using Equation (14.6) yields

$$u^\rho \nabla_\rho V^\mu = u^\rho \nabla_\rho (X^\nu \nabla_\nu u^\mu). \tag{14.12b}$$

Invoking Leibniz's rule

$$u^\rho \nabla_\rho V^\mu = \underbrace{(u^\rho \nabla_\rho X^\nu)}_{=V^\nu} \nabla_\nu u^\mu + X^\nu u^\rho \nabla_\rho \nabla_\nu u^\mu. \tag{14.12c}$$

We use commutation relations on the second term, yielding

$$u^\rho \nabla_\rho V^\mu = V^\nu \nabla_\nu u^\mu + X^\nu u^\rho \nabla_\nu \nabla_\rho u^\mu + X^\nu u^\rho \underbrace{(\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho) u^\mu}_{R_{\rho\nu}{}^\mu{}_\sigma u^\sigma} \tag{14.12d}$$

Now we observe

$$X^\nu u^\rho \nabla_\nu \nabla_\rho u^\mu = X^\nu \nabla_\nu \underbrace{(u^\rho \nabla_\rho u^\mu)}_{=0} - (X^\nu \nabla_\nu u^\rho) \nabla_\rho u^\mu. \tag{14.12e}$$

Plugging these expressions back into Equation (14.12d)

$$\begin{aligned} u^\rho \nabla_\rho V^\mu &= V^\nu \nabla_\nu u^\mu + 0 - (X^\nu \nabla_\nu u^\rho)\nabla_\rho u^\mu + X^\nu u^\rho R_{\rho\nu}{}^\mu{}_\sigma u^\sigma \\ &= X^\nu u^\rho R_{\rho\nu}{}^\mu{}_\sigma u^\sigma \end{aligned} \tag{14.12f}$$

Thus we obtain

$$\boxed{A^\mu = X^\nu u^\rho R_{\rho\nu}{}^\mu{}_\sigma u^\sigma.} \tag{14.12g}$$

Remember $\mathrm{d}x^\mu/\mathrm{d}s$ for slow bodies has

$$u^0 \sim 1, \quad \text{and} \quad u^i \sim v/c \ll 1. \tag{14.13}$$

Then the relative acceleration

$$A^i \sim R_{0\nu}{}^i{}_0 X^\nu \tag{14.14}$$

is proportional to the spatial derivative of the gradient of the potential. In classical Newtonian gravity, this gives tidal forces.

14.2 Symmetries of Riemann Tensor

Consider the curvature tensor $R_{\mu\nu\rho\sigma}$, there are some symmetries it has (or more precisely, its *indices* have). We see that

Symmetries of Riemann tensor:
(1) Skew Symmetries

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma} && (14.15a)\\ &= -R_{\mu\nu\sigma\rho} && (14.15b)\\ &= R_{\rho\sigma\mu\nu} && (14.15c)\end{aligned}$$

There are 2 pairs of antisymmetric indices, and those pairs are symmetric. Thus in 4-dimensions, there are only 21 independent components of the Riemann tensor. We also have the Jacobi identity in the last 3 indices:

(2) Jacobi Identity

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0. \tag{14.16}$$

We have, in 4-dimensions, only 20 independent components.

The last identity is known as the *Bianchi Identity*:

(3) Bianchi Identity

$$\nabla_{[\mu} R_{\nu\rho]\sigma\tau} = 0 \tag{14.17a}$$

or equivalently

$$\nabla_\mu R_{\nu\rho\sigma\tau} + \nabla_\nu R_{\rho\mu\sigma\tau} + \nabla_\rho R_{\mu\nu\sigma\tau} = 0. \tag{14.17b}$$

This actually follows from

$$\big[\nabla_\mu, [\nabla_\nu, \nabla_\rho]\big] + \big[\nabla_\nu, [\nabla_\rho, \nabla_\mu]\big] + \big[\nabla_\rho, [\nabla_\mu, \nabla_\nu]\big] = 0. \tag{14.18}$$

Note that if we include torsion, we need to modify the Bianchi identity to include some term proportional to the torsion.

14.3 Related Tensors

We should note that the only nontrivial contraction for the Riemann curvature tensor is

Ricci Tensor $R_{\mu\nu}$

$$g^{\mu\nu} R_{\mu\alpha\nu\beta} = R_{\alpha\beta}. \tag{14.19}$$

We call it the Ricci tensor. It follows from

$$R_{abcd} = R_{cdab} \tag{14.20}$$

that the Ricci tensor is symmetric. We can contract again to get

Scalar Curvature R

$$R = g^{\mu\nu} R_{\mu\nu} \tag{14.21}$$

which is the *Scalar Curvature* (sometimes called the Ricci scalar). It turns out that Einstein tensor (i.e., the traceless Ricci tensor)

Einstein Tensor $G_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{14.22}$$

is interesting, since contracting the Bianchi identities gives

$$\nabla_{\nu}G^{\mu\nu} = 0. \tag{14.23}$$

This is called the *contracted Bianchi identity.*

The Ricci tensor gives us the volume changing aspects of curvature. More precisely, given a "geodesic ball" in a manifold, the Ricci curvature tells us how it differs from a "flat ball". For General Relativity, the Ricci curvature determines the degree to which matter will tend to converge or diverge in time[9]. The remainder of the Riemann tensor is known as the Weyl tensor, which gives us the shear.

The last tensor worth mentioning is the Weyl tensor. In n-dimensions, we have

Weyl Tensor $C_{\alpha\beta\mu\nu}$

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}(g_{\rho[\mu}g_{\nu]\sigma}R). \tag{14.24}$$

Note that the Weyl tensor vanishes in 3-dimensions[10]. One interesting property the Weyl tensor possess is conformal invariance, i.e., under local rescalings

$$g_{\mu\nu} \to e^{2f}g_{\mu\nu} \tag{14.25}$$

where f is an arbitrary smooth function, the Weyl tensor remains the same.

EXERCISES

▸ **Exercise 16** (Bases and connections)**.** Given a basis of tangent vectors e_a, the connection $\Gamma^{b}_{\mu a}$ can be determined by that prescription that

$$\frac{\mathrm{d}e_a}{\mathrm{d}s} = e_b\Gamma^{b}_{\mu a}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s} \tag{14.26}$$

In general, this has to be either given from the start or determined from the metric, but it's useful to look at a simple example where it can be determined by what you already know.

A coordinate basis for Cartesian coordinates in two dimensions is

$$e_x = \frac{\partial}{\partial x}, \quad e_y = \frac{\partial}{\partial y} \tag{14.27}$$

The corresponding coordinate basis for polar coordinates is

$$\begin{aligned} e_r &= \frac{\partial}{\partial r} = \frac{x}{r}e_x + \frac{y}{r}e_y \\ e_\theta &= \frac{\partial}{\partial \theta} = -ye_x + xe_y. \end{aligned} \tag{14.28}$$

a. Show that the relation between e_x, e_y and e_r, e_θ given above is correct, that is, that $\partial/\partial r$ and $\partial/\partial\theta$ can be expressed in terms of $\partial/\partial x$ and $\partial/\partial y$ as shown.

b. Suppose that $\mathrm{d}e_x/\mathrm{d}s = \mathrm{d}e_y/\mathrm{d}s = 0$. (This means the space is flat; we'll see this later.) Using Equation (14.26), find the connection coefficients $\Gamma^{a}_{\mu b}$ in the polar coordinate basis.

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[9] This is precisely Raychaudhuri's equation, see Baez [**3**], Kar and SenGupta [**32**], Eric Poisson [**Poi,** §2], Hawking and Ellis [**HawEll,** §4.1].

[10] The Cotton tensor is used instead. See Garcia et al. [**25**].

c. A different, "noncoordinate" basis for polar coordinates is

$$e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{1}{r}\frac{\partial}{\partial \theta} \tag{14.29}$$

(This basis is useful in part because it's orthonormal; we'll see this later in the course.) Find the connection coefficients in this basis.

▸ **Exercise 17** (Killing vectors). A Killing vector χ^μ is a vector that satisfies the Killing equation

$$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = 0 \tag{14.30}$$

a. Show that for any Killing vector

$$\nabla^\mu \nabla_\mu \chi^\rho = -R^\rho{}_\sigma \chi^\sigma \tag{14.31}$$

(Hint: the way to get a curvature tensor in this kind of equation is by commuting covariant derivatives somewhere.)

b. Show that if the connection is the standard Christoffel connection, then the Killing Equation (14.30)

$$g_{\mu\rho}\partial_\nu \chi^\rho + g_{\nu\rho}\partial_\mu \chi^\rho + \chi^\rho \partial_\rho g_{\mu\nu} = 0 \tag{14.32}$$

▸ **Exercise 18** (Christoffel connection and curvature in two dimensions). Any Lorentzian metric on a two-dimensional manifold M can be locally put in the form

$$\mathrm{d}s^2 = e^{2\phi}(-\mathrm{d}t^2 + \mathrm{d}x^2) \tag{14.33}$$

by a suitable choice of coordinates in an open set on M. Here, ϕ is an arbitrary function of x and t. Find the Christoffel connection for this metric (in a coordinate basis). Find the curvature tensor. What condition must ϕ satisfy for the curvature to vanish?

(Hint: use the symmetry of the curvature tensor to see that there is only one independent nonzero component in two dimensions. This will save a lot of work.)

▸ **Exercise 19** ("Moving frames" in two dimensions). Starting with the metric of exercise 18 find an orthonormal basis of one-forms, and use the Cartan structure equations to find the connection one-form and the curvature two-form. Compare your result to exercise 18. (The answers had better be equivalent!)

Part III
General Relativity

Lecture 15. Deriving Field Equations.

Recall that the geodesic equation for nearby geodesics has the relative acceleration

$$A^\mu = R^\mu{}_{\sigma\rho\nu} u^\rho X^\nu u^\sigma. \tag{15.1}$$

Lets examine the Newtonian approximation $u^0 \approx 1$, $u^i \approx 0$. We see the spatial components of acceleration is

$$A^i \approx R^i{}_{00\nu} X^\nu = R^i{}_{00j} X^j. \tag{15.2}$$

Note that

$$R^i{}_{00j} = \partial_0 \Gamma^i_{0j} - \partial_j \Gamma^i_{00} + \underbrace{\Gamma^i_{0\mu}\Gamma^\mu_{0j} - \Gamma^i_{j\mu}\Gamma^\mu_{00}}_{\approx 0} \tag{15.3}$$

and

$$\partial_0 \Gamma^i_{0j} = 0 \tag{15.4}$$

since Γ^{i}_{0j} is time-independent. Thus

$$R^{i}{}_{00j} \approx -\partial_j \Gamma^{i}_{00} \tag{15.5}$$

for our approximation.

In Newtonian gravity, the acceleration for a particle is

$$\ddot{x}^i = -\partial^i \Phi(x) \tag{15.6}$$

Consider a separation vector $\vec{X} = X^j$, where we have one test particle described by x^i and another described by $x^i + X^i$. The second particle's acceleration is

$$\frac{\mathrm{d}^2(x^i + X^i)}{\mathrm{d}t^2} = -g^{ij}\partial_j \Phi(x^k + X^k) \tag{15.7}$$

We Taylor expand to find

$$\partial_j \Phi(x^k + X^k) = \partial_j \Phi(x^i) + \partial_k \partial_j \Phi(x^i) X^k + \dots \tag{15.8}$$

Substitute the Taylor expansion into Equation (15.7), and subtract out the acceleration of the first particle described in Equation (15.6), we obtain

$$\frac{\mathrm{d}^2 X^i}{\mathrm{d}t^2} = -\delta^{ij}\left(\partial_j \partial_k \Phi\right) X^k. \tag{15.9}$$

Compare this to Equation (15.2), we find

$$-R^{i}{}_{00k} = -\delta^{ij}\partial_j \partial_k \Phi \tag{15.10}$$

Observe that we can find one component of the Ricci tensor:

$$\begin{aligned} R_{00} &\approx R^{i}{}_{00i} \\ &= \nabla^2 \Phi = 4\pi G \rho \end{aligned} \tag{15.11}$$

where ρ is the mass-density.

This is wonderful, but we really want to consider some action

$$I = \int \mathcal{L}\,\mathrm{d}^4 x \tag{15.12}$$

such that its vanishing first variation $\delta I = 0$ yields the equations of motion. This should also be coordinate-independent. We recall that an n-form $L_{0\dots(n-1)}\,\mathrm{d}x^0 \wedge \dots \wedge \mathrm{d}x^{n-1}$ is coordinate-independent, and we may integrate it over an n-dimensional manifold. Since it is totally antisymmetric, we know it has one component. We can write this out as

$$\begin{aligned} \mathcal{L} &= (\text{Scalar})\sqrt{-g} \\ &= L\sqrt{-g}. \end{aligned} \tag{15.13}$$

We also have a few other requirements. A derivation of the action is given in Box 1, but the resulting Lagrangian is

$$\mathcal{L}_{EH} = (g^{\mu\nu} R_{\mu\nu} - 2\Lambda)\sqrt{-g} \tag{15.14}$$

giving us the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu} \tag{15.15}$$

where Λ is the cosmological constant, and $T_{\mu\nu}$ is the stress-energy tensor (describing the distribution of energy-momentum density).

Box 1. Einstein–Hilbert Action

Starting principles:

1. Action should be a coordinate-independent integral of a local Lagrangian.
2. Gravitational part should depend on metric only (no "background structure")
3. Geometry should be pseudo-Riemannian (no torsion or nonmetricity)
4. Field equations should contain no more than two derivatives of metric.

In four dimensions, most general action obeying these principles is

$$I_{EH} = \frac{1}{\kappa^2}\int \sqrt{-g}(R-2\Lambda)\,\mathrm{d}^4x = \frac{1}{\kappa^2}\int \sqrt{-g}(g^{\mu\nu}R_{\mu\nu}-2\Lambda)\,\mathrm{d}^4x \tag{15.16}$$

where κ and Λ are constants.

Variation of the action:

$$\delta I_{EH} = \frac{1}{\kappa^2}\int [\delta(\sqrt{-g})(R-2\Lambda) + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}]\,\mathrm{d}^4x \tag{15.17}$$

Look at three terms separately:

1. δg: basic relationship $\ln\det M = \operatorname{Tr}\ln M$

$$\delta\ln\det M = (\delta\det M)/\det M = \operatorname{Tr}\delta\ln M = \operatorname{Tr}(M^{-1}\delta M) = -\operatorname{Tr}(M\delta M^{-1}) \tag{15.18}$$

So $\delta g = -g_{\mu\nu}\delta g^{\mu\nu}, \quad \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$.

2. second term is already in right form, $\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}$.
3. $\delta R_{\mu\nu}$: first note that although the connection is not a tensor, $\delta\Gamma^\rho_{\mu\nu}$ *is* a tensor.

(To see this, consider the difference between two covariant derivatives, one with connection $\Gamma^\rho_{\mu\nu}$ and one with connection $\Gamma^\rho_{\mu\nu}+\delta\Gamma^\rho_{\mu\nu}$.)

Next check that

$$\delta R_{\mu\nu} = \nabla_\rho\delta\Gamma^\rho_{\mu\nu} - \nabla_\mu\delta\Gamma^\rho_{\nu\rho} \tag{15.19}$$

(You can check the variation explicitly, or you can look in Riemann normal coordinates, where $\Gamma^\rho_{\mu\nu}=0$ at some chosen point.)

Hence

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}) - \nabla_\mu(g^{\mu\nu}\delta\Gamma^\rho_{\rho\nu}) = \nabla_\rho[g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\rho\nu}\delta\Gamma^\sigma_{\nu\sigma}] \\ &= \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}\left[g^{\mu\nu}\Gamma^\rho_{\mu\nu} - g^{\rho\nu}\delta\Gamma^\sigma_{\nu\sigma}\right]\right) \end{aligned} \tag{15.20}$$

where the last step uses the fact that for a vector $\nabla_\mu v^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}v^\mu)$.

Now combine the three terms. The last term gives a total derivative,

$$\partial_\rho\left(\sqrt{-g}\left[g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\rho\nu}\delta\Gamma^\sigma_{\nu\sigma}\right]\right) \tag{15.21}$$

which integrates to zero as long as $\delta\Gamma$ goes to zero fast enough at any boundaries. That leaves the first two terms:

$$\delta I_{EH} = \frac{1}{\kappa^2}\int \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda)\right]\delta g^{\mu\nu}\,\mathrm{d}^4x \tag{15.22}$$

Now assume there is an additional "matter" contribution I_m to the action, and *define*

$$\delta I_m = -\frac{1}{2}\int \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\,\mathrm{d}^4x \tag{15.23}$$

Then

$$\delta I_{total} = \int \sqrt{-g}\left\{\frac{1}{\kappa^2}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda)\right] - \frac{1}{2}T_{\mu\nu}\right\}\delta g^{\mu\nu}\,\mathrm{d}^4x \tag{15.24}$$

and the field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{\kappa^2}{2}T_{\mu\nu} \tag{15.25}$$

Remark 15.1 (Further Reading). We have discussed the Einstein–Hilbert action, but there are other Lagrangians out there. Additionally, we have not even discussed other choice of variables, nor have we discussed the Hamiltonian formalism. These are reviewed in Peldan's "Actions for Gravity" [**40**]. We will discuss the Hamiltonian formalism in a follow up paper.

Lecture 16. Symmetries and Killing Vectors.

We will assume the cosmological constant vanishes $\Lambda = 0$. We see the field equations look like

$$G_{\mu\nu} = \frac{\kappa^2}{2}T_{\mu\nu} \tag{16.1}$$

Ten components of the curvature tensor directly depend on $T_{\mu\nu}$. The Weyl tensor indirectly depends on it. The $G_{\mu\nu}$ vanishes for flat space, yet the Weyl tensor describes the free propagation of gravity waves.

Recall in electromagnetism, the source of the electric field is charge e or in the field equations charge density ρ_e. With Lorentz transformation (viz. length contraction), charge density increases because volume decreases:

$$\begin{aligned}\rho_e &\to \frac{1}{\sqrt{1-v^2}}\rho_e\\ \implies \rho_e &\sim J^0\end{aligned} \tag{16.2}$$

where J^μ is the 4-current.

We know mass is responsible for gravity, but rest mass or total mass? Observationally, all forms of energy contributes to the gravitational field. So the energy E has energy density ρ_m. How does this transform? Well, we see:

$$\begin{aligned}E &\to \frac{1}{\sqrt{1-v^2}}E\\ \rho_m &\to \left(\frac{1}{1-v^2}\right)\rho_m\end{aligned} \tag{16.3}$$

This is what happens for a 00 components of a rank-2 tensor. So

$$\rho \approx T^{00} \tag{16.4a}$$

and similarly

$$\begin{aligned}T^{0i} &\approx \text{“Energy Current”}\\ &\approx \text{“Momentum Density”}\end{aligned} \tag{16.4b}$$

and

$$T^{ij} \approx \text{“Pressure”}. \tag{16.4c}$$

The field equations were thought of as

$$R_{\mu\nu} + g_{\mu\nu}R \propto T_{\mu\nu} \tag{16.5}$$

just as for Newtonian gravity

$$\nabla^2\Phi = 4\pi G\rho_m. \tag{16.6}$$

Einstein at one point proposed

$$R_{\mu\nu} = T_{\mu\nu} \tag{16.7}$$

but we can't change coordinates correctly, as these equations are under-determined. We know in special relativity the conservation of energy states

$$\partial_\mu T^{\mu\nu} = 0 \tag{16.8}$$

So by the comma-goes-to-semicolon rule, we expect

$$\nabla_\mu T^{\mu\nu} = 0. \tag{16.9}$$

But only

$$\nabla_\mu G^{\mu\nu} = 0 \tag{16.10}$$

whereas

$$\nabla_\mu R^{\mu\nu} \neq 0. \tag{16.11}$$

In general relativity, we use I for the action and S for the entropy (when we Wick rotate $t \to \tau = -\mathrm{i}t$, the Euclidean action for a black hole describes *is* its entropy). We have the action

$$I = \int \sqrt{-g}L\,\mathrm{d}^4x \tag{16.12}$$

and its variation is

$$\delta I = \int \sqrt{-g}E_{\mu\nu}\delta g^{\mu\nu}\,\mathrm{d}^4x \tag{16.13}$$

This action is diffeomorphism-invariant. If $\delta g^{\mu\nu}$ is just a coordinate transformation, then $\delta I = 0$ identically. This is true "on shell" (when the equations of motion are satisfied). What is $\delta g^{\mu\nu}$ under a change of coordinates? Consider

$$x^\mu \to x^\mu + \zeta^\mu \tag{16.14}$$

We see then that

$$g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu \to g_{\mu\nu}(x+\zeta)\big(\mathrm{d}x^\mu + \partial_\rho\zeta^\mu\mathrm{d}x^\rho\big)\big(\mathrm{d}x^\nu + \partial_\sigma\zeta^\nu\mathrm{d}x^\sigma\big) \tag{16.15}$$

where we Taylor expand to first order the metric

$$g_{\mu\nu}(x+\zeta) = g_{\mu\nu}(x) + \zeta^\tau\partial_\tau g_{\mu\nu}(x). \tag{16.16}$$

Observe this tells us how the metric changes, after some index gymnastics we obtain

$$\begin{aligned} g_{\mu\nu} &\to g_{\mu\nu} + (g_{\mu\rho}\partial_\nu\zeta^\rho + g_{\rho\nu}\partial_\mu\zeta^\rho + \zeta^\rho\partial_\rho g_{\mu\nu}) \\ &= g_{\mu\nu} + \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu \end{aligned} \tag{16.17}$$

Thus

$$\delta_\zeta g_{\mu\nu} = \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu. \tag{16.18}$$

If the right hand side vanishes, we have a Killing vector (c.f., Exercise 17). If we say the metric is time-independent, then this is equivalent to stating there exists some time-like Killing vector. Similarly, spherical symmetry means that we have Killing vectors generate the spherical symmetries.

So, we see that

$$\delta_\zeta g^{\mu\nu} = -\nabla^\mu\zeta^\nu - \nabla^\nu\zeta^\mu \tag{16.19}$$

So under a coordinate transformation

$$\delta I = -\int \sqrt{-g}E_{\mu\nu}(\nabla^\mu\zeta^\nu + \nabla^\nu\zeta^\mu)\,\mathrm{d}^4x. \tag{16.20}$$

We have $E_{\mu\nu} = E_{\nu\mu}$ which simplifies the integrand

$$\delta I = -2\int \sqrt{-g}E_{\mu\nu}\nabla^\mu\zeta^\nu\,\mathrm{d}^4x \tag{16.21}$$

since we're summing over dummy indices and E is symmetric. Now we may write this as

$$\delta I = -2\int \sqrt{-g}\Big[\nabla^\mu(E_{\mu\nu}\zeta^\nu) - (\nabla^\mu E_{\mu\nu})\zeta^\nu\Big]\,\mathrm{d}^4x \tag{16.22}$$

Recall

$$\nabla_\mu v^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}v^\mu) \tag{16.23}$$

thus the first term in the integrand becomes

$$-2\int \sqrt{-g}\nabla^\mu(E_{\mu\nu}\zeta^\nu)\,\mathrm{d}^4x = -2\int \partial_\mu(\sqrt{-g}E^\mu{}_\nu\zeta^\nu)\,\mathrm{d}^4x \tag{16.24}$$

which we can always do, since the metric's covariant derivative vanishes. This is a surface integral! Thus if $\zeta\to 0$ "fast enough" (or, equivalently, $\zeta = 0$ on the boundary), the first term vanishes.

Therefore

$$\int \sqrt{-g}(\nabla^\mu E_{\mu\nu})\zeta^\nu\,\mathrm{d}^4x = 0 \tag{16.25}$$

which is true for arbitrary ζ. This implies

$$\nabla^\mu E_{\mu\nu} = 0 \tag{16.26}$$

which is a conservation law. But we cannot change it into integral form. This is a special case of Noether's theorem. We can run this backwards to get the equations of motion.

Lecture 17. Stress-Energy Tensor.

Consider the stress energy tensor for a point particle. We have

$$x^\mu = z^\mu(u) \tag{17.1}$$

where u is some parameter, m be the particle's mass (at rest). We should recall the geodesic action is

$$\begin{aligned} I_{geod} &= m\int \mathrm{d}s \\ &= m\int \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\frac{\mathrm{d}x^\nu}{\mathrm{d}u}}\,\mathrm{d}u. \end{aligned} \tag{17.2}$$

Thus the matter action is

$$I_m = m\iint \delta^{(4)}\Big(x - z(u)\Big)\sqrt{g_{\mu\nu}\frac{\mathrm{d}z^\mu}{\mathrm{d}u}\frac{\mathrm{d}z^\nu}{\mathrm{d}u}}\,\mathrm{d}^4x\,\mathrm{d}u. \tag{17.3}$$

We're fixing z but varying $g_{\mu\nu}$, this is reasonably easy to do. Also note that

$$\int \delta^{(4)}(x)\,\mathrm{d}^4x = 1. \tag{17.4}$$

We'll get the stress-energy tensor for may particles, the consider the continuum limit.

First lets consider a single particle. We see that by varying the matter action with respect to the metric we obtain

$$\delta I = \frac{1}{2}\iint \left(g_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}u}\frac{\mathrm{d}x^\rho}{\mathrm{d}u}\right)^{1/2}\delta g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\frac{\mathrm{d}x^\nu}{\mathrm{d}u}\delta^4\Big(x - z(u)\Big)\,\mathrm{d}^4x\,\mathrm{d}u \tag{17.5}$$

Now we take $u = s$ as the parameter, and we find

$$\delta I = \frac{1}{2}\iint \delta g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\delta^4\Big(x - z(s)\Big)\,\mathrm{d}^4x\,\mathrm{d}s \tag{17.6}$$

Comparing this variation to

$$\begin{aligned}\delta I &= \frac{-1}{2}\int \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\mathrm{d}^4x \\ &= \frac{1}{2}\int \sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}\mathrm{d}^4x\end{aligned} \tag{17.7}$$

we find

Stress-Energy tensor for a point-particle

$$T^{\mu\nu} = \int m\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\delta^{(4)}\Big(x - z(s)\Big)\frac{1}{\sqrt{-g}}\,\mathrm{d}s. \tag{17.8}$$

This describes the stress-energy tensor for a point-particle.

Lets consider what happens in flat spacetime, just to get some intuition underpinning the components of a stress-energy tensor. We see

$$T^{0\mu} = \int m\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s}\delta^4\big(x - z(s)\big)\,\mathrm{d}s \tag{17.9a}$$

$$= \int m\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\delta\big(t - z^0(s)\big)\delta^3\big(x^i - z^i(s)\big)\,\mathrm{d}t \tag{17.9b}$$

$$= m\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\delta^3\big(x^i - z^i(s)\big)\Big|_{t=z^0(s)}. \tag{17.9c}$$

The time-time component reads

$$T^{00} = m\frac{\mathrm{d}t}{\mathrm{d}s}\delta^3\big(x^i - z^i(s)\big) \tag{17.10a}$$

$$= \frac{m}{\sqrt{1-v^2}}\delta^3\big(x^i - z^i(s)\big) \tag{17.10b}$$

$$= E\delta^3\big(x^i - z^i(s)\big) \tag{17.10c}$$

$$= \begin{pmatrix}\text{Energy}\\ \text{Density}\end{pmatrix}$$

and similarly

$$\begin{aligned}T^{0i} &= p^i\delta^3\big(x^i - z^i(s)\big) \\ &= \begin{pmatrix}\text{Momentum}\\ \text{Density}\end{pmatrix}\end{aligned} \tag{17.11}$$

This is just for a single particle, however.

Stress-Energy Tensor for Dust

For many non-interacting particles without charge (which relativists confusingly call "dust"), we have

$$T^{\mu\nu}_{(dust)} = \sum T^{\mu\nu}_{(particles)} \tag{17.12}$$

or taking the continuum limit

$$T^{\mu\nu}_{(dust)} = \rho u^\mu u^\nu \tag{17.13}$$

where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$. We take the continuum limit when we have a continuous collection of noninteracting particles. For most practical purposes in cosmology, this is good enough.

Stress-Energy Tensor for Perfect Fluid

A perfect fluid with pressure p experiences a stress-energy tensor

$$T^{\mu\nu} = (p+\rho)u^\mu u^\nu - pg^{\mu\nu}. \tag{17.14}$$

Observe that taking $p \to 0$ recovers the dust stress-energy tensor. The cosmological constant could be thought of as a perfect fluid term.

Now, lets see a miracle! Consider dust

$$T^{\mu\nu}_{\text{dust}} = \rho u^\mu u^\nu \tag{17.15}$$

and plug this into Einstein's field equation

$$G^{\mu\nu} = T^{\mu\nu}_{\text{dust}}. \tag{17.16}$$

We see that

$$\nabla_\mu G^{\mu\nu} = 0 \implies \nabla_\mu T^{\mu\nu}_{\text{dust}} = 0. \tag{17.17}$$

We will show that dust moves along geodesics. First we note

$$u^\mu u_\mu = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 1. \tag{17.18}$$

Now we consider

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho u^\mu)u^\nu + \rho u^\mu \nabla_\mu u^\nu. \tag{17.19}$$

We recall the conditions for a geodesic states

$$u_\mu \nabla_\nu u^\mu = 0. \tag{17.20}$$

Thus

$$u_\nu \nabla_\mu T^{\mu\nu} = \underbrace{u_\nu u^\nu}_{=1} \nabla_\mu(\rho u^\mu) + \rho u^\mu \underbrace{u_\nu \nabla_\mu u^\nu}_{=0} \tag{17.21}$$

which becomes

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho u^\mu). \tag{17.22}$$

This is a conservation equation. More explicitly, we can rewrite it as

$$\nabla_\mu(\rho u^\mu) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\rho u^\mu) = 0 \tag{17.23}$$

This is a very nice conservation law for mass which looks exactly like the conservation of electric charge. We can now go back and find

$$\rho u^\mu \nabla_\mu u^\nu = 0. \tag{17.24}$$

If $\rho \neq 0$ (i.e. in regions where the particles are present), then we necessarily have

$$u^\mu \nabla_\mu u^\nu = 0. \tag{17.25}$$

Field equations always give the motion of sources.

EXERCISES

▸ **Exercise 20** (Conservation and equations of motion)**.** The tensorial description of the electromagnetic field fits the electric field **E** and the magnetic field **B** together in an antisymmetric type 2 tensor $F_{\mu\nu}$, with field equations

$$\nabla_\nu F^{\mu\nu} = J^\mu, \quad \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0 \tag{17.26}$$

Consider a cloud of charged particles, with a mass density μ and a charge density ρ. The electromagnetic current for such a system at a point x is

$$J^\mu(x) = \rho(x)u^\mu(x) \tag{17.27}$$

where $u^\mu(x)$ is the four-velocity of the particle at point x. The stress-energy tensor consists of two pieces, a "dust" part

$$T^{\mu\nu}_{dust} = \mu u^\mu u^\nu \tag{17.28}$$

as discussed and an electromagnetic part

$$T^{\mu\nu}_{EM} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{17.29}$$

a. Show that the covariant conservation law

$$\nabla_\nu(T^{\mu\nu}_{dust} + T^{\mu\nu}_{EM}) = 0 \tag{17.30}$$

that follows from the Einstein field equations implies that

$$\mu u^\nu \nabla_\nu u^\mu = \rho F^\mu{}_\rho u^\rho. \tag{17.31}$$

(Note: you will have to use both sets of Maxwell's equations and the anti-symmetry of F.)

b. Suppose that the particles all have mass m and charge e, so $\rho/\mu = e/m$. Using the expression for $F_{\mu\nu}$ in terms of the electric and magnetic fields, show that in a flat spacetime, the equation derived in part (a) is just the Lorentz force law, $F = e(E + v \times B)$.

▸ **Exercise 21** (Massive Neutral Scalar Field)**.** The Lagrangian density for a massive neutral scalar field is

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 \tag{17.32}$$

where m is the mass, and ϕ is the scalar field.

1. Find the equations of motion from the Euler-Lagrange equation.
2. Find the stress-energy tensor for the scalar field.
3. Find the equations of motion from Einstein's field equation.

▸ **Exercise 22** ([**LPPT**])**.** Show the stress-energy tensor for source-less electromagnetism $T^{\mu\nu}_{\text{EM}}$ has zero trace.

Lecture 18. Linearized Gravity.

The weak field equations. The most important test of general relativity is that it gives us back Newtonian gravity. Lets consider a weak field, which can be thought of as a perturbation of a background $\eta_{\mu\nu}$. So

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{18.1}$$

and the inverse metric is given[11] by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2). \tag{18.2}$$

[11] We should recall that the Neumann series gives us $(I+X)^{-1} = I + X + X^2 + \cdots + X^n + \ldots$, which is employed here.

We raise and lower indices with η in this approximation. The Christoffel connection

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \\ &= \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(h^2).\end{aligned} \tag{18.3}$$

For the Ricci tensor, we only have the $\partial\Gamma$ terms to worry about, since $\Gamma\Gamma \sim \mathcal{O}(h^2)$. The components of the Ricci tensor are

$$R_{\mu\nu} = \frac{1}{2}(\partial^\sigma\partial_\mu h_{\sigma\nu} + \partial^\sigma\partial_\nu h_{\sigma\mu} - \partial^\sigma\partial_\sigma h_{\mu\nu} - \partial_\mu\partial_\nu h^{\sigma}{}_{\sigma}). \tag{18.4}$$

If we write

$$h = h^{\sigma}{}_{\sigma} \tag{18.5}$$

for the trace, then the "trace-reversed h" is

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{18.6}$$

Observe

$$\begin{aligned}\bar{h} &= \eta^{\mu\nu}\bar{h}_{\mu\nu} \\ &= h - 2h = -h.\end{aligned} \tag{18.7}$$

The Einstein tensor becomes

$$G_{\mu\nu} = \frac{1}{2}(-\Box\bar{h}_{\mu\nu} + \partial_\mu\partial^\sigma\bar{h}_{\sigma\nu} + \partial_\nu\partial^\sigma\bar{h}_{\sigma\mu} - \eta_{\mu\nu}\partial^\sigma\partial^\tau\bar{h}_{\sigma\tau}). \tag{18.8}$$

We can choose coordinates such that

$$G_{\mu\nu} = \frac{-1}{2}\Box\bar{h}_{\mu\nu}. \tag{18.9}$$

When

$$x^\mu \to x^\mu + \zeta^\mu \tag{18.10}$$

for infinitesimal ζ, then

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu \tag{18.11}$$

and the perturbation transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu + \mathcal{O}(h^2). \tag{18.12}$$

Observe this implies

$$\partial^\sigma\bar{h}_{\sigma\nu} \to \partial^\sigma\bar{h}_{\sigma\nu} + \Box\zeta_\nu. \tag{18.13}$$

We may choose ζ so that

$$\partial^\sigma\bar{h}_{\sigma\nu} \to 0. \tag{18.14}$$

We just have to solve

$$\Box\zeta_\nu = -\partial^\sigma\bar{h}^{(\text{old})}_{\sigma\nu} \tag{18.15}$$

which is the wave equation with a source. We know how to solve that! See, e.g., Jackson's electrodynamics text. We can choose coordinates such that

$$\partial^\sigma\bar{h}_{\sigma\nu} = 0 \tag{18.16}$$

which we call the harmonic gauge, Fock gauge, Lorenz gauge, de Donder gauge, etc.

Remark 18.1 (Physical Ramifications of Choice of Coordinates). There is no physical meaning for this choice of gauge (i.e., this particular choice of coordinates), nor does any other choice have physical meaning unless there exists strong symmetries which enable a canonical choice.

The harmonic gauge has Einstein's field equations read

$$\frac{-1}{2}\Box\bar{h}_{\mu\nu} = \frac{\kappa^2}{2}T_{\mu\nu}. \tag{18.17}$$

Among other things in life, this tells us (1) there exists gravity waves, (2) they travel at the speed of light because of the D'Alembertian.

18.1 Newtonian Limit

Lets consider the Newtonian limit, when

$$v/c \lll 1 \tag{18.18}$$

where v is the velocity of gravitating bodies. For non-interacting matter ("dust") we had the stress energy tensor

$$T^{\mu\nu} = \rho u^{\mu}u^{\nu} \tag{18.19}$$

which has components

$$T^{00} \approx \rho \tag{18.20}$$

and

$$T^{ij} \approx T^{i0} \approx T^{0j} \approx 0. \tag{18.21}$$

So we have

$$\Box\bar{h}^{00} = -\kappa^2\rho. \tag{18.22}$$

Observe the D'Alembertian is written

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{18.23}$$

but in the Newtonian approximation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} \approx 0. \tag{18.24}$$

Thus

$$\Box \approx -\nabla^2. \tag{18.25}$$

Our field equation becomes

$$-\nabla^2\bar{h}_{00} = -\kappa^2\rho. \tag{18.26}$$

This is precisely Poisson's equation for Newtonian gravity! That is

$$\nabla^2\Phi = 4\pi G\rho \tag{18.27}$$

thus by inspection

$$\bar{h}_{00} = \frac{\kappa^2}{4\pi G}\Phi \tag{18.28}$$

We see that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{18.29}$$

Hence

$$h_{\mu\nu}\frac{\kappa^2}{8\pi G}\Phi\eta_{\mu\nu}. \tag{18.30}$$

The line element in the Newtonian approximation is

$$\mathrm{d}s^2 = \left(1 + \frac{\kappa^2}{8\pi G}\Phi\right)\mathrm{d}t^2 - \left(1 - \frac{\kappa^2}{8\pi G}\Phi\right)\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}. \tag{18.31}$$

This is good: General Relativity contains Newtonian gravity at appropriate limits!

Remark 18.2. When we have very light masses moving close to the speed of light, we need to include other components of h; but we can still use the weak field approximation!

We now know that the field equations are

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{18.32}$$

we pull out our copy of Jackson, or Afkren (or whatever), and use the Green's function for the D'Alembertian

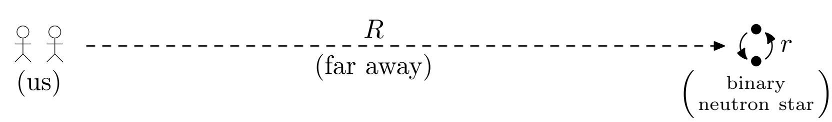
$$\begin{aligned}\bar{h}_{\mu\nu}(\mathbf{x},t) &= 4G\int\frac{T_{\mu\nu}(\vec{y},t-|\vec{x}-\vec{y}|)}{|\vec{x}-\vec{y}|}\mathrm{d}^3y\\ &= 4G\int\frac{T^{(\mathrm{ret})}_{\mu\nu}}{|\vec{x}-\vec{y}|}\mathrm{d}^3y\end{aligned} \tag{18.33}$$

We can interpret the next order corrections as gravity's coupling to the stress-energy tensor. To conclude our discussion, we will write a table comparing the multipole expansion in electromagnetism[12] and in gravity:

Multipole Term	Electromagnetism	Gravity
Monopole Moment	The total charge q; charge conserved, monopole moment is constant	The total mass m; Newtonian limit has mass conserved, fixed field unchanging in time.
Dipole Moment	$\sum q_i r_i$, $\dot{D} = \sum q_i\dot{r}_i$; Fix two charges to the ends of a spring and oscillate.	$\sum m_i r_i$, $\sum m_i\dot{r}_i = \sum p_i = 0$ in the center of mass frame. Try to oscillate total momentum, but this is fixed!
Magnetic Dipole	$\sum q_i\vec{v}_i\times\vec{r}_i$	$\sum m_i\vec{v}_i\times\vec{r}_i = \vec{L} =$ constant Gravity has no mass dipole or magnetic dipole by conservation laws.
Quadrapole	(None)	$\sum m_i {r_i}^{\mu}{r_i}^{\nu}$ This is the lowest order radiation for gravity, but we have to take the appropriate number of derivatives. The power is $\sim v/c^8$ (This is a strong restriction on corrections to gravity!)

Lecture 19. Gravitational Radiation.

So lets consider a binary neutron star. Let r be the radius of the binary neutron star, and we are a distance R away from the center of mass. We can doodle the situation:



If $R \ggg r$, then we can approximate this as

$$\bar{h}_{\mu\nu} \approx \frac{4G}{R}\int T_{\mu\nu}(\vec{y},t-R)\,\mathrm{d}^3y. \tag{19.1}$$

Remember we're in a gauge where

$$\partial_\mu\bar{h}^{\mu\nu} = 0 \tag{19.2}$$

[12] C.f., Jackson's *Classical Electrodynamics* [**Jack,** p.145 *et seq.*].

which implies $h^{t\mu}$ is determined by h^{ij}. So for all practical purposes, we can just compute h^{ij}.

Now, for some tricks:

$$\begin{aligned}\partial_k(x^i T^{jk}) &= \delta^i{}_k T^{jk} + x^i \partial_k T^{jk} \\ &= T^{ij} + x^i \partial_k T^{jk} \\ &= T^{ij} - x^i \partial_t T^{jt} + \mathcal{O}(h).\end{aligned} \tag{19.3}$$

Thus we have

$$T^{ij} = \partial_t(x^i T^{tj}) + \begin{pmatrix}\text{total}\\ \text{derivative}\end{pmatrix} + \mathcal{O}(h) \tag{19.4}$$

Using Stoke's theorem, the integral of a total derivative is zero, so we have

$$\bar{h}_{ij} = \frac{2G}{R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \underbrace{y^i y^j T^{tt}(\vec{y}, t-R)}_{\text{mass quadrapole moment}}\,\mathrm{d}^3y \tag{19.5}$$

which confirms the handwaving arguments from the last lecture, which we justified with conservation laws.

Observe the quadrapole moment behaves as

$$I_{ij} \sim mr^2 \tag{19.6}$$

for a binary star with comparable masses. So we see

$$\ddot{I}_{ij} \sim mv^2 \tag{19.7}$$

If this is a gravitationally bound system, it works out... but this means that

$$h \sim G\frac{mv^2}{R}. \tag{19.8}$$

Further we know for gravitational systems

$$v^2 \sim G\frac{m}{r} \tag{19.9}$$

and thus

$$h \sim v^4\left(\frac{r}{R}\right). \tag{19.10}$$

We discuss these things in detail in the following box. A binary neutron star affects the distance by about 1/1000 of the diameter of a nucleus, though.

Box 2. Some Simplifications for Weak Gravitational Radiation

We saw that to first order in perturbation theory

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = 4G\int \frac{T_{\mu\nu}(\mathbf{y}, t-|\mathbf{y}-\mathbf{x}|)}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}^3y \tag{19.11}$$

Let us concentrate on the purely spatial components $\bar{h}_{ij}$ since the remaining components $\bar{h}_{0\mu}$ may be obtained by using the harmonic gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$.

First, suppose an isolated source is at a distance R, and has linear size $r \lll R$. Then to a good approximation,

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{R}\int T_{\mu\nu}(\mathbf{y}, t-|\mathbf{y}-\mathbf{x}|)\,\mathrm{d}^3y \tag{19.12}$$

Now, to lowest order in h, energy conservation implies that

$$\partial_\mu T^{\mu\nu} = 0 = \partial_i T^{i\nu} + \partial_t T^{t\nu} \tag{19.13}$$

We can now use a trick. Note the identities

$$\begin{aligned} \partial_k(x^i T^{kj}) &= \delta^i_k T^{kj} + x^i \partial_k T^{kj} \\ &= T^{ij} - x^i \partial_t T^{tj} \end{aligned} \tag{19.14}$$

$$\begin{aligned} \partial_\ell(x^i x^j T^{t\ell}) &= \delta^i_\ell x^j T^{t\ell} + \delta^j_\ell x^i T^{t\ell} + x^i x^j \partial_\ell T^{t\ell} \\ &= x^j T^{ti} + x^i T^{tj} - x^i x^j \partial_t T^{t\ell} \end{aligned} \tag{19.15}$$

Solving (19.14) for T^{ij}, using the symmetry of T^{ij}, and inserting (19.15), we see that

$$\begin{aligned} T^{ij} &= x^i \partial_t T^{tj} + \partial_k(x^i T^{kj}) \\ &= \frac{1}{2}\partial_t(x^i T^{tj} + x^j T^{ti}) + \frac{1}{2}\partial_k(x^i T^{kj} + x^j T^{ik}) \\ &= \frac{1}{2}\partial_t\Big(\partial_\ell(x^i x^j T^{t\ell}) + x^i x^j \partial_t T^{tt}\Big) + \frac{1}{2}\partial_k\Big(x^i T^{kj} + x^j T^{ik}\Big) \\ &= \frac{1}{2}\partial_t^2(x^i x^j T^{tt}) + \frac{1}{2}\partial_\ell\Big(\partial_t(x^i x^j T^{t\ell}) + x^i T^{\ell j} + x^j T^{i\ell}\Big) \end{aligned} \tag{19.16}$$

We plug this back into Equation (19.12). By stokes theorem, the term involving ∂_ℓ integrates to zero—by assumption, the source is isolated, so the integral can be converted to a surface integral over a surface *outside* the source, where $T^{\mu\nu} = 0$. Hence

$$\begin{aligned} \bar{h}_{ij}(\mathbf{x}, t) &= \frac{2G}{R}\int \partial_t^2(y^i y^j T^{tt})\,\mathrm{d}^3y \\ &= \frac{2G}{R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int y^i y^j T^{tt}(\mathbf{y}, t - |\mathbf{y} - \mathbf{x}|)\,\mathrm{d}^3y. \end{aligned} \tag{19.17}$$

The integral is the quadrupole moment; thus, the metric perturbation goes as the second time derivative of the quadrupole moment.

For an isolated system of a few gravitating bodies (say, stars) with masses of order m and velocities of order v, the quadrupole moment is $\sim mr^2$, and thus $\bar{h} \sim Gmv^2/R$. Furthermore, if the system is gravitationally bound, $v^2 \sim Gm/r$, so $\bar{h} \sim v^4 r/R$.

For a typical binary neutron star, $r \sim 10^7$ km and $v^2 \sim 10^{-7}$; for such a system at a distance of a kiloparsec, this gives $\bar{h} \sim 10^{-21}$.

Nevertheless, in the next five years we will detect these things. LIFO has a photon running around in a pipe with length $L \sim 10^3$ m about 10^3 times which is effectively $L_{eff} \sim 10^9$ m. So there would be constructive or destructive interference.

If we are lucky, we'll see results in a year (in 2010); huge upgrades are due in 2009. Interesting quantum effects decreasing uncertainty in error wavelength; we are nearly saturating uncertainty at this point.

There is something of note: the weak field approximation of the Einstein tensor gives us

$$G_{\mu\nu} = \frac{-1}{2}\Box\bar{h}_{\mu\nu} + (\partial h)(\partial h) \tag{19.18}$$

where

$$(\partial h)(\partial h) = \begin{pmatrix}\text{self-contribution} \\ \text{term}\end{pmatrix} =: t^{(\text{grav})}_{\mu\nu}. \tag{19.19}$$

It's not really a tensor! The energy carried off by the radiation can be found by identifying power $\sim t^{0i}$. So the total power radiated is

$$P \sim t^{0i} R^2. \tag{19.20}$$

Remember that

$$h \sim \frac{G}{R}\ddot{I} \tag{19.21}$$

thus

$$\begin{aligned} t &\sim \frac{1}{G}\dot{h}^2 \\ &\sim G(\dddot{I}/R)^2. \end{aligned} \tag{19.22}$$

Hence the power looks like

$$P \sim R^2 t \sim G\dddot{I}^2. \tag{19.23}$$

Remember

$$I \sim mr^2, \quad \text{so} \quad \dddot{I} \sim mva \tag{19.24}$$

and by Newton's Laws

$$\dddot{I} \sim mv^3/r. \tag{19.25}$$

Then the power looks like

$$\begin{aligned} P &\sim G(mv^3/r)^2 = Gm^2v^6/r^2 \\ &\sim mv^8/r \end{aligned} \tag{19.26}$$

where the last manipulation is again by Newton's Laws. Remember we set $c = 1$, so the power is really small.

EXERCISES

▸ **Exercise 23** (Detecting gravitational radiation I)**.** We considered a coordinate system ("gauge") for weak fields in which $\partial_\mu \bar{h}^{\mu\nu} = 0$. In a region in which the stress-energy tensor $T_{\mu\nu}$ is zero, we can make a further coordinate transformation such that $h_{0\mu} = 0$ and $h = \eta^{\mu\nu} h_{\mu\nu} = 0$. In such a coordinate system, a gravitational plane wave moving along the z axis has a metric (see, e.g., Carroll [**Carroll**] section 7.4)

$$g_{\mu\nu} = \eta_{\mu\nu} + C_{\mu\nu}\cos\bigl(\omega(t-z)\bigr), \quad \text{with} \quad C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{19.27}$$

where h_+ and $h_\times$ are constants.

1. Under a rotation in the x-y plane, the coordinates transform as

   $$\begin{aligned} x &= \bar{x}\cos\theta + \bar{y}\sin\theta \\ y &= \bar{y}\cos\theta - \bar{x}\sin\theta \end{aligned} \tag{19.28}$$

   Find the transformations for h_+ and $h_\times$. For what angle are the two polarizations interchanged?

2. Consider a mass initially located at position $(x_0, 0, 0)$ with vanishing initial velocity, $\mathrm{d}x^i/\mathrm{d}s = 0$. Find the geodesic equation for this object, with the metric (19.27), to first order in h. Show that the object will remain at rest at $(x_0, 0, 0)$. (Note that "at rest" is a coordinate-dependent statement. That's OK, though: for this problem, the gauge conditions have implicitly determined a unique coordinate system.)

3. Consider two mirrors, at rest along the x axis at $(0,0,0)$ and $(L,0,0)$. Using the metric (19.27), compute the round-trip time Δt for a light pulse starting at the origin at time t_0, moving along the x axis, reflecting from the mirror at $x = L$, and returning to the origin. For this computation you can assume that $\Delta t \ll 1/\omega$, so the quantity ωt can be treated as a constant. Your answer should depend on t_0; if it doesn't, you've made a mistake.

4. For flat spacetime, the round-trip time Δt of part 3 is $2L$. The effect of the gravitational wave is the same as if the light traveled a slightly different distance $2L + \Delta L$. For a "strain" of $h \sim 10^{-21}$ (a reasonable estimate for astrophysical sources of gravitational radiation) and a mirror separation $L \sim 4\,\mathrm{km}$ (the length of an arm of the LIGO detector), estimate the maximum value of ΔL. Compare this to the size of a typical atomic nucleus of about 1 fm. Tiny as it is, this change in distance should be detectable in an interferometer!

▸ **Exercise 24** (Detecting gravitational radiation II)**.** Another way to construct a gravitational wave detector is to use a metal bar isolated from external sources of noise. When a gravitational wave passes, the two ends of the bar will accelerate at different rates, setting up oscillations. The relative acceleration of the two ends is determined by the equation of geodesic deviation. The deviation vector X can be interpreted as the distance between the two ends of the rod—since spacetime is assumed to be nearly flat, it makes sense to talk about Cartesian coordinates and distances. In this problem, we will model the bar of metal by two masses at the ends of a spring.

A weak gravitational wave is given by the metric of problem 2. Consider two equal masses on a spring in the x-y plane, initially separated by a distance L, so

$$X_0 = X(t=0) \approx (L\cos\theta, Lsin\theta, 0) \tag{19.29}$$

Say that the spring has natural frequency ω_0, that is, that it exerts a restoring acceleration $a = \omega_0^2(X - X_0)$ when the ends are displaced from their initial positions. We shall look at the effect of a gravitational wave as a driving force for this oscillator (ignoring its tendency to rotate the spring).

Start again with a gravitational wave moving along the z axis, with polarization $h_\times$. Since the wave is assumed to be weak, we can write

$$X(t) = X_0 + \zeta(t) \tag{19.30}$$

with ζ small, and work to lowest order.

1. From the geodesic deviation equation, find the component of gravitational acceleration along the direction of the spring in terms of $h_\times$, k, and—to lowest order—X_0.

2. Solve the equations of motion for ζ subject to this acceleration and the restoring force of the spring. (Strictly speaking, by ζ I mean here "the component of ζ in the direction of the spring," since we're ignoring rotation of the spring in the x-y plane.) Explain the dependence on θ, the angle at which the bar lies in the x-y plane.

Lecture 20. Spherically Symmetric Solutions, Black Holes.

The weakfield equations' advantage: reduces Einstein's equations to linear, uncoupled differential equations. But the problem is it doesn't tell us everything. On the other hand, the full Einstein equation has for each component some 50000 terms if we don't use summation. If we use symmetry, we can reduce the number of terms.

We first look at a static (time independent) and spherically symmetric solution. (In general, if we assume "homogeneous", then we cannot assume "static".)

The technical way to deal with time independence is to say there exists a timelike vector ζ^μ such that the transformation

$$x^\mu \to x^\mu + \zeta^\mu \tag{20.1}$$

doesn't change the metric. We saw in Equation (16.18) the metric changes under transformations of this sort as

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu. \tag{20.2}$$

We have

$$\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0 \tag{20.3}$$

the Killing equation, and ζ^μ is the Killing vector. We showed in Exercise 17 that

$$\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = g_{\mu\rho}\partial_\nu \zeta^\rho + g_{\nu\rho}\partial_\mu \zeta^\rho + \zeta^\rho \partial_\rho g_{\mu\nu}. \tag{20.4}$$

We just take coordinates $\zeta^t = 1$, and $\zeta^i = 0$ (i.e., rescale the time components). Then we have our Killing equation reduce to

$$\zeta^t \partial_t g_{\mu\nu} = \partial_t g_{\mu\nu} = 0 \tag{stationary metric}$$

which is a coordinate independent expression telling us the metric is time independent. There is a subtlety here: a rotating object doesn't appear to change. We introduce another condition, a new symmetry as $t \to -t$. A stationary metric which satisfies is said to be "**Static**". In these coordinates, this is equivalent to

$$g_{it} = 0. \tag{20.5}$$

So

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{20.6}$$

(If we cannot eliminate g_{it}, it's an indicator of a moving system.)

We will examine the static, spherically symmetric metric. The metric shouldn't "change" when moving along an entire loop on a 2-sphere; in some sense there is an invariance. The angular dependence is just

$$\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2$$

so

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 - R^2(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2) - g_{rr}\,\mathrm{d}r^2 \tag{20.7}$$

if we think of the spacetime as foliated spheres, the r determines which spherical disc we're on. We know that g_{tt}, R, g_{rr} depend on r but not on the angles or we wouldn't have spherical symmetry, nor does it have a dependency on t.

Remark 20.1. See Wald [**Wald**] for a good discussion of SO(3) symmetry in General Relativity.

We still have one coordinate degree of freedom—r. We can still choose many different coordinate systems.

If we fix r, we can do it several different ways. The laziest way is to choose r satisfying

$$g_{rr} = 1 \tag{20.8}$$

which happens when r is the proper distance. We can also choose r to satisfy instead

$$g_{rr} = R^2/r^2 \tag{20.9}$$

which then gives us

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 - g_{rr}\underbrace{\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right)}_{\text{usual flat metric}}. \tag{20.10}$$

With this choice we have isotropic coordinates. Both of these make the field equations a wee bit complicated. But there is a third choice! We fix

$$R = r \tag{20.11}$$

which are "areal coordinates" describing a 2-sphere at r with area $4\pi r^2$.

For the Schwarzschild solution, we choose areal coordinates. (Originally Schwarzschild chose coordinates where $\det(g) = 1$.) We then have

$$\mathrm{d}s^2 = A(r)\,\mathrm{d}t^2 - B(r)\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{20.12}$$

In the Einstein vacuum equation we have

$$A = B^{-1} = 1 - \frac{2m}{r} \tag{20.13}$$

But really by integration, the constant term (the "1") is an integration constant.

What if $r \approx 2m$? It gets mildly interesting. At $r = 2m$, something goes horribly awry since $A \to 0$ but $B \to \infty$, and $\mathrm{d}s^2 \to$ **??** This was not understood for a longtime. Back in the 1920s, Panlieve et al. wrote papers with novel coordinate systems but this was largely ignored. Is this singularity from poor choice of coordinates, or from something deep and not easily understandable in nature?

Lets examine as an example

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2, \qquad \text{let}\quad x = \frac{1}{u-1} \\ &= \frac{\mathrm{d}u^2}{(u-1)^2} + \mathrm{d}y^2 \end{aligned} \tag{20.14}$$

at $u = 1$ we have a singularity!

We can try to look at coordinate independent quantities as a first step. For example

$$R = 0 \tag{20.15a}$$

$$R_{\mu\nu}R^{\mu\nu} = 0. \tag{20.15b}$$

but

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}. \tag{20.15c}$$

At $r = 2m$, nothing scary happens! It turns out every scalar we can form from the curvature behaves unsuspiciously at $r = 2m$. Physically, there doesn't appear locally anything new and scary.

On the other hand, for $r < 2m$, the temporal component and radial component switch. That is

$$A(r) < 0, \quad \text{and} \quad B(r) < 0 \tag{20.16}$$

so spacelike becomes timelike, and timelike becomes spacelike.

Lecture 21. Eddington–Finkelstein, Kruskal–Szekeres Coordinates.

We have the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \tag{21.1}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \tag{21.2}$$

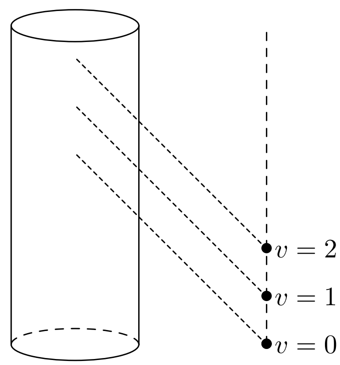
is the usual notation for the metric on $(n-2)$-sphere.

Theorem 21.1 (Birkhoff)**.** *Spherically symmetric vacuum field equations imply the Schwarzschild solution.*

Physically if we have a spherically symmetric field, we can treat it as concentrated at a point, all gravitation waves have to be spherical, but they're really quadrapole or higher order.

We chose t by demanding a time variable such that everything's independent of it. Our solution is still perfectly good. We would like some physical time component with some physical meaning.

We have an observer shooting off light to the cylinder of constant radius. We could equally make this baseballs instead of photons, which is useful for collapsing spherical shells. The diagram is



We have light (null geodesics) and it's only radial (so we have $d\Omega^2 = 0$). Then we have

$$ds^2 = 0 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \tag{21.3}$$

which implies

$$\pm\left(1 - \frac{2m}{r}\right)^{-1} dr = dt. \tag{21.4}$$

We introduce a coordinate r_* such that

$$dt = \pm dr_* \tag{21.5}$$

so

$$r_* = r + 2m \ln\left|\frac{r}{2m} - 1\right|. \tag{21.6}$$

Either $t - r_* = u$ or $t + r_* = v$ where u, v are constants and the same as geodesics v-labeling.

We eliminate t from the metric, so we get

advanced Eddington–Finkelstein Coordinates

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2\,dv\,dr - r^2\,d\Omega^2 \tag{21.7}$$

This is the same metric expressed in different coordinates. They are called the "**advanced Eddington–Finkelstein Coordinates**". If we used u instead of v, we'd get *retarded* Eddington–Finkelstein coordinates.

The coordinates with v yields a bit of information. The null radial geodesics satisfy

$$\left(1 - \frac{2m}{r}\right) dv^2 = 2\,dr\,dv \tag{21.8}$$

The solutions are either

$$v = \text{const.}, \quad \text{or} \qquad \frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2m}{r}\right). \tag{21.9}$$

Outgoing geodesics asymptotically approach $r = 2m$. One thing to note is that $r = 2m$ is a null geodesic (i.e., it's lightlike)!

Definition 21.2. A "**Killing Horizon**" is when a Killing vector changes from timelike to lightlike.

Kruskal–Szekeres Coordinates

The next thing to do is try replacing r with u. It's easier to first define

$$U = \exp(-u/4m), \quad \text{and} \quad V = \exp(v/4m). \tag{21.10}$$

We find (plugging these back into the Schwarzschild solution, we have

$$\mathrm{d}s^2 = \frac{32m^3}{r}\exp(-r/2m)\,\mathrm{d}U\,\mathrm{d}V - r^2\,\mathrm{d}\Omega^2 \tag{21.11}$$

where $r = r(U, V)$ is defined by

$$\left(\frac{r}{2m} - 1\right)\exp(r/2m) = -UV, \quad \text{and} \quad \frac{U}{V} = \exp(-t/2m). \tag{21.12}$$

These coordinates are called "**Kruskal–Szekeres Coordinates**". One of the nice things about these coordinates: nothing in particular goes horribly awry when the metric goes to zero, everything's nicely behaved.

Consider $r = 2m$, in our new coordinates this is

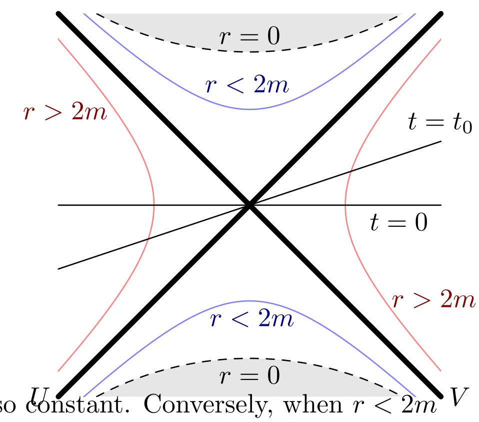
$$UV = 0. \tag{21.13}$$

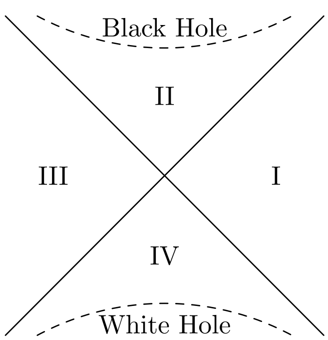
Our event horizon has two solutions

$$U = 0, \quad \text{or} \quad V = 0. \tag{21.14}$$

These are null geodesics, so $U = 0$, $V = 0$ gives two lines are 45° angles.

When $r = 0$, we have $UV = 1$. This is a hyperboloid. The hyperboloid is drawn to the right with dashed lines to denote a genuine singularity (usually, it's with a "squiggly" line). We also have the situation when r is a constant and $r > 2m$; then $UV < 0$ is also constant. Conversely, when $r < 2m$ is a constant, we have $UV > 0$ be constant. These situations are drawn in red and blue to the right.





We have 4 regions labeled as shown on the left. Regions I and IV are outside of the black hole. Regions II and III are inside of the black hole. If we enter these regions, we necessarily hit the singularity (we'd need to travel faster than light to escape the region). Note that a "white hole" is just a time-reversed black hole. The regions relevant for black holes are I and II, whereas I and III are relevant for white holes.

We see black holes but not white holes. Why? Well, we're working with $T^{\mu\nu} = 0$. We're working with matter collapsing, all we really have for the vacuum is part of region I and part of region II.

We think of white hole/black hole as an eternal black hole perhaps formed by early fluctuations of the young universe (Hsu suggests something along these lines [**30**]), perhaps this is a wrong intuition.

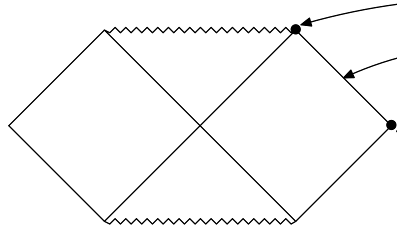
There is no solution of the vacuum with an isometry which takes region III into any other region. Presumably the white holes radiate away.

Penrose Diagrams

Now we had

$$\mathrm{d}s^2 = (\dots)\mathrm{d}U\,\mathrm{d}V - r^2\,\mathrm{d}\Omega^2. \tag{21.15}$$

If we multiply by a function, it doesn't change null geodesics. Penrose invented a trick to make $r = \infty$ into a finite distance, a doodle called a "**Penrose Diagram**". For the Schwarzschild metric, we have the diagram:



This distorts area but preserves the causal structure[13].

EXERCISES

▸ **Exercise 25** (Black holes and trapped surfaces)**.** The Schwarzschild metric in Kruskal-Szekeres coordinates is

$$\mathrm{d}s^2 = \frac{32m^2}{r}\mathrm{e}^{-r/2m}\left(-\mathrm{d}T^2 + \mathrm{d}X^2\right) + r^2\,\mathrm{d}\Omega^2 \tag{21.16}$$

where r is viewed as a function of X and T:

$$\left(\frac{r}{2m} - 1\right)\mathrm{e}^{r/2m} = X^2 - T^2$$

a. Show that radial null geodesics emitted from the two-sphere $(T_0, X_0, \theta, \varphi)$ are described by the equation of motion

$$X - X_0 = \epsilon(T - T_0), \quad \theta = \text{const.}, \quad \varphi = \text{const.} \tag{21.17}$$

where $\epsilon = 1$ for outgoing geodesics and $\epsilon = -1$ for ingoing geodesics.
b. Consider the new two-sphere formed by the wave front at time T of these radial geodesics. Show that the area of this sphere is $A = 4\pi r^2(X, T)$. (Hint: this is not completely obvious; you need to think about how area is defined in a curved spacetime.)
c. In region I, $X_0 > 0$ and $-X_0 < T_0 < X_0$. By considering $\mathrm{d}A/\mathrm{d}T$, show that the the area A increases with T for outgoing geodesics, and decreases for ingoing geodesics.
d. In region II (inside the event horizon), $T_0 > 0$ and $-T_0 < X_0 < T_0$. Show that in this region, A decreases with T for both ingoing and outgoing geodesics. This is the condition that the initial sphere $(T_0, X_0, \theta, \varphi)$ is a trapped surface.

Lecture 22. Brief Cosmology.

We observe over long distances the universe is homogeneous and isotropic. "Homogeneous" means if we take two regions of space, we cannot distinguish them. Certainly this is not true at small distances (e.g., compare a human being and a rock). "Isotropic" says if you're at one point, every direction appears the same.

What is a homogeneous non-isotropic shape? A cylinder!

A sphere with a variable density depending on the distance from the equator is isotropic but non-homogeneous.

Lets look at metrics that are homogeneous and isotropic. First thing to notice is that this is a coordinate dependent statement.

Lets Choose a time t' such that space at constant t' is homogeneous. So

$$\mathrm{d}s^2 = A\,(\mathrm{d}t')^2 + 2B_i\,\mathrm{d}x^i\,\mathrm{d}t' + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{22.1}$$

Homogeneity and isotropy requires $B_i = 0$, otherwise B_i picks out a direction. So we throw it away. Similarly, A must be a function of time only.

[13]This is because the "causal structure" is determined by the angles between intersecting curves; it's a conformal transformation.

We choose a t such that

$$\mathrm{d}t = \sqrt{-A(t')}\,\mathrm{d}t'. \tag{22.2}$$

This locally rescales the t' coordinate. We now can write

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{22.3}$$

Lets examine the (fixed time t) spatial metric's curvature:

$$R^{ij}{}_{kl} \to R^A{}_B \tag{22.4}$$

where $A = 12, 13, 23 = ij$. We have a 3×3 matrix we can look at its eigenvalues λ_1, λ_2, λ_3 and its eigenvectors. We claim they are all equal, otherwise we could pick out the largest eigenvector $v^B = v^{ij}$ then obtain a 3-vector $\varepsilon_{kij}v^{ij}$ and this picks out a direction. But we cannot allow this! So we have

$$R^A{}_B = k\delta^A{}_B \tag{22.5}$$

and k cannot depend on spatial directions (otherwise its gradient picks out some preferred direction). At best we have $k = k(t)$ be a function of time. So

$$R_{ij\ell m} = k(g_{i\ell}g_{jm} - g_{mi}g_{j\ell}) \tag{22.6}$$

and we have a "space of constant curvature". We then find

$$g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-k^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{22.7}$$

where $k = -1, 0, 1$. For $k = 1$, we write $r = \sin\psi$ and thus we obtain

$$g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{22.8}$$

which describes S^3. But $k = 0$ gives us

$$g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{22.9}$$

which is flat. Last $k = -1$ we pick $r = \sinh(\psi)$ and obtain

$$g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \mathrm{d}\psi^2 + \sinh^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{22.10}$$

which describes hyperbolic 3-space $\mathbb{H}^3$.

We have the metric as (at fixed time)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \underbrace{a^2(t)}_{\text{scale factor as function of time}}\widetilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{22.11}$$

What about the stress-energy tensor? We have

$$T^{0i} = 0 \tag{22.12a}$$

otherwise we'll have a preferred direction, and

$$T^{00} = \rho(t) \tag{22.12b}$$

$$T^i{}_j = p(t)\delta^i{}_j \tag{22.12c}$$

otherwise if we have $T^i{}_j$ be a function of position, its gradient would determine some preferred direction. Note

$$\begin{aligned}\rho &= \text{energy density}\\ p &= \text{pressure.}\end{aligned}$$

We plug this into the Einstein field equation (using units where $G_N = 1$), we end up with the Friedmann equations: *Friedmann Equations*

$$\frac{3\dot{a}^2}{a^2} = 8\pi\rho - \frac{3k}{a^2} \tag{22.13a}$$

$$\frac{3\ddot{a}}{a} = -4\pi(\rho + 3p). \tag{22.13b}$$

We need a third equation, which is an equation of state, i.e., looks like *Equation of State*

$$p = p(\rho). \tag{22.14}$$

The simplest choice is

$$p = w\rho \tag{22.15}$$

for some constant of proportionality w. If $w = 0$, we have dust; for $w = 1/3$ we have radiation (it follows from Maxwell's equations); and if $w = -1$, then we have a cosmological constant. We can have some combination (e.g., a universe with dust, radiation, and a cosmological constant). To get a cosmology, we choose some equation of state then plug it back in.

The next step is to examine perturbations. We need to do a weak-field approximation to a background metric which is not necessarily flat. *Perturbations*

From the Friedmann equations (22.13), we see

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(3\dot{a}^2) &= 6\dot{a}\ddot{a} = -8\pi(\rho + 3p)a\dot{a}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(8\pi\rho a^2)\end{aligned} \tag{22.16}$$

thus

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0. \tag{22.17}$$

If $p = w\rho$, then this is easy to solve:

$$\rho = \rho_0 a^{-3(1+w)}. \tag{22.18}$$

If $w = 0$, this is a conservation of dust $\rho = \rho_0 a^{-3}$. For radiation ($w = 1/3$) we get an extra factor for redshift. Observe when we have a cosmological constant ($w = -1$), we have ρ be constant.

Suppose $\rho + 3p > 0$, there is an initial singularity. This isn't due to the homogeneity and isotropy conditions, years ago Hawking, Penrose and others proved if $\rho + 3p > 0$, then there is an initial singularity. If $\rho + 3p < 0$, we have a big bounce.

EXERCISES

▸ **Exercise 26** (de Sitter and anti-de Sitter space)**.** Consider a homogeneous, isotropic cosmology with a nonzero cosmological constant Λ and an otherwise vanishing stress-energy tensor. The cosmological constant can be thought of as part of the stress-energy tensor, so this setting amounts to saying that

$$\rho = -p = \frac{\Lambda}{8\pi}$$

(in units $G_N = 1$).

a. Solve the Friedmann equations (see Carroll [**Carroll**] section 8.3) for the case $\Lambda > 0$, considering all values of the spatial curvature parameter k. This solution is called de Sitter space.

b. Do the same for $\Lambda < 0$. This solution is called anti-de Sitter space.

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